

Topics on the Applied Math Qual

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Preface

This text was mostly prepared during Summer 2009 while studying for the Applied Math qualifying exam at University of Wisconsin-Madison. I then finished and polished it in my spare time during the following year. The text is largely based on my course notes from Math 703 (Fall 2008 - Prof. Leslie Smith) and Math 704 (Spring 2009 - Prof. Fabian Waleffe). The reader should be aware that the content of these courses, and the questions on the qualifying exam, vary depending on the professor.

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Math 703

1 Equilibrium

1.1 Discrete Systems

The two common examples of discrete systems are mass/spring systems, and LRC circuits. Both of these systems (and in general, other discrete systems) can be represented by a graph. In the mass/spring system, the nodes of the graph are masses and the edges are springs while in the LRC circuit system, you consider the voltage at the nodes and the current through the edges. Here, we will only consider mass/spring systems.

When setting up a mass/spring equilibrium problem, there are 6 matrices and 3 equations that arise.

Table 1: Set-up for a discrete mass/spring system

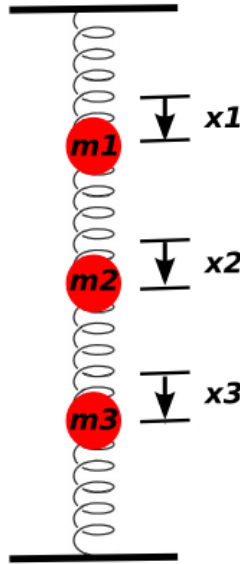
\vec{x}	$[n \times 1]$	displacement of masses	n masses
\vec{e}	$[m \times 1]$	elongation of springs	m springs
\vec{y}	$[m \times 1]$	internal forces (on springs)	
\vec{f}	$[n \times 1]$	external forces (on masses)	$\vec{e} = A\vec{x}$
A	$[m \times n]$	difference matrix	$\vec{y} = C\vec{e}$
C	$[m \times m]$	constitutive law	$\vec{f} = \pm A^T\vec{y}$

NOTE: The sign in the third equation depends on how the matrix C is defined and whether \vec{y} is measuring the force *on* the springs, or the force exerted *by* the springs. In practice, it is convenient to take \vec{y} to be the force on the springs, and let C be a positive definite matrix. In this case, the third equation should be $\vec{f} = A^T\vec{y}$

Combining the three equations yields

$$\boxed{A^T C A \vec{x} = \vec{f}} \quad (1.1)$$

Example. (A 2008 #3b) Consider the system of four springs and three masses shown below:



- Let x_i denote the displacement of mass m_i after the masses are inserted between the springs.
- Let c_i denote the spring constants and assume that a linear Hooke's law applies for each spring.
- Let y_i denote the force on spring i .
- Assume that $f_i = m_i g$ is the force of gravity for each mass, and that g is the gravitational constant.

Write a system of equations relating

- \vec{x} to \vec{y}
- \vec{y} to \vec{f}
- \vec{x} to \vec{f}
- Rewrite (iii) as a minimization problem for a quadratic functional $P(x)$

Solution. We can find each of the required relations by setting up the matrices and equations in Table 1.

- i. We know that $\vec{e} = A\vec{x}$ and $\vec{y} = C\vec{e}$, so together they provide a relation between \vec{x} and \vec{y} : $\vec{y} = CA\vec{x}$. We just need to define the matrices A and C . From the picture, if we define the downward direction as positive (i.e. if x_1 is positive), we can see that the elongations are given by the equations

$$\begin{aligned} e_1 &= x_1 \\ e_2 &= x_2 - x_1 \\ e_3 &= x_3 - x_2 \\ e_4 &= -x_3 \end{aligned}$$

So we know that

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

For the C matrix, note that the problem defines \vec{y} as the force **on** the springs. So they have already chosen which convention to use, and you just have to make sure you understand it correctly. Since down is positive, and the force *on* the top spring is downward, we want C to have positive entries. That is,

$$A = \begin{bmatrix} c_1 & 0 & 0 & 0 \\ 0 & c_2 & 0 & 0 \\ 0 & 0 & c_3 & 0 \\ 0 & 0 & 0 & c_4 \end{bmatrix}$$

- ii. The relation between \vec{y} and \vec{f} is given by the third basic equation. And with the convention above, it is $\vec{f} = A^T\vec{y}$.
- iii. To relate \vec{x} to \vec{f} we just need to combine the previous two parts: $\vec{f} = A^TCA\vec{x}$.
- iv. As you'll see in Section 1.2, the solution (iii) is the result of minimizing the functional

$$P(\vec{x}) = \frac{1}{2}\vec{x}^T A^T C A \vec{x} - \vec{x}^T \vec{f}. \quad \square$$

1.2 Minimization and Least Squares Data Fitting

First, a definition.

Definition: A matrix square matrix A is **positive definite** if $\vec{x}^T A \vec{x} > 0$ for all vectors $\vec{x} \neq 0$. (In which case $\vec{x} = 0$ is the global minimum.)

Here is the main minimization functional for discrete systems. For the mass-spring systems, this function is the potential energy of the system. That is, the equilibrium state is exactly the state that minimizes the potential energy.

Theorem: If A is positive definite then the functional

$$P(\vec{x}) = \frac{1}{2} \vec{x}^T A \vec{x} - \vec{x}^T \vec{b} \quad (1.2)$$

is minimized where $\frac{1}{2}(A + A^T)\vec{x} = \vec{b}$.

If A is also symmetric, then this is where $A\vec{x} = \vec{b}$.

Theorem: If A is $[n \times m]$ ($n > m$) and has linearly independent columns, then

1. $A^T A$ is symmetric positive definite
2. If C is symmetric positive definite, then $A^T C A$ is symmetric positive definite.

In the preceding example, A was $[4 \times 3]$ and C was positive definite, so by the second theorem, $A^T C A$ was symmetric positive definite. So we could apply the first theorem using \vec{f} as \vec{b} , and thinking of $A^T C A$ as the square matrix A in the theorem.

Least Squares Data Fitting

This minimization principle can also be used to fit a function to an overdetermined system. For example, we may have many data points, and want to find the line that fits the data best. We choose the “best” function that fits the data by minimizing the square of the norm of the error.

Given A ($[n \times m]$ with $n > m$) with linearly independent columns, and the overdetermined system $A\vec{x} = \vec{b}$, we want to minimize $\|\vec{e}\|^2$ where $\vec{e} = A\vec{x} - \vec{b}$.

The solution is easy to derive (and also easy to memorize!):

$$\begin{aligned}
 \|\vec{e}\|^2 &= \vec{e}^T \vec{e} \\
 &= (A\vec{x} - \vec{b})^T (A\vec{x} - \vec{b}) \\
 &= (\vec{x}^T A^T - \vec{b}^T) (A\vec{x} - \vec{b}) \\
 &= \vec{x}^T A^T A\vec{x} - \vec{x}^T A^T \vec{b} - \vec{b}^T A\vec{x} + \vec{b}^T \vec{b} \\
 &= \vec{x}^T A^T A\vec{x} - 2\vec{x}^T A^T \vec{b} + \vec{b}^T \vec{b}
 \end{aligned}$$

In the last step, we note that all of the terms are scalars, and that taking the transpose of a scalar does not change it. Now, finding the \vec{x} that minimizes the above function, is the same as finding the \vec{x} that minimizes

$$\frac{1}{2} \vec{x}^T A^T A\vec{x} - \vec{x}^T A^T \vec{b}.$$

So the solution is when

$$\boxed{A^T A\vec{x} = A^T \vec{b}} \quad (1.3)$$

1.3 Continuous Systems

There are two types of continuous equilibrium systems that we encounter - the stretching rod and the bending rod.

Stretching Rods

A vertical rod that is stretching under its own weight can be thought of as infinitely many small masses connected by infinitely many springs. So the **stretching rod** system is very similar to the mass spring system shown earlier. The main difference is that the difference matrix A becomes the differential operator $\frac{d}{dx}$. The transpose of A is just $-\frac{d}{dx}$ (Also, by convention, x is used to denote the position on the rod, and $u(x)$ is used to denote the displacement of points on the rod. So the notation is different from the discrete case.)

So the continuous version of $A^T C A\vec{x} = \vec{f}$ is

$$\boxed{-\frac{d}{dx} \left[C(x) \frac{du(x)}{dx} \right] = f(x)} \quad (1.4)$$

This is a second order differential equation, so we need two boundary conditions. They come from both ends of the rod. At the fixed end of the

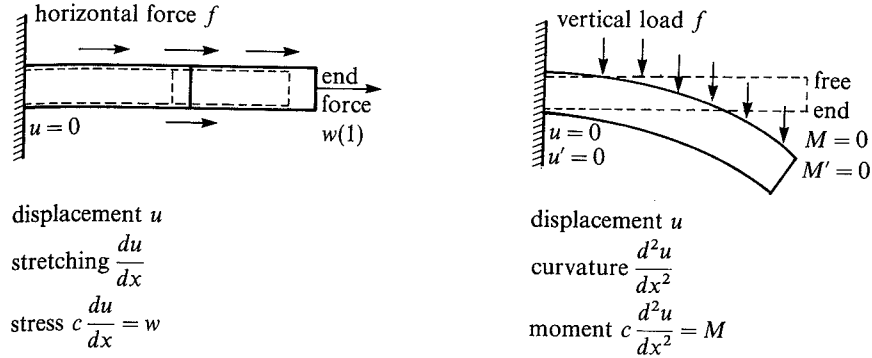


Figure 1: Stretching and Bending Rods (*Source: Strang p. 173*)

rod there can be no displacement. At the free end (here assumed to be $x = 1$) of the rod, there can be no stretching (no internal force). This gives the following boundary conditions:

$$\begin{aligned} u(0) &= 0 \\ C(1)u'(1) &= 0 \end{aligned}$$

Just like in the discrete case, this is the solution to an appropriate minimization principle. Since we are dealing with functions now, the minimization is done via calculus of variations (see Section 5). For the stretching rod, the minimized functional is

$$P[u] = \int_0^L \left(\frac{1}{2}C(x) \left[\frac{du}{dx} \right]^2 - fu \right) dx \quad (1.5)$$

Bending Rods

In the discrete version of the **bending rod**, the matrix A is a second difference matrix. So in the continuous case, it becomes $\frac{d^2}{dx^2}$. One way to think about this is that since the rod is bending, we care about its curvature (how much it bends), and curvature is related to the second derivative.

The solution for the bending rod is

$$-\frac{d^2}{dx^2} \left[C(x) \frac{d^2 u(x)}{dx^2} \right] = f(x) \quad (1.6)$$

This is a fourth order differential equation, so we need four boundary conditions. Two come from the fixed end, and two from the free end. At the free end, there is no internal force and no stretching. So we have the conditions:

$$\begin{aligned} u(0) &= 0 \\ u'(0) &= 0 \\ M(L) &= 0 \\ M'(L) &= 0 \end{aligned}$$

where $M(x) = C(x) \frac{d^2 u}{dx^2}$ is the internal force (the equation is the continuous version of $\vec{y} = C\vec{e}$).

The variational principle in this case is

$$P[u] = \int_0^L \left(\frac{1}{2} C(x) \left[\frac{d^2 u}{dx^2} \right]^2 - fu \right) dx \quad (1.7)$$

Qualifying Exam Problems: J2000 #5, J2001 #3, J2007 #1, A2007 #2, A2008 #3, J2009 #2

2 Sturm-Liouville ODEs

A Sturm-Liouville ODE is a two point boundary problem with the standard form:

$$\begin{aligned} -[p(x)y'(x)]' + q(x)y(x) &= \lambda r(x)y(x) \\ a_1 y(0) + a_2 y'(0) &= 0 \\ b_1 y(1) + b_2 y'(1) &= 0 \end{aligned} \quad (2.1)$$

such that $p(x), p'(x), q(x), r(x)$ are all real and continuous in $[0, 1]$ and a_i, b_i are real. Note: In general, the domain can be any interval including $(-\infty, \infty)$. There are usually many (infinite) values of the parameter λ that each allow a different solution. The values of λ are called eigenvalues, and the associated solutions are called eigenfunctions.

The Sturm-Liouville problem (S-L problem) is **regular** or **non-singular** if $r(x), p(x) > 0$ on $[0, 1]$. In this case, the operator, L , and boundary conditions are *Hermitian* (self-adjoint), and the S-L problem has the following properties:

1. All eigenvalues are real and simple.

2. All eigenfunctions are real.
3. The eigenfunctions can be chosen orthogonal with respect to $r(x)$.
4. The eigenfunctions are complete (i.e. they form a basis for all functions).

A S-L problem is **singular** if $r(x), p(x) \leq 0$ on $[0, 1]$. Usually the problem is that $r(x)$ or $p(x)$ is equal to zero somewhere. In some cases, it is possible to do a simple change of variable to turn the singular problem into a non-singular problem. Alternatively, if you can show that the singular S-L is Hermitian, then it will have the nice properties above. That is, you want to show that

$$\int vL[u] dx = \int uL[v] dx \quad (2.2)$$

After two integration by parts, this is equivalent to showing that

$$\boxed{p(x) [u'(x)v(x) - u(x)v'(x)]_0^1 = 0} \quad (2.3)$$

2.1 Non-Homogeneous Sturm-Liouville Problem

The standard form given above (2.1) is actually the homogeneous S-L problem. The non-homogeneous S-L problem has the form

$$\boxed{\begin{aligned} -[p(x)y'(x)]' + q(x)y(x) &= \mu r(x)y(x) + f(x) \\ a_1y(0) + a_2y'(0) &= 0 \\ b_1y(1) + b_2y'(1) &= 0 \end{aligned}} \quad (2.4)$$

where μ is a given constant, $f(x)$ is a given piece-wise continuous function, $p(x), p'(x), q(x), r(x)$ are all real and continuous in $[0, 1]$, $r(x), p(x) > 0$ on $[0, 1]$, and a_i, b_i are real.

The method for solving a non-homogeneous S-L problem is as follows:

1. Solve the homogeneous version (replace μ with λ and ignore the $f(x)$) to find eigenvalues λ_n and eigenfunctions ϕ_n .
2. Normalize the eigenfunctions so that

$$\int_0^1 r(x)\phi_n^2(x) dx = 1.$$

3. Since the ϕ_n form a complete basis for all functions, the solution to the non-homogeneous problem, $\phi(x)$, must be a linear combination of them. That is,

$$\phi(x) = \sum_{n=1}^{\infty} b_n \phi_n(x).$$

This $\phi(x)$ will satisfy the boundary conditions because all of the ϕ_n do.

4. We can also write $f(x)/r(x)$ as a linear combination of the eigenfunctions.

$$\frac{f(x)}{r(x)} = \sum_{n=1}^{\infty} c_n \phi_n(x).$$

Since the eigenfunctions are orthogonal and normalized, we can solve for the c_n .

$$\begin{aligned} \int_0^1 r(x) \phi_m(x) \sum_{n=1}^{\infty} c_n \phi_n(x) dx &= \int_0^1 r(x) \phi_m(x) \frac{f(x)}{r(x)} dx \\ \sum_{n=1}^{\infty} c_n \int_0^1 r(x) \phi_m(x) \phi_n(x) dx &= \int_0^1 \phi_m(x) f(x) dx \\ c_m &= \int_0^1 \phi_m(x) f(x) dx \end{aligned}$$

5. Now plug everything ($\phi, f/r$) into the non-homogeneous equation:

$$\begin{aligned} L[\phi] &= \mu r \phi + f \\ \sum_{n=1}^{\infty} L[b_n \phi_n] &= \mu r \sum_{n=1}^{\infty} b_n \phi_n + r \sum_{n=1}^{\infty} c_n \phi_n \\ \sum_{n=1}^{\infty} b_n L[\phi_n] &= \mu r \sum_{n=1}^{\infty} b_n \phi_n + r \sum_{n=1}^{\infty} c_n \phi_n \\ \sum_{n=1}^{\infty} b_n \lambda_n r \phi_n &= \mu r \sum_{n=1}^{\infty} b_n \phi_n + r \sum_{n=1}^{\infty} c_n \phi_n \\ \int_0^1 \phi_m \sum_{n=1}^{\infty} b_n \lambda_n r \phi_n dx &= \int_0^1 \phi_m \mu r \sum_{n=1}^{\infty} b_n \phi_n dx + \int_0^1 \phi_m r \sum_{n=1}^{\infty} c_n \phi_n dx \\ b_m \lambda_m &= \mu b_m + c_m \end{aligned}$$

This last equation determines the type of solution to the S-L problem.

6. There are two cases. **Case 1:** If $\mu \neq \lambda_n$ for all n , then the above equation can be solved for b_n ,

$$b_n = \frac{c_n}{\lambda_n - \mu}$$

and so the (*unique*) solution to the non-homogeneous S-L problem is

$$\phi(x) = \sum_{n=1}^{\infty} b_n \phi_n(x) = \sum_{n=1}^{\infty} \frac{c_n}{\lambda_n - \mu} \phi_n(x).$$

Case 2: If $\mu = \lambda_k$ for some k , then the above equation reduces to $c_k = 0$ (for that k only!). But we know the values of the c_k , so if this $c_k \neq 0$, there is no solution to the problem. If this $c_k = 0$, then b_k is arbitrary, so there are infinitely many solutions. (In this case, $f(x)$ is orthogonal to $\phi_k(x)$.)

This result is summed up by the Fredholm Alternative Theorem.

Fredholm Alternative Theorem. *For a given value of μ , either the non-homogeneous problem has a unique solution for each $f(x)$, or the homogeneous problem has a non-trivial solution.*

2.2 Estimating the Smallest Eigenvalue

There are a few times where you have to estimate the smallest eigenvalue of a homogeneous S-L problem.

$$-[p(x)y'(x)]' + q(x)y(x) = \lambda r(x)y(x)$$

If the S-L is non-singular (or you've proven that the operator is Hermitian), then you can use the *Rayleigh-Ritz* method. The basis of this method is the Rayleigh equation, which can be derived by plugging an eigenfunction, ϕ_n , into the S-L equation, multiplying the whole equation by the same eigenfunction, ϕ_n , and integrating.

$$\begin{aligned} \int_0^1 -[p\phi_n']' \phi_n + q\phi_n\phi_n \, dx &= \lambda_n \int_0^1 r\phi_n\phi_n \, dx \\ -[p\phi_n']\phi_n|_0^1 + \int_0^1 p(\phi_n')^2 + q(\phi_n)^2 \, dx &= \lambda_n \int_0^1 r(\phi_n)^2 \, dx \end{aligned}$$

Solving for λ_n gives the Rayleigh quotient:

$$\lambda_n = \frac{-[p\phi_n']\phi_n|_0^1 + \int_0^1 p(\phi_n')^2 + q(\phi_n)^2 dx}{\int_0^1 r(\phi_n)^2 dx} \quad (2.5)$$

This equation holds true for all n so (in theory) we can use it to find all of the eigenvalues (and eigenfunctions). However, they must be found in order (λ_1 first).

To estimate λ_1 , make a guess for the corresponding eigenfunction, ϕ_1 . Note that the real ϕ_1 will satisfy the boundary conditions, so your guess should as well. This will *always* be an **overestimate**, so you can improve the estimate by using multiple guesses for ϕ_1 and using the smallest λ_1 that you find.

Once you find ϕ_1 (the argmin of the quotient over all functions that satisfy the boundary conditions), you can estimate λ_2 by guessing functions, ϕ_2 , that satisfy the boundary conditions and are orthogonal to ϕ_1 .

If the S-L has a *regular singular point* (see Section 3.1) that you can't get rid of by multiplying through or changing variables, it is still possible to estimate the smallest eigenvalue. The method is as follows:

1. Expand $y(x)$ as a series solution about the regular singular point x_0

$$y(x) = (x - x_0)^r \sum_{n=0}^{\infty} a_n (x - x_0)^n.$$

2. Plug this into the ODE and get a recursion relation for the coefficients (i.e. solve for a_n).
3. Use the boundary condition and the first few terms of the series to solve for the eigenvalue.

I'm not sure why this second method gives the smallest eigenvalue, or if it will always work.

Qualifying Exam Problems: A2000 #1, J2001 #4, J2002 #1, A2004 #3, A2005 #2

3 Local Analysis Methods

Given a homogeneous linear ODE, we can examine it at different points to determine the local behavior. A point x_0 can be classified as either a regular

point (aka ordinary point), a regular singular point (RSP), or an irregular singular point (ISP).

The solution in the neighborhood of regular points and regular singular points can be expressed as a series. Around irregular singular points, we can only find the asymptotic behavior of the solution.

3.1 Singular Points

Classifying Points

The first step in classifying points is to write the ODE in standard form

$$y^{(n)}(x) + p_{n-1}(x)y^{(n-1)}(x) + p_{n-2}(x)y^{(n-2)}(x) + \dots + p_0(x)y(x) = 0$$

by dividing through by the “leading coefficient”.

Regular points: x_0 is a regular point if, for all i , $p_i(x)$ is analytic (infinitely differentiable) in a neighborhood of x_0 .

Regular singular points: x_0 is a regular singular point if it is not a regular point and for all i , $(x - x_0)^{n-i}p_i(x)$ is analytic in a neighborhood of x_0 .

Irregular singular points: x_0 is an irregular singular point if it is neither a regular point nor a regular singular point.

NOTE: To classify the point at infinity, map it to zero with the transformation $x = 1/t$. Remember that all the derivatives will change too!

Local Behavior Near Classified Points

- Near a **regular point**, the solution should be well behaved, at $x = x_0$, and can be represented by a Taylor series

$$y(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n \quad (3.1)$$

To find the coefficients a_n , plug the Taylor series into the differential equation and equate coefficients.

- Near a **regular singular point**, the solution may or may not be singular at $x = x_0$. The solution can be expressed as a “Frobenius series”

$$y(x) = (x - x_0)^r \sum_{n=0}^{\infty} a_n(x - x_0)^n, \quad r \in \mathbb{Q} \quad (3.2)$$

To find the coefficients a_n and the exponent r , plug the expression into the differential equation. Equating coefficients will give recurrence relations for the unknowns. By convention, we assume $a_0 \neq 0$ so that the value of r is well defined.

If this substitution only yields one solution, the second linearly independent solution is of the form

$$y_2 = y_1 \ln(x - x_0) + \sum_{n=0}^{\infty} b_n (x - x_0)^n. \quad (3.3)$$

Use the same process to find the coefficients b_n .

- Near an **irregular singular point**, the solution behaves asymptotically like

$$y(x) \sim e^{S(x-x_0)} \quad \text{as } x \rightarrow x_0 \quad (3.4)$$

This solution is more singular than any power of $(x - x_0)$ as $x \rightarrow x_0$. However, part of $e^{S(x-x_0)}$ can be expanded as a series

$$y(x) \sim e^{\beta(x-x_0)} \sum_{n=0}^{\infty} a_n (x - x_0)^{\alpha n} \quad \text{as } x \rightarrow x_0 \quad (3.5)$$

where α is typically not an integer. Additionally, the series is usually divergent (but asymptotic as $x \rightarrow x_0$).

To find the function $S(x - x_0)$, use the method of dominant balance (below).

NOTE: Frequently problems will only ask for the **leading behavior** of the solution. The leading behavior is $e^{\tilde{S}(x)}$ where $\tilde{S}(x)$ is all the terms of S that do not go to zero as $x \rightarrow x_0$. This usually only the first two or three terms.

Method of Dominant Balance

The method of dominant balance is one of the hardest subjects on the applied math qualifying exam. This is because there is a large amount of “fuzzy math” involved. Here are the basic steps:

1. Calculate the necessary derivatives of $y \sim e^{S(x)}$ and plug them into the differential equation. Divide through by $e^{S(x)}$. What remains is an ODE for $S(x)$.

2. Look for terms that are obviously much smaller than others, and cross them off (we'll ignore them). For example, if you have $(1 + 3x)S'$ and the singular point is $x_0 = 0$, then the $3x$ will be much much smaller than 1 as $x \rightarrow x_0$. So we can ignore the $3xS'$ and keep the S' .
3. From the remaining terms, pick the two (or three if you have to) that you think will be the dominant ones (i.e. once you solve for them, they will be much bigger than the other terms you ignore). This is a bit of trial and error. However, in practice, it seems that the $(S')^2$ and the S' terms are often dominant. (This is a good place to start anyways, since it gives a first order ODE.)
4. Solve the resulting ODE exactly.
5. Check that your guess was correct - that is, you need to make sure that the terms you ignored really were much smaller than the terms you kept. So for each ignored term, check that *ignored* \ll *kept* as $x \rightarrow x_0$.

If any of the checks fail, go back to step #3 and try again.

6. To improve the solution, we now assume that $y \sim e^{S(x)+C(x)}$ as $x \rightarrow x_0$ where $C(x) \ll S(x)$ ($C(x)$ is a 'correction term').
7. Now repeat steps #1 through #5 with the new form of the solution. This time, in step #2, we can eliminate more terms because we know that $C \ll S$ (so $C'' \ll S''$, $(C')^2 \ll (S')^2$, etc.). Additionally, you can cross off the terms that you balanced in step #3 (they cancel with each other).
8. If you want to improve the solution even further, just add another correction term: $y \sim e^{S(x)+C(x)+D(x)}$ as $x \rightarrow x_0$ where $D(x) \ll C(x) \ll S(x)$. Etc!

The following is a nice example, because it has both a regular singular point, and an irregular singular point.

Example. (A 2006 #6 also Bender and Orszag p.71) Consider the modified Bessel equation with ν a real constant.

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} - (x^2 - \nu^2)y = 0$$

- (a) Find the leading asymptotic behavior(s) for $y(x)$ as $x \rightarrow 0^+$.

(b) Find the leading asymptotic behavior(s) for $y(x)$ as $x \rightarrow +\infty$.
 Make sure to distinguish special values of ν .

Solution. (a) First, we need to classify the singularity at $x_0 = 0$. After dividing through by x^2 we get the following differential equation

$$\frac{d^2y}{dx^2} + \frac{1}{x} \frac{dy}{dx} - \frac{x^2 - \nu^2}{x^2} y = 0$$

and note that $x_0 = 0$ is a regular singular point.

Since we are dealing with a RSP, the local solution is of the form

$$y(x) = x^r \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+r}.$$

After taking two derivatives, plugging into the original equation, and simplifying a bit, we obtain

$$(n+r)^2 \sum_{n=0}^{\infty} a_n x^{n+r} - \sum_{n=0}^{\infty} a_n x^{n+r+2} - \nu^2 \sum_{n=0}^{\infty} a_n x^{n+r} = 0 \quad (3.6)$$

Now we start equating coefficients of x_i to get equations for the a_n and r . The smallest power of x that appears in (3.6) is x^r . So we start there:

$$\begin{aligned} x^r &: (r^2 - \nu^2)a_0 = 0 \\ x^{r+1} &: ((1+r)^2 - \nu^2)a_1 = 0 \\ x^{r+n} &: ((n+r)^2 - \nu^2)a_n = a_{n-2} \quad \text{for } n \geq 2 \end{aligned}$$

- The first coefficient equation says that either $a_0 = 0$ or $r^2 = \nu^2$. By convention, we assume $a_0 \neq 0$, so we can conclude that $r = \pm\nu$.
- The second coefficient equation also yields two possibilities: either $a_1 = 0$ or $(1+r)^2 - \nu^2 = 0$.
- The third coefficient equation lets us relate every a_{2n} to a_0 and every a_{2n+1} to a_1 - we'll come back to the exact formulas later.

Let us first consider the case where $a_1 = 0$. Then we know that $0 = a_1 = a_3 = a_5 = \dots$. However, we still have two possible values for r , $r = \nu$ and $r = -\nu$. These give two linearly independent solutions (except when

$\nu = 0$ - see Bender and Orszag). To find the coefficients for $r = \nu$, we use the third relation:

$$\begin{aligned} a_{2n} &= \frac{1}{(2n + \nu)^2 - \nu^2} a_{2n-2} = \frac{1}{2^2 n(n + \nu)} a_{2n-2} \\ &= \frac{1}{2^4 n(n-1)(n + \nu)(n + \nu - 1)} a_{2n-4} = \dots \\ &= \frac{\Gamma(\nu + 1)}{2^{2n} n! \Gamma(n + \nu + 1)} a_0 \end{aligned}$$

So the first solution is

$$y_1 = a_0 \Gamma(\nu + 1) x^\nu \sum_{n=0}^{\infty} \frac{x^{2n}}{2^{2n} n! \Gamma(n + \nu + 1)}$$

The coefficients for the second solution (when $r = -\nu$) are given by

$$a_{2n} = \frac{\Gamma(\nu + 1)}{2^{2n} n! \Gamma(n - \nu + 1)} a_0$$

so the second solution is

$$y_1 = a_0 \Gamma(\nu + 1) x^\nu \sum_{n=0}^{\infty} \frac{x^{2n}}{2^{2n} n! \Gamma(n - \nu + 1)}$$

although here we have a problem if $\nu \in \mathbb{N}$ (see Bender and Orszag).

Note: The constant a_0 can be absorbed into the constants A, B in $y = Ay_1 + By_2$.

Now we have the second case: $a_1 \neq 0$. In this case, $(1 + r)^2 - \nu^2 = 0$. We can use $r = \pm\nu$ to simplify this to $(r + 1)^2 - r^2 = 0$. Then the solution is $r = -1/2, \nu = \pm 1/2$. We plug these in, and find the coefficients:

$$\begin{aligned} a_{2n} &= \frac{a_0}{(2n)!} \\ a_{2n+1} &= \frac{a_1}{(2n + 1)!} \end{aligned}$$

So

$$y = a_0 \frac{1}{\sqrt{x}} \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} + a_1 \frac{1}{\sqrt{x}} \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n + 1)!}.$$

Since the two sums do not involve the same powers of x , they are the two linearly independent solutions.

Special values of ν in this part are $\nu = 0, \pm 1/2$.

(b) To classify the point at infinity, we need the transformation

$$\begin{aligned}
 t &= \frac{1}{x} \\
 \frac{dy}{dx} &= \frac{dy}{dt} \frac{dt}{dx} = \frac{dy}{dt} \frac{-1}{x^2} \\
 &= -t^2 \frac{dy}{dt} \\
 \frac{d^2y}{dx^2} &= \frac{d}{dx} \left[\frac{dy}{dt} \frac{-1}{x^2} \right] = \frac{d}{dx} \left[\frac{dy}{dt} \right] \frac{-1}{x^2} + \frac{dy}{dt} \frac{2}{x^3} \\
 &= \frac{dy}{dt^2} \frac{dt}{dx} \frac{-1}{x^2} + \frac{dy}{dt} \frac{2}{x^3} = \frac{dy}{dt^2} \frac{1}{x^4} + \frac{dy}{dt} \frac{2}{x^3} \\
 &= t^4 \frac{dy}{dt^2} + 2t^3 \frac{dy}{dt}
 \end{aligned}$$

So the differential equation becomes

$$\frac{d^2y}{dt^2} + \frac{1}{t} \frac{dy}{dt} - \frac{1 + \nu^2 t^2}{t^4} y = 0$$

and we can see that $t = 0$ is an irregular singular point.

To find the leading behavior, we start with $y \sim e^{S(t)}$. Substituting into the above, we get

$$t^4(S'' + (S')^2) + t^3 S' - (1 - \nu^2 t^2) \sim 0 \quad \text{as } t \rightarrow 0 \quad (3.7)$$

Since $t \rightarrow 0$, the term $\nu^2 t^2$ is much smaller than 1, so we can ignore it. But that is the only term we can easily ignore. From the remaining, we can probably ignore $t^4 S''$ (based on experience). You can solve the remaining three, using the quadratic formula and a few more estimates. However, you can also just use $t^4 (S')^2 \sim 1$. The solution is $S' \sim \pm 1/t^2$ or $S \sim \mp 1/t$.

Now we need to check that the terms ignored were smaller than the two terms we kept. In this case, they all check:

$$\begin{aligned}
 t^4 S'' &\sim t \ll 1 \\
 t^3 S' &\sim t \ll 1 \\
 \nu^2 t^2 &\sim t^2 \ll 1
 \end{aligned}$$

So far, we have found that $y \sim e^{\pm 1/t}$ as $t \rightarrow 0$. The \pm indicates the two solutions (it is a second order differential equation, after all). To improve the approximation, we repeat the process again with $y \sim e^{S+C}$ where $C \ll S$.

Then the new version of (3.7) is

$$t^4(S'' + C'' + (S')^2 + (C')^2 + 2S'C') + t^3(S' + C') - (1 - \nu^2 t^2) \sim 0 \quad \text{as } t \rightarrow 0$$

Now we can ignore $t^4(S')^2$ and -1 because we balanced those two terms in the previous step. We can also ignore the $t^4 C''$, $t^4(C')^2$, and $t^3 C'$ terms because they are each smaller than their S counterparts. We'll also take a chance and ignore the $\nu^2 t^2$ term. This leaves:

$$\begin{aligned} \mp 2t \pm 2t^2 C' &\sim \mp t \\ 2t^2 C' &\sim t \\ C' &\sim \frac{1}{2t} \\ C &\sim \ln \sqrt{t} \end{aligned}$$

Check:

$$\begin{aligned} t^4 C'' &\sim t^2 \ll t \\ t^4 (C')^2 &\sim t^2 \ll t \\ t^3 C' &\sim t^2 \ll t \\ \nu^2 t^2 &\sim t^2 \ll t \end{aligned}$$

Everything works, so our guesses were right.

Now we have

$$\begin{aligned} y_1 &\sim A\sqrt{t}e^{1/t} \quad \text{as } t \rightarrow 0 \\ y_2 &\sim B\sqrt{t}e^{-1/t} \quad \text{as } t \rightarrow 0 \end{aligned}$$

This is probably the leading behavior, but you need to do one more correction to check (exercise). For the final answer, we switch back to x .

$$\begin{aligned} y_1 &\sim \frac{A}{\sqrt{x}}e^x \quad \text{as } x \rightarrow \infty \\ y_2 &\sim \frac{B}{\sqrt{x}}e^{-x} \quad \text{as } x \rightarrow \infty \end{aligned} \quad \square$$

Qualifying Exam Problems: J2003 #1a, J2004 #2, J2005 #1c, J2006 #1c, A2006 #6

4 Global Analysis Methods

4.1 Boundary Layer Theory

In some differential equations involving a small parameter ε , there is a boundary layer. This means that the solution when $\varepsilon = 0$ is qualitatively different than the general solution as $\varepsilon \rightarrow 0$.

The difficult part of boundary layer theory is determining when there is a boundary layer, and where it is. There are only a few solid results. If the differential equation is

$$\varepsilon y'' + a(x)y' + b(x)y = 0 \quad x \in [0, 1]$$

then

- if $a(x) > 0$ on $[0, 1]$ then there is a boundary layer near $x = 0$ of width ε ,
- if $a(x) < 0$ on $[0, 1]$ then there is a boundary layer near $x = 1$ of width ε ,
- there is no internal boundary layer at $x_0 \in (0, 1)$ unless $a(x_0) = 0$.

In other situations, you can use distinguished limits (see BO) to determine the location and width of any boundary layers. Alternatively, you can start by finding the outer solution, and see if that solution will satisfy both boundary conditions. If it doesn't, then there is a boundary layer (somewhere).

How to Solve a Boundary Layer Problem

1. Expand the outer solution as $y_{\text{out}} = y_0(x) + \varepsilon y_1(x) + \varepsilon^2 y_2(x) + \dots$ and solve for as many orders as desired.
2. Switch to the inner variable (an order one variable): $z = x/\delta$ or $z = (1-x)/\delta$. Here, δ is the width of the boundary layer. If the width is not known already, you can do dominant balance in the next step to find the relation between ε and δ .
3. Expand inner solution as $y_{\text{in}} = y_0(z) + \varepsilon y_1(z) + \varepsilon^2 y_2(z) + \dots$ (we abuse notation here, you may want to use $Y_i(z)$ instead). Solve for as many orders as desired.
4. Match the inner and outer solutions, order by order. For a boundary layer at $x = 0$ this is

$$y_{\text{match},i} = \lim_{x \rightarrow 0} y_{\text{out},i} = \lim_{z \rightarrow \infty} y_{\text{in},i}.$$

If the boundary layer is at $x = 1$ the only difference is that the first limit is $x \rightarrow 1$. The matching process will also determine any remaining constants of integration.

5. After all this, the final solution is

$$y = y_{\text{out}} + y_{\text{in}} - y_{\text{match}}.$$

Example. Solve the boundary layer problem:

$$\varepsilon y'' + (1+x)y' + y = 0, \quad y(0) = 1 \quad y(1) = 1$$

Solution. First, note that $1+x > 0$ for all $x \in [0, 1]$ so there is a boundary layer near $x = 0$ and the width of the boundary layer is ε . We will find only the leading order solution. So the outer solution, y_{out} , is given by

$$(1+x)y'_{\text{out}} + y_{\text{out}} = 0 \quad y_{\text{out}}(1) = 1.$$

The solution here is

$$y_{\text{out}} = \frac{2}{1+x}.$$

Now for the inner solution. Since the boundary layer is near $x = 0$ with width ε , we make the change of variable $z = x/\varepsilon$. Then the differential equation is

$$y'' + (1+\varepsilon z)y' + \varepsilon y = 0$$

where the primes now denote differentiation with respect to z . So the leading order inner solution, y_{in} , is given by

$$y'' + y' = 0 \quad y_{\text{in}}(0) = 1.$$

The solution is

$$y_{\text{in}} = 1 + C(e^{-z} - 1).$$

Now for the matching:

$$y_{\text{match}} = \lim_{x \rightarrow 0} \frac{2}{1+x} = 2$$

$$y_{\text{match}} = \lim_{z \rightarrow \infty} 1 + C(e^{-z} - 1) = 1 - C$$

So $y_{\text{match}} = 2$ and $C = -1$. Then the solution is

$$\begin{aligned} y &= y_{\text{out}} + y_{\text{in}} - y_{\text{match}} \\ &= \frac{2}{1+x} + 1 - 1(e^{-z} - 1) - 2 \\ &= \frac{2}{1+x} + e^{-x/\varepsilon}. \end{aligned}$$

□

Qualifying Exam Problems: A2002 #3, A2003 #3

4.2 WKB Theory

The WKB method is more general than boundary layer techniques. It can handle any homogeneous differential equation where the highest order term is multiplied by a small parameter, ϵ . Typically, the qualifying exam only has second order equations - i.e. equations of the form

$$\epsilon y'' + p(x)y' + q(x)y = 0.$$

At the heart of the WKB method is the following ansatz

$$y \sim \exp \left[\frac{1}{\delta} \sum_{n=0}^{\infty} \delta^n S_n(x) \right] \quad \text{as } \delta \rightarrow 0 \quad (4.1)$$

There are two common truncations of the infinite series. The first is the *Geometrical Optics Approximation* which uses only the first term of the series, $S_0(x)$. This approximation is typically not very accurate. The second is the *Physical Optics Approximation*. The Physical Optics Approximation keeps only the first two terms, $S_0(x)$ and $S_1(x)$. This approximation can be motivated by writing the exponential of the tail of the series as its own series expansion. That is,

$$\begin{aligned} y(x) &\sim \exp \left[\frac{1}{\delta} S_0(x) + S_1(x) + \delta S_2(x) + \dots \right] \quad \text{as } \delta \rightarrow 0 \\ &\sim \exp \left[\frac{1}{\delta} S_0(x) + S_1(x) \right] [1 + \delta S_2(x) + \dots] \end{aligned}$$

So this approximation is valid as long as $\delta S_2(x) \ll 1$.

There is some additional terminology related to the WKB method. Define

$$S(x) = \sum_{n=0}^{\infty} \delta^n S_n(x).$$

When $S(x)$ is real, there is an exponentially growing or decaying solution - this is *dissipative behavior*. When $S(x)$ is imaginary there is *dispersive behavior*.

The WKB method can be summarized as follows:

1. Calculate the necessary derivatives of $y \sim \exp \left[\frac{1}{\delta} S_0(x) + S_1(x) + \dots \right]$ and substitute them into the differential equation. (Cancel the exponential in each term.)

2. Use dominant balance to find the relationship between ε and δ . Substitute this into the equation.
3. Equate powers of ε and solve for the $S_n(x)$. At each step, check that $S_n \ll S_{n+1}$.

Qualifying Exam Problems: J2002 #1, J2003 #1b, J2009 #1

5 Calculus of Variations

Calculus of Variations is used when trying to minimize or maximize a functional. Common examples are minimizing distance between points, finding the shape of a hanging rope (minimizing potential energy), and minimizing surface area of a soap film.

The functional usually takes the form of an integral, e.g.

$$P[u(x)] = \int_0^1 G(u, u', x) dx \quad (5.1)$$

with some boundary conditions (e.g. $u(0) = a, u(1) = b$).

The general idea is to assume that u minimizes $P[u]$. Then any small perturbation of u should make the functional bigger. That is, for every v and small ε , $P[u + \varepsilon v] \geq P[u]$. From here, we can derive the Euler-Lagrange Equations.

5.1 Euler-Lagrange Equations

The Euler-Lagrange equations are involved in the solving of any Calculus of Variations problem. They can be derived for an individual problem without much difficulty, or simply used directly. Here, we'll derive the Euler-Lagrange equation for the simple example.

$$P[u(x)] = \int_0^1 G(u, u', x) dx$$

$$u(0) = a, u(1) = b$$

We then want to work with the relation $P[u + \varepsilon v] \geq P[u]$ or $P[u + \varepsilon v] - P[u] \geq 0$, where $v(0) = 0$ and $v(1) = 0$ (v will always have the same type of

boundary conditions as u , but they will always be homogeneous).

$$\begin{aligned}
 P[u + \varepsilon v] - P[u] &= \int_0^1 G(u + \varepsilon v, (u + \varepsilon v)', x) - G(u, u', x) \, dx \\
 &= \int_0^1 G(u, u', x) + \varepsilon v \frac{\partial G}{\partial u} + \varepsilon v' \frac{\partial G}{\partial u'} - G(u, u', x) \, dx + O(\varepsilon^2) \\
 &= \int_0^1 \varepsilon v \frac{\partial G}{\partial u} \, dx - \varepsilon v \frac{\partial G}{\partial u'} \Big|_0^1 - \int_0^1 \varepsilon v \frac{d}{dx} \left[\frac{\partial G}{\partial u'} \right] \, dx + O(\varepsilon^2) \\
 &= \int_0^1 \varepsilon v \frac{\partial G}{\partial u} - \varepsilon v \frac{d}{dx} \left[\frac{\partial G}{\partial u'} \right] \, dx + O(\varepsilon^2) \\
 &= \varepsilon \frac{\delta P}{\delta u} + O(\varepsilon^2)
 \end{aligned}$$

where $\delta P/\delta u$ is called the first variation.

Important Note: We really need the boundary term in the third line to be zero. It is here because we have $v(0) = v(1) = 0$. If you are just using the result and not going through the derivation, you should check that is term is in fact zero for the boundary conditions given in your problem.

We want $P[u + \varepsilon v] - P[u] \geq 0$ for all v , but the first variation is *odd* in v . Therefore, the first variation must be equal to zero. (Though not shown here, the second variation is *even* in v , so it need not be zero.) For the first variation to be zero for all v , it must be that

$$\boxed{\frac{\partial G}{\partial u} - \frac{d}{dx} \left[\frac{\partial G}{\partial u'} \right] = 0} \tag{5.2}$$

This is the *Euler-Lagrange equation* also known as the *strong form*.

There are several different types of functionals that often show up. For example, the functional may involve more derivatives of u , it may depend on two functions, or may involve functions of two independent variables. Table 2 shows various functionals and their associated Euler-Lagrange equations.

Special Case: When the function does not explicitly depend on the independent variable, the Euler-Lagrange equation has a special form. This is sometimes called the *Hamiltonian Equation* and can be quite handy when solving calculus of variations problems.

Given the functional

$$P[u(x)] = \int F(u, u') \, dx,$$

Table 2: Euler-Lagrange equations for various functionals

$P[u(x)] = \int G(x, u, u') dx$	$\Rightarrow \frac{\partial G}{\partial u} - \frac{d}{dx} \left[\frac{\partial G}{\partial u'} \right] = 0$
$P[u(x)] = \int G(x, u, u', u'') dx$	$\Rightarrow \frac{\partial G}{\partial u} - \frac{d}{dx} \left[\frac{\partial G}{\partial u'} \right] + \frac{d^2}{dx^2} \left[\frac{\partial G}{\partial u''} \right] = 0$
$P[x(t), y(t)] = \int G(t, x, y, x', y') dx$	$\Rightarrow \frac{\partial G}{\partial x} - \frac{d}{dt} \left[\frac{\partial G}{\partial x'} \right] = 0$ $\frac{\partial G}{\partial y} - \frac{d}{dt} \left[\frac{\partial G}{\partial y'} \right] = 0$
$P[u(x, y)] = \iint G(x, y, D_0u, D_1u, \dots) dx dy$	$\Rightarrow \sum_i D_i^T \left[\frac{\partial F}{\partial D_i u} \right] = 0$

where D_i is a differential operator (e.g. $\partial/\partial x$) and

$$D_i^T = \begin{cases} D_i & \text{for even derivatives} \\ -D_i & \text{for odd derivatives} \end{cases}$$

the Euler-Lagrange equation is

$$\begin{aligned}
 0 &= \frac{\partial F}{\partial u} - \frac{d}{dx} \left[\frac{\partial F}{\partial u'} \right] \\
 &= F_u - \frac{d}{dx} (F_{u'}) \\
 &= F_u - F_{u'u} u' - F_{u'u'} u'' \\
 &= F_u u' - F_{u'u} (u')^2 - F_{u'u'} u'' u' \\
 &= F_u u' + F_{u'u'} u'' - F_{u'u} u'' - F_{u'u} (u')^2 - F_{u'u'} u'' u' \\
 &= \frac{d}{dx} (F - u' F_{u'}).
 \end{aligned}$$

Integrating both sides with respect to x yields the Hamiltonian Equation

$$\boxed{F - u' \frac{\partial F}{\partial u'} = \text{constant}} \tag{5.3}$$

5.2 Constrained Problems

This method can also be used to solve problems where a functional needs to be minimized (or maximized), but subject to a *constraint*. (For example, minimizing the potential energy of a hanging rope such that the length of the rope is a fixed (given) value.) The constraint is incorporated into the functional by way of a *Lagrange multiplier*. If the constraint is global, the Lagrange multiplier is a constant; if the constraint depends on an independent variable, the constraint is a function of that variable.

In practice, usually only problems with global constraints show up. Furthermore, the constraint usually involves an integral of the same form as the functional and can therefore be easily incorporated into it.

Example. (*J 2008 #1*) A rope of length l with constant line density ρ hangs between two points in an (x, y) plane, say $x = a, y = y_0$ and $x = b, y = y_1$. Let $y(x)$ denote the rope configuration, with $y(x) > 0$. The total potential energy of the configuration is

$$J(y) = \int_0^l \rho g y \, ds$$

where s is the arclength and g is the acceleration of gravity. The actual configuration minimizes potential energy subject to the fact that the rope has fixed length l .

The constraint is that the rope has fixed length l , i.e.

$$\int_0^l ds = l.$$

We add this constraint to the functional as follows

$$\begin{aligned} P[y] &= \int_0^l \rho g y \, ds + m \left(\int_0^l ds - l \right) \\ &= \int_0^l (\rho g y + m) \, ds - ml \end{aligned}$$

The constant ml can be ignored because it will not affect the result of the minimization. So for solving this problem, the functional in question is

$$\begin{aligned} P[y] &= \int_0^l (\rho g y + m) \, ds \\ &= \int_a^b (\rho g y + m) \sqrt{1 + (y')^2} \, dx \end{aligned}$$

5.3 Three Formulations of Classical Mechanics

Calculus of variations also plays a role in the three formulations of classical mechanics - Newtonian, Lagrangian, and Hamiltonian.

Newtonian: The Newtonian formulation is the standard $F = ma$.

Lagrangian: The basis of the Lagrangian formulation is minimizing action. The action is defined as

$$\boxed{A = \int_{t_0}^{t_1} \text{KE} - \text{PE} dt = \int_{t_0}^{t_1} L(q, \dot{q}) dt} \quad (5.4)$$

where KE is the kinetic energy, PE is the potential energy, and q is a generalized coordinate (depending on the specific problem).

Hamiltonian: The Hamiltonian formulation defines a new coordinate p , and the Hamiltonian $H(p, q)$ as follows.

$$\boxed{p := \frac{\partial L}{\partial \dot{q}} \quad H(p, q) := p\dot{q} - L(q, \dot{q})} \quad (5.5)$$

Then the solution is given by the system:

$$\boxed{\frac{\partial H}{\partial p} = \dot{q} \quad \frac{\partial H}{\partial q} = -\dot{p}} \quad (5.6)$$

It is easy to see that the three formulations are equivalent by using all three to find the ODE that determines the motion of a system. Classic examples are a falling object, a swinging pendulum, and a mass on a spring. (The potential energy due to a spring is given by $kx^2/2$ where x is the displacement from rest length.)

Example. (*Pendulum*) Suppose a pendulum of mass m is swinging on a string of length l , and that at any given time, the angle from vertical is θ . Then $l\theta$ is the displacement of the pendulum from its rest position (hanging straight down). Then the acceleration is the second time derivative of $l\theta$ and the force (due to gravity) is $F = -mg \sin(\theta)$. So $F = ma$ becomes

$$-mg \sin(\theta) = m \frac{d^2}{dt^2} (l\theta)$$

This is the equation that determines the motion of the pendulum.

For the Lagrangian formulation, let $q = l\theta$. The kinetic energy is given by $\text{KE} = 1/2mv^2 = 1/2m\dot{q}^2$. The potential energy due to gravity is $\text{PE} = mgh = mg(l - l \cos(q/l))$. So the action is

$$A = \int_{t_0}^{t_1} \frac{1}{2}m\dot{q}^2 - mg \left(l - l \cos\left(\frac{q}{l}\right) \right) dt.$$

The Euler-Lagrange equation for this functional is

$$\frac{-mgl \sin\left(\frac{q}{l}\right)}{l} - \frac{d}{dt} [m\dot{q}] = 0$$

or

$$-mg \sin(\theta) = m \frac{d^2}{dt^2} (l\theta)$$

which matches the result from the Newtonian formulation.

In the Hamiltonian formulation, $p = \partial L / \partial \dot{q} = m\dot{q}$, so $\dot{q} = p/m$. Then

$$H(p, q) = p\dot{q} - L(q, \dot{q}) = \frac{p^2}{2m} + mg \left(l - l \cos\left(\frac{q}{l}\right) \right)$$

The solution in the Hamiltonian formulation is given by

$$\frac{\partial H}{\partial p} = \dot{q} \quad \frac{\partial H}{\partial q} = -\dot{p}.$$

From the first equation, we see that $p = m\dot{q}$ or $\dot{p} = m\ddot{q}$. The second equation says that

$$\frac{mgl \sin(\theta)}{l} = -\dot{p} = -m\ddot{q}$$

or

$$mg \sin(\theta) = -m \frac{d^2}{dt^2} (l\theta)$$

which again matches the previous results. □

Qualifying Exam Problems: A2000 #3, J2000 #1, J2005 #5, J2006 #3, J2007 #2, J2008 #1, A2008 #2, J2009 #3

6 Green's Functions

Finding the Green's Function is an efficient way to solve a 2nd order linear ODE. Let L be a general linear, 2nd order operator

$$L = \frac{d^2}{dx^2} + p(x) \frac{d}{dx} + q(x),$$

and suppose we want to solve the ODE

$$L[y(x)] = f(x) \quad b \leq x \leq c \quad (6.1)$$

with given *homogeneous* boundary conditions. (If the boundary conditions are not homogeneous, do a change of (dependent) variable: $y(x) = w(x) + \alpha + \beta x$ for appropriate α and β .)

If we can find a function, $G(x; a)$ such that $L[G(x; a)] = \delta(x - a)$ (with the same boundary conditions), then the solution to (6.1) is

$$y(x) = \int_b^c f(a)G(x; a) da. \quad (6.2)$$

This $G(x, a)$ is called the Green's Function. (See Appendix ?? for more about $\delta(x - a)$.)

To find the Green's Function, first solve the homogeneous equation, $L[y(x)] = 0$ to find two linearly independent solutions, $y_1(x), y_2(x)$. Now when $x < a$, $\delta(x - a) = 0$ so $L[G(x; a)] = \delta(x - a) = 0$. Therefore at these points, the Green's function satisfies the homogeneous equation, hence $G(x; a) = A_1 y_1(x) + A_2 y_2(x)$. Similarly, when $x > a$, $G(x; a) = B_1 y_1(x) + B_2 y_2(x)$. That is,

$$G(x; a) = \begin{cases} A_1 y_1(x) + A_2 y_2(x) & \text{if } x < a, \\ B_1 y_1(x) + B_2 y_2(x) & \text{if } x > a \end{cases} \quad (6.3)$$

Around $x = a$, $G(x; a)$ satisfies

$$\int_{a-\varepsilon}^{a+\varepsilon} \frac{d^2}{dx^2} G(x; a) + p(x) \frac{d}{dx} G(x; a) + q(x) G(x; a) dx = \int_{a-\varepsilon}^{a+\varepsilon} \delta(x - a) dx$$

Recall that the first derivative of the Ramp Function is the Heaviside Function, and the second derivative is the Dirac Delta Function. Since the RHS is the integral of the Dirac Delta function, the integrand on the LHS must have the same level of discontinuity. Higher derivatives are more

discontinuous, so it should be that $G''(x; a)$ is like $\delta(x - a)$. Then $G'(x; a)$ should be like the Heaviside Function (with a jump of exactly 1 when $x = a$), and $G(x; a)$ should be like the Ramp Function (continuous at $x = a$).

To summarize, we know that

1. $G'(x; a)$ must have a jump of exactly 1:

$$\lim_{\varepsilon \rightarrow 0^+} \frac{dG}{dx} \Big|_{x=a+\varepsilon} - \frac{dG}{dx} \Big|_{x=a-\varepsilon} = 1$$

So from (6.3),

$$\boxed{B_1 y_1'(a) + B_2 y_2'(a) - A_1 y_1'(a) + A_2 y_2'(a) = 1} \quad (6.4)$$

2. $G(x; a)$ must be continuous:

$$\lim_{\varepsilon \rightarrow 0^+} G(x; a) \Big|_{a-\varepsilon}^{a+\varepsilon} = 0$$

So using (6.3) again, we see that

$$\boxed{B_1 y_1(a) + B_2 y_2(a) - A_1 y_1(a) + A_2 y_2(a) = 0} \quad (6.5)$$

These two equations, along with the two boundary conditions, allow us to solve for the four constants A_1, A_2, B_1, B_2 . Then the solution is

$$\begin{aligned} y(x) &= \int_b^c f(a)G(x; a) da \\ &= \int_b^x f(a)[B_1 y_1(x) + B_2 y_2(x)] da + \int_x^c f(a)[A_1 y_1(x) + A_2 y_2(x)] da. \end{aligned}$$

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