# Topological Entropy and Secondary Folding

Sarah Matz

Specialty Exam - April 27, 2010

University of Wisconsin-Madison

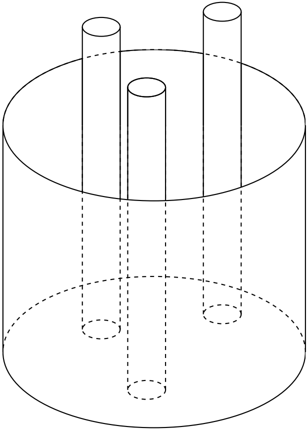
Advisor: Jean-Luc Thiffeault
Committee: Joel Robbin and James Rossmanith

# Mixing with Rods

How do you mix two (viscous) fluids together? You need to stir them somehow.

A simple stirring device is a container with vertical rods. The rods move horizontally in a specified way (called stirring protocol).

We just look at the cross-section – a disk with holes.



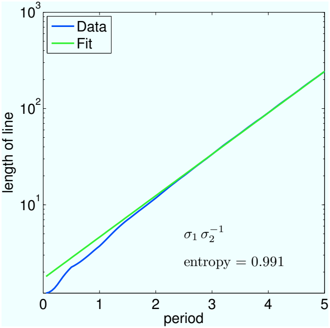
[movie 1] [movie 2]

[Boyland *et al.* (2000); Thiffeault & Finn (2006)]

# Topological Entropy

How can we tell how well the map is mixing the fluid?

The topological entropy, $h$, can give us a measure of this. Higher entropy indicates better mixing.




For piecewise smooth maps, the entropy is related to the rate of growth of the length of a material line.

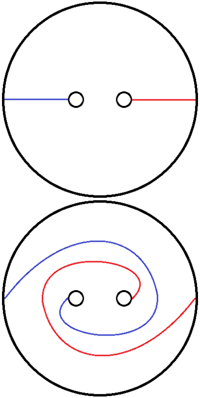
$$h = \sup_{\gamma} \limsup_{n \to \infty} \frac{1}{n} \ln \ell(\gamma_n)$$

where $\ell(\gamma_n)$ is the length of the $n^{th}$ iterate of the material line $\gamma$.

[Newhouse (1988)]

# Isotopy Classes

- The rod motion induces a homeomorphism, which belongs to an isotopy class. (we allow rotation of the boundaries).
- When the domain is a disk with holes, the isotopy classes are labeled by braids.
- The isotopy class gives a lower bound on the entropy of the homeomorphism. (via the associated braid)
- But it might be a bad bound!

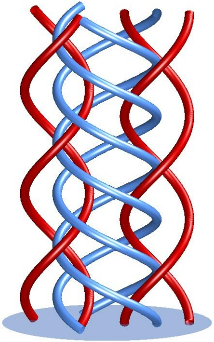


Isotopic to the identity

[Boyland *et al.* (2000); Thiffeault & Finn (2006)]

# Isotopy Classes

- The rod motion induces a homeomorphism, which belongs to an isotopy class. (we allow rotation of the boundaries).
- When the domain is a disk with holes, the isotopy classes are labeled by braids.
- The isotopy class gives a lower bound on the entropy of the homeomorphism. (via the associated braid)
- But it might be a bad bound!

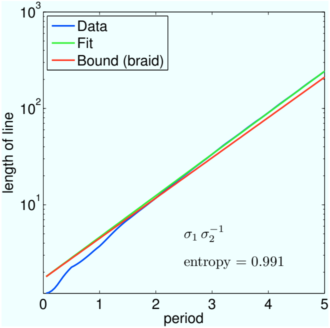


[Finn & Thiffeault (2010)]

[Boyland *et al.* (2000); Thiffeault & Finn (2006)]

# Isotopy Classes

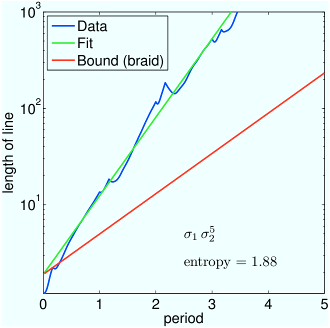
- The rod motion induces a homeomorphism, which belongs to an isotopy class. (we allow rotation of the boundaries).
- When the domain is a disk with holes, the isotopy classes are labeled by braids.
- The isotopy class gives a lower bound on the entropy of the homeomorphism. (via the associated braid)
- But it might be a bad bound!




[Boyland *et al.* (2000); Thiffeault & Finn (2006); Finn *et al.* (2006)]

# Isotopy Classes

- The rod motion induces a homeomorphism, which belongs to an isotopy class. (we allow rotation of the boundaries).
- When the domain is a disk with holes, the isotopy classes are labeled by braids.
- The isotopy class gives a lower bound on the entropy of the homeomorphism. (via the associated braid)
- But it might be a bad bound!




[Boyland *et al.* (2000); Thiffeault & Finn (2006); Finn *et al.* (2006)]

# Goals

- Can we predict when the lower bound is sharp? (capturing close to 100% of the entropy.)
- What causes it not to be sharp?
- Can we find a better estimate?

To answer these questions, we'll look at mixers with three rods and linked twist maps.

# Goals

- Can we predict when the lower bound is sharp? (capturing close to 100% of the entropy.)
- What causes it not to be sharp?
- Can we find a better estimate?

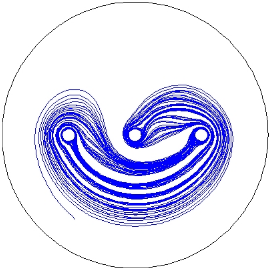
To answer these questions, we'll look at mixers with three rods and linked twist maps.

# Describing the rod motion

We can describe any rod motion using generators of the braid group [Birman (1975)]:

- $\sigma_1$ is the clockwise interchange of the first and second rods;
- $\sigma_2$ is the clockwise interchange of the second and third rods.

We consider only protocols of the form $\sigma_1^n \sigma_2^{-m}$.



Two types:

- counter-rotating ($nm > 0$)
- co-rotating ($nm < 0$)

This protocol has $n = m = 1$:

# Action on homology

The braid describing the rod motion gives us a lower bound on the entropy.

To find the entropy associated with the braid, we look at the homology group on the orientable double cover of the disk with holes, $H_1(T^2; \mathbb{Z})$.

Use Burau matrix representation for braid generators [Burau (1936); Birman (1975)]:

$$[\sigma_1] = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \qquad [\sigma_2] = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

$$[\sigma_1^n \sigma_2^{-m}] = \begin{bmatrix} 1 + nm & n \\ m & 1 \end{bmatrix}$$

These matrices tell us about how loops are transformed.

# Topological entropy

$[\sigma_1^n \sigma_2^{-m}]$ is a hyperbolic matrix (|largest eigenvalue| $> 1$) if

$$|2 + nm| > 2$$

If $[\sigma_1^n \sigma_2^{-m}]$ is hyperbolic, then the entropy is

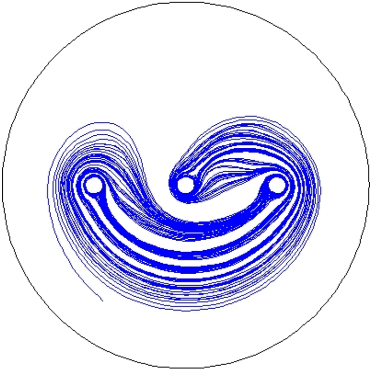
$$h = \log |\text{largest eigenvalue}| > 0$$

For instance, if $n = m = 1$ (counter-rotating) or $n = 1$, $m = -5$ (co-rotating), then $|2 + nm| = 3$, and

$$h = \log |\text{largest eigenvalue}| = \log(\tfrac{1}{2}(3 + \sqrt{5})) \approx 0.962$$

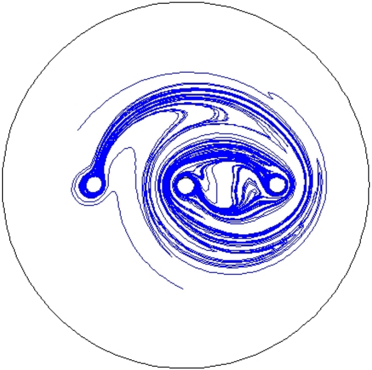
The only difference is that in the counter-rotating case the eigenvalue of the matrix $[\sigma_1^n \sigma_2^{-m}]$ is positive, while for the co-rotating case it is negative.

# Material Lines



$\sigma_1\sigma_2^{-1}$

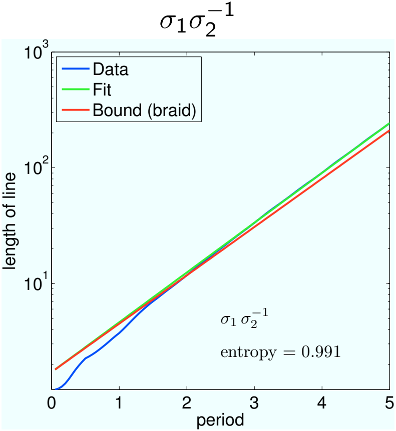
[movie 3]

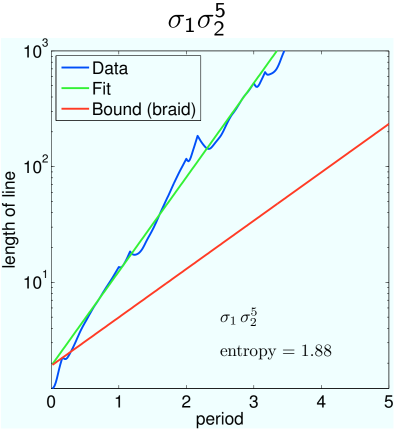


$\sigma_1\sigma_2^5$

[movie 4]

# Line Growth



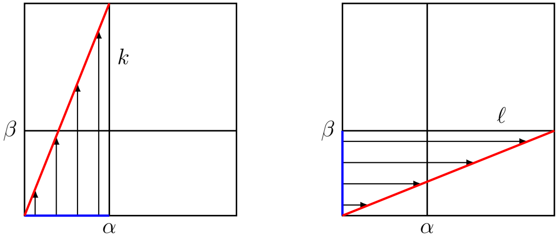




The lower bound is much worse in the co-rotating case.
What's the difference?

# Toral LTM

Compare to the toral linked twist map (LTM).

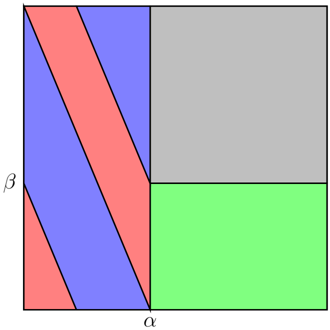
- Domain: two overlapping strips on the torus
- Map: vertical shear ($k$) followed by a horizontal shear ($\ell$)
  - $k, \ell \in \mathbb{Z}$ so the map is continuous.




If $k\ell > 0$ it is counter-rotating*; if $k\ell < 0$ it is co-rotating*.

[Sturman *et al.* (2006); Devaney (1980); Burton & Easton (1980); Przytycki (1983),etc.]

# Relation to Anosov Map




Four different regions:

- Gray - Fixed
- Green - Horiz. shear only
- Blue: - Vert. shear only
- Red: - Both shears

In the (red) interaction region the map is given by

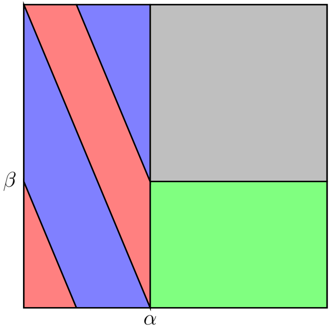
$$\begin{bmatrix} 1 + k\ell\alpha^{-1}\beta^{-1} & \ell\beta^{-1} \\ k\alpha^{-1} & 1 \end{bmatrix}$$

When $\alpha = \beta = 1$ this is the same as the Anosov map.

$$\begin{bmatrix} 1 + k\ell & \ell \\ k & 1 \end{bmatrix}$$

The LTM and the Anosov map are semi-conjugate. [Franks (1970)]

# Relation to Anosov Map




Four different regions:

Gray - Fixed

Green - Horiz. shear only

Blue: - Vert. shear only

Red: - Both shears

In the (red) interaction region the map is given by

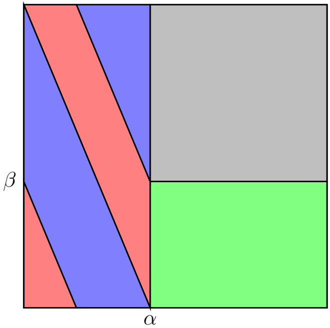
$$\begin{bmatrix} 1 + k\ell\alpha^{-1}\beta^{-1} & \ell\beta^{-1} \\ k\alpha^{-1} & 1 \end{bmatrix}$$

When $\alpha = \beta = 1$ this is the same as the Anosov map.

$$\begin{bmatrix} 1 + k\ell & \ell \\ k & 1 \end{bmatrix}$$

The LTM and the Anosov map are semi-conjugate. [Franks (1970)]

# Relation to Anosov Map




Four different regions:

Gray - Fixed

Green - Horiz. shear only

Blue: - Vert. shear only

Red: - Both shears

In the (red) interaction region the map is given by

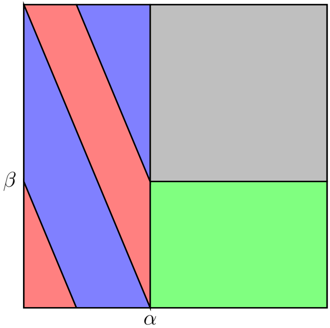
$$\begin{bmatrix} 1 + k\ell\alpha^{-1}\beta^{-1} & \ell\beta^{-1} \\ k\alpha^{-1} & 1 \end{bmatrix}$$

When $\alpha = \beta = 1$ this is the same as the Anosov map.

$$\begin{bmatrix} 1 + k\ell & \ell \\ k & 1 \end{bmatrix}$$

The LTM and the Anosov map are semi-conjugate. [Franks (1970)]

# Relation to Anosov Map




Four different regions:

Gray - Fixed

Green - Horiz. shear only

Blue: - Vert. shear only

Red: - Both shears

In the (red) interaction region the map is given by

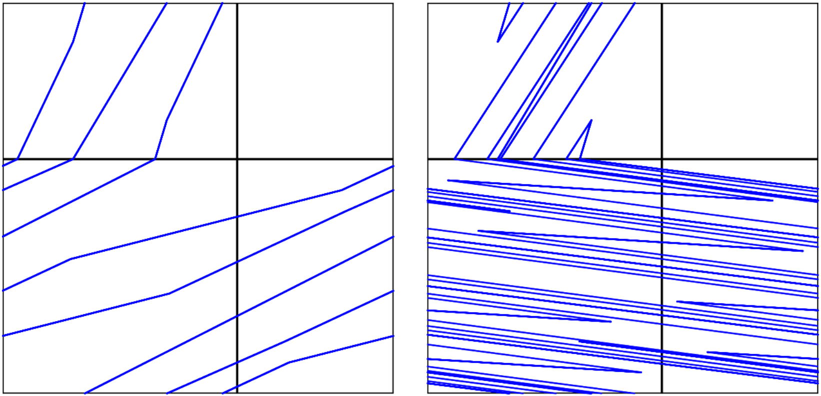
$$\begin{bmatrix} 1 + k\ell\alpha^{-1}\beta^{-1} & \ell\beta^{-1} \\ k\alpha^{-1} & 1 \end{bmatrix}$$

When $\alpha = \beta = 1$ this is the same as the Anosov map.

$$\begin{bmatrix} 1 + k\ell & \ell \\ k & 1 \end{bmatrix}$$

The LTM and the Anosov map are semi-conjugate. [Franks (1970)]

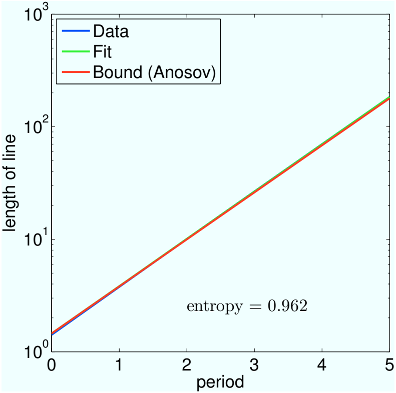
# Material Lines and Secondary Folding

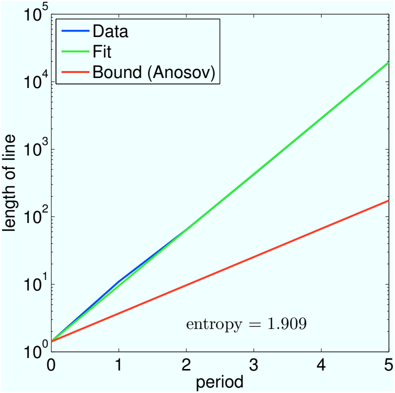


counter-rot. $(k\ell = 1 > 0)$
$(\alpha = \beta = 0.6)$

co-rot. $(k\ell = -5 < 0)$
$(\alpha = \beta = 0.6)$

# Line Growth




counter-rot. $(k\ell = 1 > 0)$
entropy $\simeq$ .962
$(\alpha = \beta = 0.5)$

co-rot. $(k\ell = -5 < 0)$
entropy $\simeq$ 1.91 $>$ .962
$(\alpha = \beta = 0.5)$

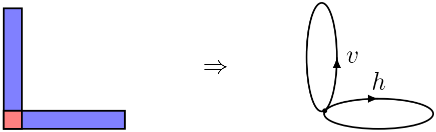
# Spine

Can we find a better bound for the co-rotating case?

Yes, for the limit $\alpha, \beta \to 0$. [work with Phil Boyland]

Extra entropy comes from secondary folding. (Material lines not being 'pulled tight') For the small $\alpha, \beta$ limit, we can calculate how much longer these extra folds will make the material lines.

Since $\alpha, \beta$ are very small, the material lines are nearly vertical/horizontal, and we can squish them onto the spine ($S^1 \vee S^1$).



# Derivation of the Bound

Consider how the LTM map affects $\pi_1(S^1 \vee S^1)$:

| Vertical shear: | Horizontal shear: | Composition: |
| --- | --- | --- |
| $h \mapsto v^k h$ | $h \mapsto h$ | $h \mapsto (h^\ell v)^k h$ |
| $v \mapsto v$ | $v \mapsto h^\ell v$ | $v \mapsto h^\ell v$ |

Notice that if $k\ell < 0$ then as the map iterates, there will be some cancellations. So modify the map to avoid cancellations:

$$h \mapsto (h^{|\ell|} v)^{|k|} h$$
$$v \mapsto h^{|\ell|} v$$

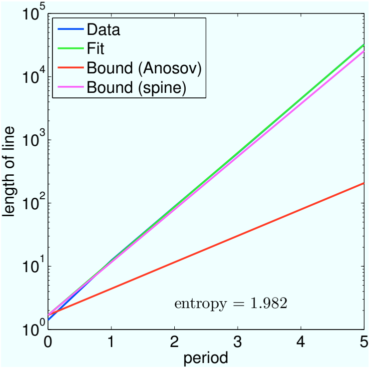
# Result

This abelianizes to

$$\begin{bmatrix} h \\ v \end{bmatrix} \mapsto \begin{bmatrix} 1 + |k||\ell| & |k| \\ |\ell| & 1 \end{bmatrix} \begin{bmatrix} h \\ v \end{bmatrix}$$

The entropy of this map (max eigenvalue) is a lower bound for the entropy of the LTM as $\alpha, \beta \to 0$.

For $k\ell = -5$, the 'spine bound' is 1.92.




co-rot. $(k\ell = -5 < 0)$
entropy $\simeq 1.982$
$(\alpha = \beta = 0.01)$

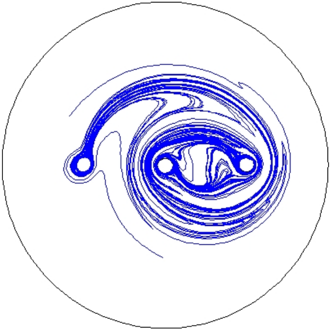
# Another Look at the Three Rod Mixer

Can we do the same thing for the three rod mixer? Yes and No.

No spine here, but we can still take absolute value:

$$\begin{bmatrix} 1+|n||m| & |n| \\ |m| & 1 \end{bmatrix}$$

Meaning: The secondary folds don't get pulled back.



$\sigma_1\sigma_2^5$

This is not a bound, but it works surprisingly well as an estimate!

For $nm = -5$, this 'spine estimate' has entropy 1.92. (fit was 1.88)

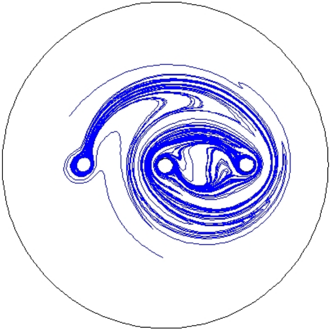
# Another Look at the Three Rod Mixer

Can we do the same thing for the three rod mixer? Yes and No.

No spine here, but we can still take absolute value:

$$\begin{bmatrix} 1 + |n||m| & |n| \\ |m| & 1 \end{bmatrix}$$

Meaning: The secondary folds don't get pulled back.



$\sigma_1 \sigma_2^5$

This is not a bound, but it works surprisingly well as an estimate!

For $nm = -5$, this 'spine estimate' has entropy 1.92. (fit was 1.88)

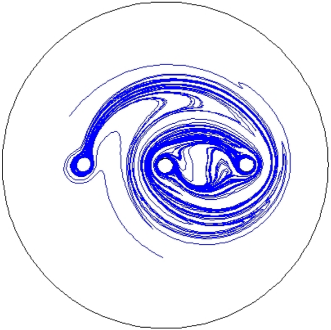
# Another Look at the Three Rod Mixer

Can we do the same thing for the three rod mixer? Yes and No.

No spine here, but we can still take absolute value:

$$\begin{bmatrix} 1+|n||m| & |n| \\ |m| & 1 \end{bmatrix}$$

Meaning: The secondary folds don't get pulled back.



$\sigma_1 \sigma_2^5$

This is not a bound, but it works surprisingly well as an estimate!

For $nm = -5$, this 'spine estimate' has entropy 1.92. (fit was 1.88)

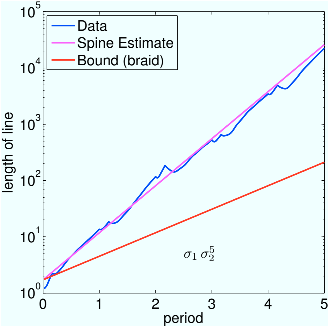
# Another Look at the Three Rod Mixer

Can we do the same thing for the three rod mixer? Yes and No.

No spine here, but we can still take absolute value:

$$\begin{bmatrix} 1+|n||m| & |n| \\ |m| & 1 \end{bmatrix}$$

Meaning: The secondary folds don't get pulled back.




$\sigma_1 \sigma_2^5$

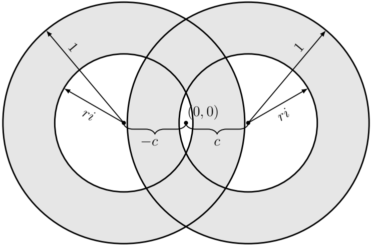
This is not a bound, but it works surprisingly well as an estimate!

For $nm = -5$, this 'spine estimate' has entropy 1.92. (fit was 1.88)

# Planar Linked Twist Map

System

- Two overlapping annuli;
- Scale outer radius to 1;
- Choose inner radius ($ri$) and offset ($c$) to have two distinct interaction regions.




Planar LTM

- More closely related to physical systems.

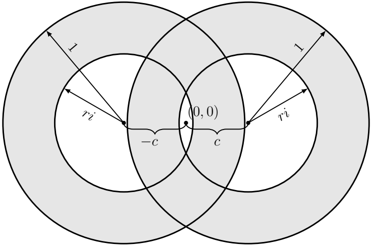
- Harder to analyze:
  - curvature;
  - two interaction regions.

[Sturman *et al.* (2006); Devaney (1978); Springham & Wiggins (2008)]

# Planar Linked Twist Map

System

- Two overlapping annuli;
- Scale outer radius to 1;
- Choose inner radius ($ri$) and offset ($c$) to have two distinct interaction regions.



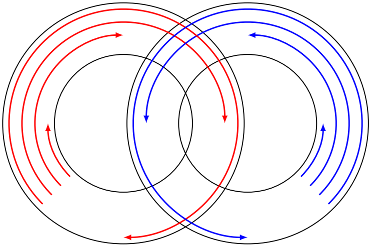

Planar LTM

- More closely related to physical systems.

- Harder to analyze:
  - curvature;
  - two interaction regions.

[Sturman *et al.* (2006); Devaney (1978); Springham & Wiggins (2008)]

# Map



$(k = -1, \ell = 1)$

Each iteration of the map is

- linear shear in the left annulus;
- then linear shear in the right annulus.

Each shear fixes the inner circle, and rotates the outer circle an integer number of times $(k, \ell)$.

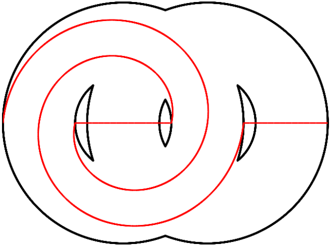
If $k$ and $\ell$ have the same sign, then the system is co-rotating; if they have opposite signs then it is counter-rotating.

# Isotopy Class

A single shear on the left annulus ($k = 1$) is isotopic to the braid $\sigma_1^2$.

Similarly, a single shear on the right annulus ($\ell = 1$) is isotopic to $\sigma_2^2$.

So the planar LTM is isotopic to $\sigma_1^{2k}\sigma_2^{2\ell}$.



The braid $\sigma_1^n\sigma_2^{-m}$ had positive entropy when $|2 + nm| > 2$.

So $\sigma_1^{2k}\sigma_2^{2\ell}$ has positive entropy when $|2 - 4k\ell| > 2$.

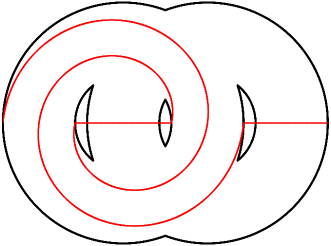
For integers $k, \ell$, the smallest is $|2 - 4k\ell| = 6$.
($k\ell = -1$ or $k\ell = 2$)

# Isotopy Class

A single shear on the left annulus ($k = 1$) is isotopic to the braid $\sigma_1^2$.

Similarly, a single shear on the right annulus ($\ell = 1$) is isotopic to $\sigma_2^2$.

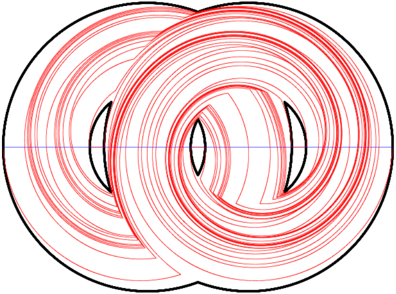
So the planar LTM is isotopic to $\sigma_1^{2k}\sigma_2^{2\ell}$.



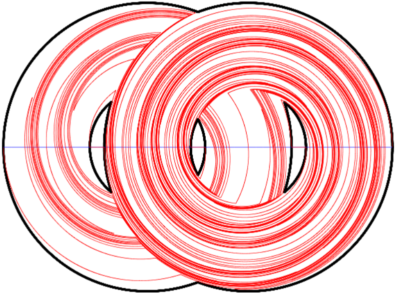
The braid $\sigma_1^n\sigma_2^{-m}$ had positive entropy when $|2 + nm| > 2$.
So $\sigma_1^{2k}\sigma_2^{2\ell}$ has positive entropy when $|2 - 4k\ell| > 2$.

For integers $k, \ell$, the smallest is $|2 - 4k\ell| = 6$.
($k\ell = -1$ or $k\ell = 2$)

# Material Lines and Secondary Folding

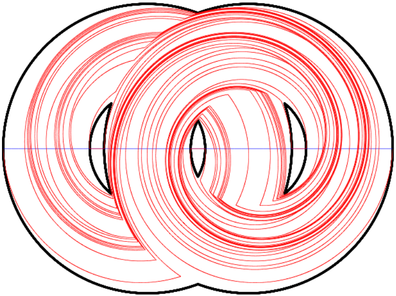


counter-rotating
$(k\ell = -1 < 0)$

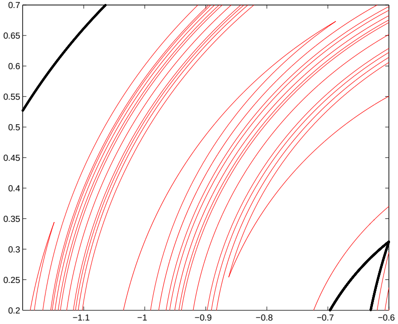


co-rotating
$(k\ell = 2 > 0)$

# Material Lines and Secondary Folding

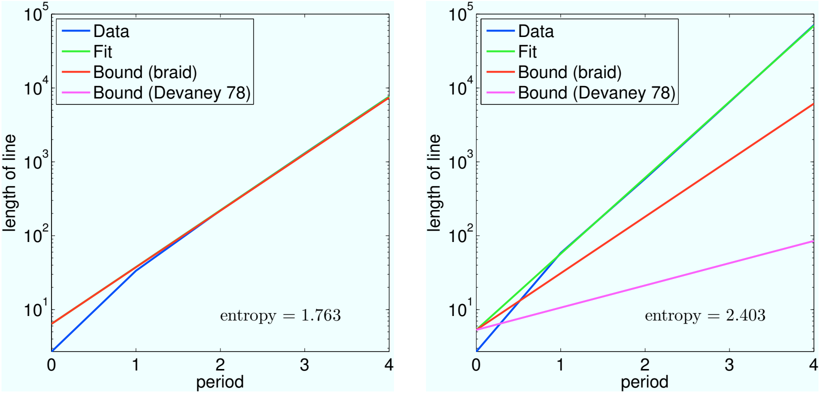


counter-rotating
$(k\ell = -1 < 0)$




co-rotating
$(k\ell = 2 > 0)$

# Entropy Results




counter-rotating
$(k\ell = -1 < 0)$

co-rotating
$(k\ell = 2 > 0)$

[Devaney (1978)]

# Conclusion

Summary and ideas for future work:

1. We've seen three systems where the 'homological bound' is sharp for counter-rotating, but bad for co-rotating.
   - Instead of co-rotating vs. counter-rotating, is it really just the sign of the eigenvalue that makes the difference?
2. Why is the bound so bad?
   - The homology doesn't 'see' the secondary folding.
   - In one case, we have another bound ('spine bound') that can 'see' the secondary folding.
     - This same method (taking absolute values) gives a surprisingly good estimate of the entropy for the hydrodynamic system.
3. Other ways to find better bounds/estimates:
   - Generalize the 'spine bound' – get some measure of the secondary folding in general systems.
   - Include orbits of periodic points ($\Rightarrow$ more complex braid).

# References


Birman, J. S. 1975. *Braids, Links, and Mapping Class Groups*. Annals of Mathematics Studies, Princeton University Press, Princeton, NJ.

Boyland, P. L., Aref, H. & Stremler, M. A. 2000. Topological fluid mechanics of stirring. J. Fluid Mech. 403, 277–304.

Burau, W. 1936. Über Zopfgruppen und gleichsinnig verdrilte Verkettungen. Abh. Math. Semin. Hamburg Univ. 11, 171–178.

Burton, R. & Easton, R. W. 1980. Ergodicity of linked twist maps. In *Global theory of dynamical systems (Proc. Internat. Conf., Northwestern Univ., Evanston, Ill., 1979)* vol. 819, of *Lecture Notes in Math.* pp. 35–49. Springer Berlin.

Devaney, R. L. 1978. Subshifts of finite type in linked twist mappings. Proc. Amer. Math. Soc. 71 (2), 334–338.

Devaney, R. L. 1980. Linked twist mappings are almost Anosov. In *Global theory of dynamical systems (Proc. Internat. Conf., Northwestern Univ., Evanston, Ill., 1979)* vol. 819, of *Lecture Notes in Math.* pp. 121–145. Springer Berlin.

Donnay, V. J. 1992. Physical examples of linked twist maps with chaotic dynamics. In *Twist mappings and their applications* vol. 44, of *IMA Vol. Math. Appl.* pp. 95–117. Springer New York.

Finn, M. D. & Thiffeault, J.-L. Topological optimisation of rod-stirring devices. (preprint).

Finn, M. D., Thiffeault, J.-L. & Gouillart, E. 2006. Topological chaos in spatially periodic mixers. Phys. D, 221 (1), 92–100.

Franks, J. 1970. Anosov diffeomorphisms. In *Global Analysis (Proc. Sympos. Pure Math., Vol. XIV, Berkeley, Calif., 1968)* pp. 61–93. Amer. Math. Soc. Providence, R.I.

Newhouse, S. E. 1988. Entropy and volume. Ergodic Theory Dynam. Systems, $8^*$ (Charles Conley Memorial Issue), 283–299.

Przytycki, F. 1983. Ergodicity of toral linked twist mappings. Ann. Sci. École Norm. Sup. (4), 16 (3), 345–354 (1984).

Przytycki, F. 1986. Periodic points of linked twist mappings. Studia Math. 83 (1), 1–18.

Springham, J. & Wiggins, S. A bernoulli linked-twist map in the plane. (www.arXiv.org).

Sturman, R., Ottino, J. M. & Wiggins, S. 2006. *The Mathematical Foundations of Mixing: The Linked Twist Map as a Paradigm in Applications: Micro to Macro, Fluids to Solids*. Cambridge University Press, Cambridge, U.K.

Thiffeault, J.-L. & Finn, M. D. 2006. Topology, braids and mixing in fluids. Philos. Trans. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. 364 (1849), 3251–3266.

Wojtkowski, M. 1980. Linked twist mappings have the $K$-property. In *Nonlinear dynamics (Internat. Conf., New York, 1979)* vol. 357, of *Ann. New York Acad. Sci.* pp. 65–76. New York Acad. Sci. New York.