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# MODIFICATIONS OF THE HARMONIC SERIES

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# Abstract

While the harmonic series itself diverges, there exist modifications of the series that converge. This paper explores the modifications that are formed by deleting certain terms from the harmonic series. The most common examples are the convergent series formed by removing all of the terms where the denominator contains a certain digit.

In addition to a brief history of the subject, we show a method for estimating these ‘forbidden digit’ modifications from above and below. With the aid of a computer program, the convergence values are calculated to several decimal places. We also explore modifications of the harmonic series in different bases and trends among their convergence values.

Modifications formed in ways other than forbidding digits are also explored, with the specific example of removing all terms that are reciprocals of composite numbers. Two theorems that serve as convergence tests for modifications of any series are explored and used to test several modifications of the harmonic series in base ten as well as in other bases.

In the last section, we move into the $p$-adic realm, and show that neither the harmonic series, nor any of its sub-series converges in $\mathbb{Q}_p$. Finally we examine the ‘harmonic’ sequence and it’s subsequences in $\mathbb{Q}_p$, and show that the terms are dense in $\mathbb{Q}_p$.

Keywords: Harmonic series, modifications, subseries, convergence

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# 1. History

It is well known that the harmonic series diverges. However, by removing some of the terms in the series, it can be made to converge. The most common example is to remove all of the terms where the denominator contains the digit 9, but the series can be modified in other ways as well. Modifications of the harmonic series have been explored on and off since 1914. Between 1914 and 1919 there was a lively interchange between A. J. Kempner, F. Irwin, and E. J. Moulton. Afterward, the idea was ignored until I.E. Perlin published a paper in 1941 and P. Erdös and I. Niven published a paper in 1945. The topic was again brought up in 1975 by A. D. Wadhwa and by R. Baillie in 1979.

## 1.1 Kempner, Irwin, and Moulton

In February 1914, A. J. Kempner published a small paper showing that if the harmonic series is modified by removing all terms whose denominator contains the digit 9, then the modified series converges [1]. His method of proof was to establish an upper bound on the modified series – it converges to some number less than $90$. Kempner also stated that the same proof would work if any digit in $\{1,2,\ldots,9\}$ was removed, but it would not work for the digit $0$.

A year later, in 1915, Kempner posed a question: if the harmonic series is modified by removing all terms that contain a combination of figures, such as ‘37’, does it converge or diverge [2]? Then in March 1916 he asked if the digit 0 could be removed to make a convergent series [3].

Frank Irwin published a paper in May 1914 that looked at two types of modifications [4]. The first modification removes all terms where the denominator contains the digit 9 at least $a$ times, the digit 8 at least $b$ times, and so on, through the digit 0 at least $j$ times. Irwin proved that this series is convergent. The second part of the paper returned to Kempner’s modification – removing any term with the digit 9 – but showed that much closer estimates to the sum of the series could be obtained. His method of estimating from above and below is discussed in Section 2.1.

Just a few months later, E. J. Moulton’s solution to Kempner’s question about removing zeros was published [5]. Although Irwin’s paper had addressed the removal of the digit 0 in its second section, Moulton’s solution had been received before Irwin’s paper was published. Then in June 1919, Irwin provided a solution to Kempner’s question about removing combinations of digits – the series converges [6].

## 1.2 Perlin, Erdos and Niven

In 1941, I. E. Perlin published a paper titled “Series with Deleted Terms” [7]. In it, he establishes conditions for a given modification of *any* given series to converge.  The resulting theorems are extremely useful for exploring modifications of the harmonic series and are discussed in detail in Section 3.2.

P. Erdös and I. Niven published a paper in 1945 that explored a different aspect of the harmonic series [8].  Here the harmonic series is not modified by removing terms, but rather by switching the signs of alternating “blocks” of terms. This variation of the harmonic series converges for certain sets of blocks.

## 1.3 Wadhwa and Baillie

In November 1975, A. D. Wadhwa published a small paper about the modification made by removing all terms that contain the digit 0 [9].  There, a different method of estimating the series from above and below was shown, and some machine calculations were done to calculate the exact value of the series to two decimal places.  Wadhwa also mentioned that modifications can be made to the series in different bases, with similar results.  Wadhwa published another paper in 1978 that explored modifications where every term has exactly $n$ zeros in the denominator [10].  All of these modifications converge, but Wadhwa also explored relationships between the modifications.

Finally, in May 1979, Robert Baillie published a paper that calculated the sums of some modifications out to 20 decimal places [11].  The 10 modified series that he calculated were those that forbid terms with the digits $0,1,2,...,9$ in the denominators.  This method of calculation is discussed in Section 2.2.

# 2. Removing Digits

## 2.1 Estimates from Above and Below

We will first look at the modification of the harmonic series made by removing all terms whose denominators contain the digit 9. This modified harmonic series has the form

$$\begin{aligned} S = \frac{1}{1} + \frac{1}{2} + \ldots + \frac{1}{8} + \frac{1}{10} + \frac{1}{11} + \ldots + \frac{1}{18} + \frac{1}{20} + \frac{1}{21} \ldots \\ + \frac{1}{78} + \frac{1}{80} + \frac{1}{81} + \ldots + \frac{1}{88} + \frac{1}{100} + \frac{1}{101} + \ldots + \frac{1}{108} + \ldots \end{aligned}.$$

Here we show a method for estimating the sum from above and below, based on the method by Frank Irwin [4].

**Estimating from Above:** First, let $a_i$ be finite sub-series of $S$ such that each denominator in $a_i$ has exactly $i$ digits. Then

$$\begin{aligned} a_1 &= 1 + \frac{1}{2} + \cdots + \frac{1}{8} \\ a_2 &= \frac{1}{10} + \frac{1}{11} + \cdots + \frac{1}{88} \\ a_3 &= \frac{1}{100} + \frac{1}{101} + \cdots + \frac{1}{888} \\ &\vdots \end{aligned}.$$

Next, compare each $a_i$ with $a_1$. The first nine terms of $a_2$, are each less than or equal to $1/10$, so together they are less than $9/10$. The next nine terms are each less than or equal to $1/20$ and together less than $9/20 = (9/10)\cdot(1/2)$. Continue this process, and it is clear that $a_2 < 9/10 \cdot a_1$. Similarly, $a_3 < 81/100 \cdot a_1 = (9/10)^2 \cdot a_1$. Furthermore, $a_i < (9/10)^{i-1} \cdot a_1$. Then we can estimate the modified series from above with

$$S < a_1 + \left( \frac{9}{10} + \left(\frac{9}{10}\right)^2 + \left(\frac{9}{10}\right)^3 + \ldots \right) a_1 = 10a_1 \approx 27.18 .$$

**Estimating from Below:** To estimate the series from below, we use a method similar to the one for estimating from above. Here we partition the series differently and estimate each finite sub-series from the other side. We shift the partitions that we used above by one term, so that

$$\begin{aligned} b_1 &= \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{8} + \frac{1}{10} \\ b_2 &= \frac{1}{11} + \frac{1}{12} + \cdots + \frac{1}{88} + \frac{1}{100} \\ b_3 &= \frac{1}{101} + \frac{1}{102} + \cdots + \frac{1}{888} + \frac{1}{1000} \\ &\vdots \end{aligned} .$$

We divide the terms of $b_2$ again into groups of nine, and compare them to the terms in $b_1$. The first group consists of $1/11 + 1/12 + \ldots + 1/18 + 1/20$. Each of these terms is greater than or equal to $1/20$, and they are together greater than $9/20$. Likewise, the next nine terms are greater than or equal to $9/30 = (9/10)\cdot(1/3)$, and so on. Then $b_2 > 9/10 \cdot b_1$. Similarly, $b_3 > 81/100 \cdot b_1 = (9/10)^2 \cdot b_1$. We can see, then, that $b_i > (9/10)^{i-1} \cdot b_1$. So we can estimate the series from below by

$$S > 1 + b_1 + \left( \frac{9}{10} + \left(\frac{9}{10}\right)^2 + \left(\frac{9}{10}\right)^3 + \ldots \right) = 1 + 10b_1 \approx 19.17.$$

We have found that $19.17 < S < 27.18$. We can make the bounds closer by comparing the parts of the series to $a_2$ and $b_2$ instead of $a_1$ and $b_1$. In this case, the estimation from above becomes

$$S < a_1 + a_2 + \left( \frac{9}{10} + \left(\frac{9}{10}\right)^2 + \left(\frac{9}{10}\right)^3 + \ldots \right) a_2 = a_1 + 10a_2 \approx 23.3$$

The estimation from below follows similarly, and we get the following estimates for the modified series:

$$22.4 < S < 23.3.$$

Notice that these estimates are much closer together than the previous ones. If we keep going, and compare everything to $a_3$ and $b_3$ instead, then we have the following bounds:

$$22.8 < S < 23.0.$$

The bounds are very close together now. Going further, comparing the terms to $a_4$ and $b_4$ or $a_5$ and $b_5$ will not yield much improvement.

This method can be used without modification for removing, or forbidding, other digits from the series. With some small changes, it can also be used for finding bounds on a series modified by forbidding digits from the denominators in different base representations. For example, the modified series where no denominator contains the digit 2 in its base 3 representation would look like this:

$$S = \frac{1}{1} + \frac{1}{10} + \frac{1}{11} + \frac{1}{100} + \frac{1}{101} + \frac{1}{110} + \frac{1}{111} + \frac{1}{1000} + \dots.$$

## 2.2 "Exact" Values

These modified harmonic series converge extremely slowly, so it is impossible to compute their sums directly. However, an indirect method for calculating the sums to extreme accuracy was found by Robert Baillie [11]. His method gives the sums of the modified series up to 20 decimal places.

Baillie's method uses not only the modified harmonic series, but also the series $\sum_{x \in S} x^{-j}$ for $j \geq 1$ where $S$ is the set of positive integers that do not contain the forbidden digit. All of the series are partitioned into groups based on the number of digits in the denominator such that $s(i, j) = \sum_{x \in S} x^{-j}$ where $x$ has exactly $i$ digits. The sums of the first few groups, $i \leq 5$, are calculated directly. Then a recursion formula is used to calculate the sum of the next group, $s(i+1, j)$, by summing the previous groups, $s(i, j+n)$ with $0 \leq n \leq 10$. (Increasing the upper bound on $n$ would increase the accuracy of the final answer.) After values have been calculated for $i \leq 30$, the tail of the modified harmonic series is estimated. Then the calculated sum of the modified harmonic series is the sum of the $s(i,1)$, with $1 \leq i \leq 30$, plus the estimated tail.

In his paper, Baillie published values for series with the digits 0 through 9 forbidden– see Table 1.

Table 1

| Forbidden Digit | Value of Series |
|---|---|
| 0 | 23 . 10344 79094 20541 61603 |
| 1 | 16 . 17696 95281 23444 26657 |
| 2 | 19 . 25735 65328 08071 11453 |
| 3 | 20 . 56987 79509 61230 37107 |
| 4 | 21 . 32746 57995 90036 68663 |
| 5 | 21 . 83460 08122 96918 16340 |
| 6 | 22 . 20559 81595 56091 88416 |
| 7 | 22 . 49347 53117 05945 39817 |
| 8 | 22 . 72636 54026 79370 60283 |
| 9 | 22 . 90267 66192 64150 34816 |

Notice that if the digit 0 is considered as a ‘digit 10’, then the values of the modified series strictly increase as the value of the removed digit increases. This can be logically verified by thinking about the terms removed. Numbers with the digit 9 will be larger than numbers with the digit 8, so their inverses will be smaller. Then, when the digit 9 is forbidden, you are removing smaller terms than if you had forbid the digit 8, and therefore making the sum of the modified series larger.

## 2.3 Removing Digits in Other Bases

We now explore the possibility that the values of the modified series continue to follow this increasing trend across the bases. We programmed Baillie’s method into *Mathematica* to calculate the values of the modified series with each digit removed in bases 2 through 13. Values were calculated to 8 decimal digits and are listed in Table 2. The data is also presented in Graph 1 and Graph 2. The code for the *Mathematica* program can be found in the Appendix.

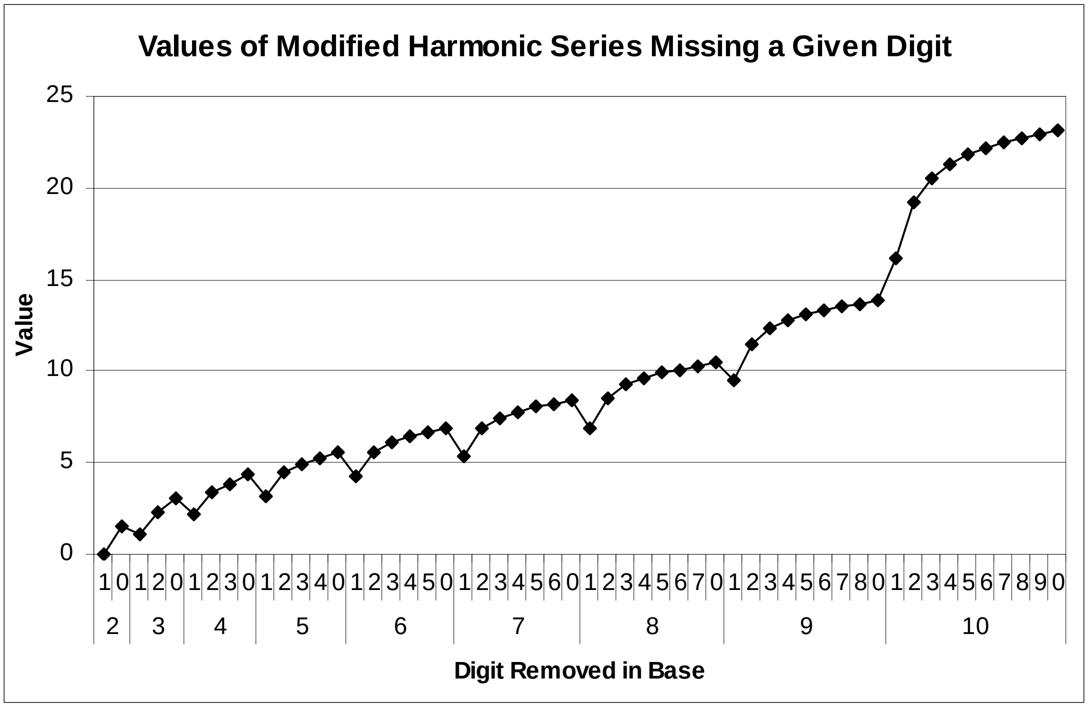
Looking at the data, we notice the following:

1. Within the bases, the values do indeed increase as the value of the removed digit increases. The same reasoning used for base ten at the end of Section 2.2 applies for each base here as well.
2. The changes in the values when moving from digit 0 in one base to digit 1 in the next are not strictly increasing. In the switches from bases 2 to 3 through 8 to 9 the values actually decrease. After that, the values increase during all of the base switches. Moreover, after base 10, the values start to increase rapidly between the bases. The value increases by approximately 28 between bases 10 and 11, by 277 between bases 11 and 12, and by 2526 between bases 12 and 13. We are unsure why the decrease turns to a rapid increase at base 10.

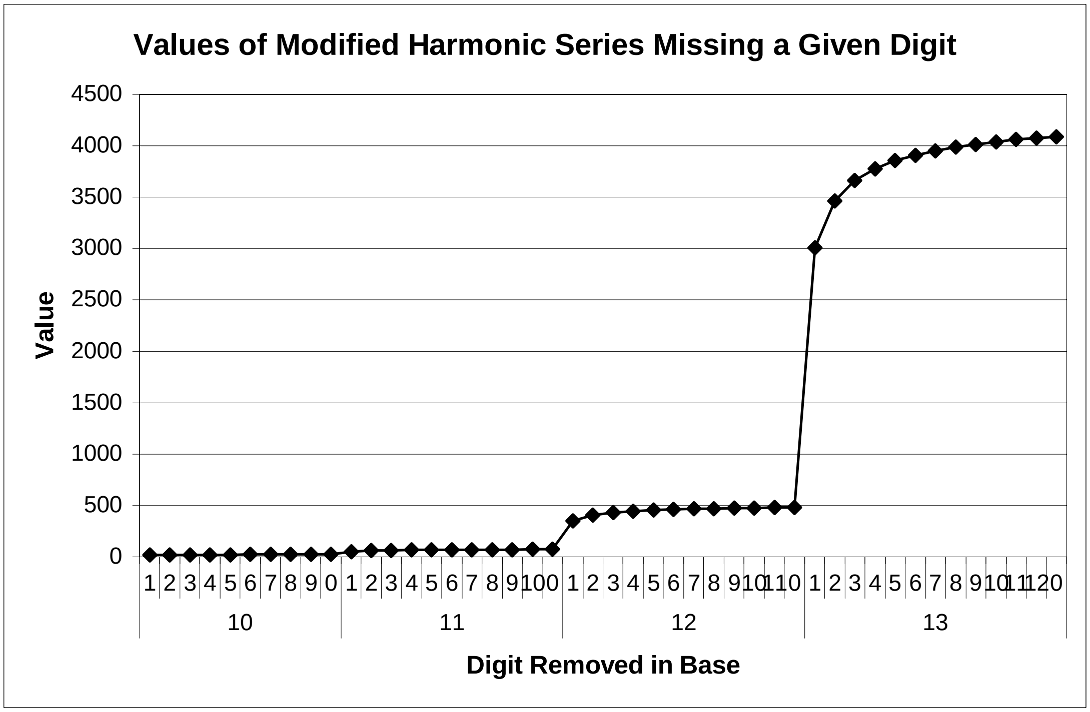
Table 2

| Base | Digit | Value |
|---|---|---|
| 2 | 1 | 0.00000000 |
| | 0 | 1.54907649 |
| 3 | 1 | 1.12918765 |
| | 2 | 2.25837531 |
| | 0 | 3.00667994 |
| 4 | 1 | 2.17021598 |
| | 2 | 3.35226509 |
| | 3 | 3.86343163 |
| | 0 | 4.34487537 |
| 5 | 1 | 3.17646972 |
| | 2 | 4.42434293 |
| | 3 | 4.95476277 |
| | 4 | 5.26816709 |
| | 0 | 5.61944553 |
| 6 | 1 | 4.21119134 |
| | 2 | 5.54405867 |
| | 3 | 6.10420079 |
| | 4 | 6.43125373 |
| | 5 | 6.65360550 |
| | 0 | 6.92967461 |
| 7 | 1 | 5.38187441 |
| | 2 | 6.83382553 |
| | 3 | 7.44036578 |
| | 4 | 7.79152738 |
| | 5 | 8.02843456 |
| | 6 | 8.20318757 |
| | 0 | 8.43169885 |
| 8 | 1 | 6.91431322 |
| | 2 | 8.55507667 |
| | 3 | 9.24001165 |
| | 4 | 9.63454989 |
| | 5 | 9.89929279 |
| | 6 | 10.09361321 |
| | 7 | 10.24491575 |
| | 0 | 10.44247078 |
| 9 | 1 | 9.46514637 |
| | 2 | 11.47336126 |
| | 3 | 12.31641363 |
| | 4 | 12.80136383 |
| | 5 | 13.12584476 |
| | 6 | 13.36333249 |
| | 7 | 13.54774311 |
| | 8 | 13.69713229 |
| | 0 | 13.87671762 |
| 10 | 1 | 16.17696953 |
| | 2 | 19.25735653 |
| | 3 | 20.56987795 |
| | 4 | 21.32746580 |
| | 5 | 21.83460081 |
| | 6 | 22.20559876 |
| | 7 | 22.49347531 |
| | 8 | 22.72636540 |
| | 9 | 22.92067662 |
| | 0 | 23.10344791 |
| 11 | 1 | 51.80277847 |
| | 2 | 60.69852516 |
| | 3 | 64.56346908 |
| | 4 | 66.80652815 |
| | 5 | 68.31066024 |
| | 6 | 69.41147501 |
| | 7 | 70.26559795 |
| | 8 | 70.95642496 |
| | 9 | 71.53268572 |
| | 10 | 72.02490144 |
| | 0 | 72.30689270 |
| 12 | 1 | 350.12882598 |
| | 2 | 406.12298007 |
| | 3 | 430.54795503 |
| | 4 | 444.70024409 |
| | 5 | 454.16259573 |
| | 6 | 461.06558898 |
| | 7 | 466.40502104 |
| | 8 | 470.71123935 |
| | 9 | 474.29399031 |
| | 10 | 477.34710303 |
| | 11 | 479.99856356 |
| | 0 | 481.07330330 |
| 13 | 1 | 3007.14710139 |
| | 2 | 3465.00907859 |
| | 3 | 3663.74869646 |
| | 4 | 3778.30562627 |
| | 5 | 3854.51113468 |
| | 6 | 3909.83900231 |
| | 7 | 3952.44611417 |
| | 8 | 3986.67040668 |
| | 9 | 4015.04130271 |
| | 10 | 4039.13884323 |
| | 11 | 4060.00439897 |
| | 12 | 4078.35448108 |
| | 0 | 4085.27647107 |

Graph 1



Graph 2



# 3. Convergence and Divergence of Other Modifications

Removing terms that contain certain digits is not the only way to modify the harmonic series. In this section we look at one specific example; removing all terms with composite denominators. We will also discuss a method for determining when any given modification converges.

## 3.1 Reciprocals of Primes

We've seen that by forbidding a certain digit from the denominators of the harmonic series, the resulting series will converge. At first glance, the prime numbers may seem much rarer, much more spaced out than numbers that do not contain a certain digit. However, this does not mean that we can modify the harmonic series to contain only reciprocals of prime numbers and have it converge. The series $\sum \frac{1}{p}$ where $p$ is prime actually diverges. This was first shown by Euler in 1737. Here, we present a proof by contradiction due to Paul Erdös (1938) [12].

PROOF: Assume that the series converges. Then there exists $k \in \mathbb{N}$ such that

$$\sum_{i>k} \frac{1}{p_i} < \frac{1}{2}$$

where $p_i$ is the $i^{th}$ prime. Now define $\lambda_k(n)$ as the number of integers less than $n$ whose prime factors are all less than or equal to $p_k$. We also define $\mu_k(n)$ as the number of integers less than $n$ whose prime factors contain at least one $p_i$, $i > k$. Then by definition,

$$\lambda_k(n) + \mu_k(n) = n \tag{1}$$

Note that any of the numbers enumerated by $\lambda_k(n)$ can be represented as $p_1^{A_1} \cdots p_k^{A_k} \cdot s^2$, $A_i \in \{0,1\}$. There are $2^k$ ways to choose the square free part, so $\lambda_k(n) \le 2^k \sqrt{n}$. Now the number of integers less than $n$ that are divisible by $p_i$ is $\left\lfloor \frac{n}{p_i} \right\rfloor$. Then

$$\mu_k(n) \le \sum_{i>k} \frac{n}{p_i} < \frac{n}{2}. \tag{2}$$

Combining equations (1) and (2) we get,

$$\frac{n}{2} < \lambda_k(n) \le 2^k \sqrt{n} \quad \forall n \in \mathbb{N}.$$

But values of $n$ greater than or equal to $n = 2^{2k+2}$ give a contradiction. □

## 3.2 General Conditions for Convergence

In his 1941 paper, I. E. Perlin established sufficient conditions for a modification of any series to converge [7]. We will present his results here, along with some of his applications to the harmonic series, and further, we will apply the results to several other modifications of the harmonic series.

Given a series $\sum_{n=1}^{\infty} a_n$, partition the terms into groups, with the $i$th group containing $N_i$ terms. Perlin then defines the following:

| | |
|---|---|
| $\{M_i\}$ | a positive sequence s.t. $\forall a_{n,i}$ in the $i$th group, $\lvert a_{n,i} \rvert \le M_i$ |
| $\{a_{n_j}\}$ | the sequence of terms of the series being deleted |
| $k_i$ | the number of terms deleted from the $i$th group |
| $p_i$ | $1 - k_i / N_i$ |
| $\sum_{n=1}^{\infty}{}' a_n$ | the series after deleting the terms $\{a_{n_j}\}$; the modified series |

For many modifications, it is easier to count the number of terms *remaining* in each group, rather than the ones removed. If we define $r_i$ to be the number of terms remaining in the $i$th group, then we get $p_i = 1 - k_i / N_i = r_i / N_i$.

The following two theorems test for the convergence of the modified series. See [7] for the proofs.

THEOREM 1: If both $\sum_{i=1}^{\infty} p_i$ and $\sum_{i=1}^{\infty} \lvert M_{i+1}N_{i+1} - M_i N_i \rvert$ converge, then $\sum_{n=1}^{\infty}{}' a_n$ converges.

THEOREM 2: If $\sum_{i=1}^{\infty} p_i$ converges and $\{M_i N_i\}$ is monotonically decreasing, then $\sum_{n=1}^{\infty}{}' a_n$ converges.

We can now apply these two theorems as convergence tests to various modifications of the harmonic series. For all of the modifications discussed, the series will be partitioned in the same manner as in Section 2.1 – the $i$th group will be the terms that have exactly $i$ digits in the denominator. Then we can calculate $N_i = 9 \cdot 10^{i-1}$, and choose $M_i = 10^{i-1}$. It is easy to see that $\sum |M_{i+1}N_{i+1} - M_i N_i|$ converges, and $\{M_i N_i\}$ is monotonically decreasing. So any modification that uses this partitioning will automatically have the second conditions of both Theorem 1 and Theorem 2 satisfied.

If we modify the series by removing the terms that contain the digit nine, then, using the above partitioning, $r_i = 8 \cdot 9^{i-1}$ and therefore $p_i = \frac{8}{9} \cdot \left(\frac{9}{10}\right)^{i-1}$. So, using either Theorem 1 or Theorem 2, we see that this modification converges – as we have already shown.

Another modification is to remove all of the terms that have two consecutive digits alike [7]. For example, the terms $1/22$ and $1/75548$ would be removed, but the term $1/8384$ would remain. To check for convergence we use the same partitioning and calculate $r_i = 9^i$ and $p_i = \left(\frac{9}{10}\right)^{i-1}$. Since $\sum p_i$ converges, this modified series also converges.

We can apply these theorems to many other modifications of the harmonic series. Several modifications are listed in Table 3, along with their $r_i$ and $p_i$ values and the result of using Theorem 1 and Theorem 2 as a convergence test. The two modifications discussed above are also included for completion. Note that all the convergence tests for the modifications below use the same partitioning as above and therefore have $\sum |M_{i+1}N_{i+1} - M_i N_i|$ converging, and $\{M_i N_i\}$ monotonically decreasing.

Table 3

| Modification | $r_i$ | $p_i$ | $\sum p_i$ converges | $\sum{}' a_n$ converges |
|---|---|---|---|---|
| The digit 9 is forbidden | $8 \cdot 9^{i-1}$ | $\frac{8}{9} \cdot \left(\frac{9}{10}\right)^{i-1}$ | Yes | Yes |
| The digit 0 is forbidden | $9^i$ | $\left(\frac{9}{10}\right)^{i-1}$ | Yes | Yes |
| No consecutive digits are the same number | $9^i$ | $\left(\frac{9}{10}\right)^{i-1}$ | Yes | Yes |
| No consecutive digits are consecutive numbers* | $9 \cdot 7^{i-1}$ | $\left(\frac{7}{10}\right)^{i-1}$ | Yes | Yes |
| All digits are even | $4 \cdot 5^{i-1}$ | $\frac{4}{9} \cdot \left(\frac{1}{2}\right)^{i-1}$ | Yes | Yes |
| All digits are odd | $5^i$ | $\frac{5}{9} \cdot \left(\frac{1}{2}\right)^{i-1}$ | Yes | Yes |
| No terms are multiples of 5 | $8 \cdot 9 \cdot 10^{i-2}$ | $\frac{8}{10}$ | No | |
| No terms are multiples of 2 | $5 \cdot 9 \cdot 10^{i-2}$ | $\frac{5}{10}$ | No | |
| All terms are multiples of 5 | $2 \cdot 9 \cdot 10^{i-2}$ | $\frac{1}{5}$ | No | |
| All consecutive digits are consecutive numbers* | $9 \cdot 3^{i-1}$ | $\frac{9}{10} \cdot \left(\frac{3}{10}\right)^{i-1}$ | Yes | Yes |

* Here, numbers are considered consecutive if they are adjacent to each other in the cyclic set $\{0,1,2,...,9\}$. Also, every number is consecutive to itself. For example, in the number $78890901233210012$ every consecutive digit is a consecutive number.

The last column only indicates if $\sum{}' a_n$ converges by the convergence test. Since the theorems do not test for divergence, if $\sum p_i$ does not converge, then the theorems cannot determine anything about the convergence or divergence of the modified series. In theses cases, other methods should be used to attempt to determine the convergence of these modifications. In fact, two of the modifications listed in Table 3 do diverge. The modification that does not contain multiples of five, and the one that does not contain multiples of two, both diverge because they have a divergent subsequence – namely the reciprocals of primes that was shown to diverge in Section 3.1.

We can also use these theorems for modifications of the harmonic series in different bases. Although the series will still be partitioned based on the number of digits in the denominator, the values of $N_i$ and $M_i$ will change as the base changes. However, if $b$ is the base, then $N_i = (b-1)\cdot b^{i-1}$ and we can choose $M_i = \left(\frac{1}{b}\right)^{i-1}$. Then we still have that $\sum \left| M_{i+1}N_{i+1} - M_iN_i \right|$ converges, and $\left\{ M_iN_i \right\}$ is monotonically decreasing. Table 4 shows a few modifications of the harmonic series in different bases.

Table 4

| Base | Modification** | $r_i$ | $p_i$ | $\sum p_i$ converges | $\sum{}' a_n$ converges |
|---|---|---|---|---|---|
| 3 | The digit 2 is forbidden | $1\cdot 2^{i-1}$ | $\frac{1}{2}\cdot\left(\frac{2}{3}\right)^{i-1}$ | Yes | Yes |
| 4 | The digit 2 is forbidden | $2\cdot 3^{i-1}$ | $\frac{2}{3}\cdot\left(\frac{3}{4}\right)^{i-1}$ | Yes | Yes |
| 7 | The digit 2 is forbidden | $5\cdot 6^{i-1}$ | $\frac{5}{6}\cdot\left(\frac{6}{7}\right)^{i-1}$ | Yes | Yes |
| 12 | The digit 2 is forbidden | $10\cdot 11^{i-1}$ | $\frac{10}{11}\cdot\left(\frac{11}{12}\right)^{i-1}$ | Yes | Yes |
| 3 | No consecutive digits | 0 | 0 | Yes | Yes |
| 4 | No consecutive digits | 3 | $\left(\frac{1}{4}\right)^{i-1}$ | Yes | Yes |
| 7 | No consecutive digits | $6\cdot 4^{i-1}$ | $\left(\frac{4}{7}\right)^{i-1}$ | Yes | Yes |
| 12 | No consecutive digits | $11\cdot 9^{i-1}$ | $\left(\frac{9}{12}\right)^{i-1}$ | Yes | Yes |
| 3 | Only consecutive digits | $2\cdot 3^{i-1}$ | 1 | No | |
| 4 | Only consecutive digits | $3^i$ | $\left(\frac{3}{4}\right)^{i-1}$ | Yes | Yes |
| 7 | Only consecutive digits | $6\cdot 3^{i-1}$ | $\left(\frac{3}{7}\right)^{i-1}$ | Yes | Yes |
| 12 | Only consecutive digits | $11\cdot 3^{i-1}$ | $\left(\frac{3}{12}\right)^{i-1}$ | Yes | Yes |

** The modifications described as “No consecutive digits” are modified series where the denominators have no consecutive digits that are consecutive numbers. The modifications described as “Only consecutive digits” are modified series where the denominators have every consecutive digit being a consecutive number. Here, numbers in base $b$ are considered consecutive if they are adjacent to each other in the cyclic set $\{0,1,\ldots,b-1\}$. As before, every number is consecutive to itself.

In this table, the theorems can make no determination about convergence for the modification in base three where every digit is consecutive in number. However, we know that this series diverges because it is, in fact, the entire harmonic series. Another modification of interest is the harmonic series in base three where no consecutive digits are consecutive numbers. This modification is actually finite series, consisting of the terms that have only one digit in the denominator.

# 4. The Harmonic Series in $\mathbf{Q_p}$

In the same way the rational numbers are completed to form the field of real numbers, they can also be completed to form the field of $p$-adic numbers. The process of completing relies upon the use of a norm. If we use the ‘normal’ absolute value, then when we complete $\mathbb{Q}$ we get $\mathbb{R}$. If we use a different norm instead, we get a completely different field. The field of $p$-adic numbers, $\mathbb{Q}_p$, is formed by completing $\mathbb{Q}$ using the following norm:

$$|x|_p = \begin{cases} p^{-ord_p(x)} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases},$$

where

$$ord_p x = \begin{cases} \text{the largest } t \text{ for which } p^t \mid x & \text{if } x \in \mathbb{Z} \\ ord_p a - ord_p b, & \text{if } x = a/b,\ a, b \in \mathbb{Z},\ b \neq 0 \end{cases}.$$

There are two things to note about this norm. First, $p$ must be prime. Second, the prime numbers each define a different norm, and the fields created with the different primes are not isomorphic to each other. The 2-adic numbers are quite different from the 7-adic numbers.

We will discuss only a few of the general properties of $\mathbb{Q}_p$ here, specifically, the properties needed throughout this section. For further reading on $p$-adic numbers see [13] and [14].

From the definition of the norm, we see that it can only take on discreet values. Namely, the norm of a number must be in the set $\{\dots p^{-2}, p^{-1}, 0, p^0, p^1, p^2 \dots\}$. As a result, the following two inequalities are identical:

$$|x|_p < p^2 \quad \text{and} \quad |x|_p \leq p .$$

Within $\mathbb{Q}_p$, the set $\mathbb{Z}_p = \{x \in \mathbb{Q}_p \mid |x|_p \leq 1\}$ is call the set of $p$-adic integers. Furthermore, the natural numbers, $\mathbb{N}$, are contained in $\mathbb{Z}_p$ – they are all $p$-adic integers. Additionally, the inverses of the $p$-adic integers are again $p$-adic integers. From these properties, we know that $\{1/n \mid n \in \mathbb{N}\} \in \mathbb{Z}_p$.

Since we are looking at the harmonic series, we will also need the condition for a series to converge in $\mathbb{Q}_p$. Without going into details, the condition is the following: A series $\sum_{n=1}^{\infty} a_n$ with $a_n \in \mathbb{Q}_p$ converges in $\mathbb{Q}_p$ if and only if $\lim_{n\to\infty} |a_n|_p = 0$. We will also need this fact: For a sequence $a_n$ to converge in $\mathbb{Q}_p$, it must have $\lim_{n\to\infty} |a_{n+1} - a_n|_p = 0$.

## 4.1 The p-adic Harmonic Series

It is interesting to examine the harmonic series in $\mathbb{Q}_p$ and compare it to what we see in $\mathbb{R}$. To this end, we prove several theorems.

THEOREM 3: The harmonic series, $\sum_{n=1}^{\infty} 1/n$ diverges in $\mathbb{Q}_p$.

PROOF: We look at two different types of terms. First, the terms that can be written as $1/mp^k$ where $m \in \{0,1,\ldots,p-1\}$. These terms have $p$-adic norm equal to $p^k$. So as $n \to \infty$ the $p$-adic norm of these terms goes to infinity. The rest of the terms are not divisible by any power of $p$, so their $p$-adic norm is equal to 1. Then $\lim_{n\to\infty} |a_n|_p \neq 0$, and hence the harmonic series diverges in $\mathbb{Q}_p$. □

COROLLARY: No sub-series of the harmonic series converges in $\mathbb{Q}_p$.

PROOF: From the $p$-adic norms of the terms $a_n = 1/n$, it is clear that no subsequence of $a_n$ will have the property $\lim_{n\to\infty} |a_n|_p = 0$. □

## 4.2 The p-adic Harmonic Sequence

In the previous section, we saw that the harmonic series diverges in $\mathbb{Q}_p$, and has no converging sub-series. Here, we will look instead at the 'harmonic' sequence. That is, the sequence $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4} \ldots$ in $\mathbb{Q}_p$.

THEOREM 4: The harmonic sequence diverges in $\mathbb{Q}_p$.

PROOF: For a sequence $a_n$ to converge in $\mathbb{Q}_p$, we need $\lim_{n\to\infty} |a_{n+1} - a_n|_p = 0$. We examine the subsequence $a_n = \dfrac{1}{p^n}$. Then,

$$\begin{aligned}
|a_{n+1}-a_n|_p &= \left|\frac{1}{p^{n+1}}-\frac{1}{p^n}\right|_p \\
&= \left|\frac{1}{p^n}\right|_p \left|\frac{1}{p}-1\right|_p \\
&= p^n\left|\frac{1-p}{p}\right|_p \\
&= p^{n+1}|1-p|_p \\
&= p^{n+1}
\end{aligned}$$

.

So $\lim_{n\to\infty}|a_{n+1}-a_n|_p \neq 0$ and the subsequence diverges. Hence, the harmonic sequence also diverges. □

THEOREM 5: The harmonic sequence has a converging subsequence in $\mathbb{Z}_p$.

PROOF: In $p$-adic analysis, every infinite sequence of $p$-adic integers has a convergent subsequence. Let $a_n = 1/n$ such that $n$ is not divisible by $p$. Then $a_n$ is an infinite sequence of $p$-adic integers and is also a subsequence of the harmonic sequence. Hence, there exists a converging subsequence of $a_n$ and therefore a converging subsequence of the harmonic sequence. □

The following theorem shows an interesting property regarding the density of the harmonic sequence in a subset of $\mathbb{Q}_p$.

THEOREM 6: The set $\{1, \tfrac{1}{2}, \tfrac{1}{3}, ...\}$ is dense in the set $\left\{x\in\mathbb{Q}_p \,\middle|\, |x|_p \geq 1\right\}$.

PROOF: Let $A=\left\{1/n \,\middle|\, n\in\mathbb{N}\right\}$, and let $B=\left\{1/x \,\middle|\, x\in\mathbb{Z}_p\right\}$. By properties of the $p$-adic norm, $B$ is equivalent to the set $\left\{x\in\mathbb{Q}_p \,\middle|\, |x|_p \geq 1\right\}$. Choose $1/x$ in $B$ such that $|x|_p = c \leq 1$. Since $\mathbb{N}$ is dense in $\mathbb{Z}_p$ we can choose $n$ such that $|x-n|_p < c^2\varepsilon$

for any $c > \varepsilon > 0$. Then $|x-n|_p < c^2\varepsilon < c = |x|_p$, so by the isosceles triangle property, $|x|_p = |n|_p = c$. So

$$\left|\frac{1}{n}-\frac{1}{x}\right|_p = \left|\frac{x-n}{nx}\right|_p < \frac{\varepsilon c^2}{c^2} = \varepsilon . \qquad \square$$

# Appendix

Code for *Mathematica* program: This program calculates the sum of the harmonic sub-series defined by removing a specified digit from the denominators of the terms, where the denominators can be written in a specified base. Here, the program is removing the digit 9 from base 10 representations.

```
base = 10; digit = 9;

s = Array[0, {31, 5}];

b = Function[z, Sum[x^z, {x, 0, base - 1}] - (digit^z)];
c = Function[{h, z}, ((-1)^z)*((h + z - 1)!)/(z!)/((h - 1)!)];
a = Function[{h, z}, b[z]*c[h, z]/base^(h + z)];

i = 1; j = 1;

While[i < 4,
  n = base^(i - 1); ad = 0;

  While[n < base^i,
      If[Length[Cases[IntegerDigits[n,base],digit]]==0, ad=ad+1/n^j]
      ; n++]

  s[[i, j]] = ad;
  ; i++]

i = 4; j = 1;

ad = Array[0, 5]; k = 1; While[k <= 5, ad[[k]] = 0; k++];

n = base^(i - 1);

While[n < base^i,
  If[Length[Cases[IntegerDigits[n, base], digit]] == 0,
    j = 1;
    While[j <= 5, ad[[j]] = ad[[j]] + 1/(n^j); j++]
    ]
  ; n++]
```

```
j = 1;

While[j <= 5, s[[i, j]] = ad[[j]]; j++];

i = 5;

While[i <= 30,
    j = 1;
    While[j <= 5,
      s[[i, j]] = (Sum[a[j, n]*s[[i - 1, j + n]], {n, 1,5 - j}] +
                  ((base - 1)*s[[i - 1, j]])/(10^j)); j++]
    ; i++];

s[[31, 1]] = (base - 1)*s[[30, 1]];

t = Sum[s[[i, 1]], {i, 1, 31}];
```

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