HODGE GENERA OF ALGEBRAIC VARIETIES, II.

SYLVAIN E. CAPPELL, ANATOLY LIBGOBER, LAURENTIU MAXIM, AND JULIUS L. SHANESON

Abstract. We study the behavior of Hodge-theoretic genera under morphisms of complex algebraic varieties. We prove that the additive χ_y^c-genus which arises in the motivic context satisfies the so-called "stratified multiplicative property", which shows how to compute the invariant of the source of an algebraic morphism from its values on various varieties that arise from the singularities of the map. By considering morphisms to a curve, we obtain a Hodge-theoretic analogue of the Riemann-Hurwitz formula. We also study the contribution of monodromy to the χ_y-genus of a family of compact complex manifolds, and prove an Atiyah-Meyer type formula for twisted χ_y-genera, both in the algebraic and the analytic context. This formula measures the deviation from multiplicativity of the χ_y-genus, and expresses the correction terms as higher-genera associated to the period map; these higher-genera are Hodge-theoretic extensions of Novikov higher-signatures to analytic and algebraic settings. By making use of Saito's theory of mixed Hodge modules, we also obtain formulae of Atiyah-Meyer type for the corresponding Hirzebruch characteristic classes.

Contents

Date: April 21, 2009.

2000 *Mathematics Subject Classification.* Primary 57R20, 32S20, 14C30, 32S35, 32S50, 14D05, 14D06, 14D07, 55N33; Secondary 57R45, 32S60, 13D15, 16E20.

A. Libgober partially supported by an NSF grant. S. Cappell and J. Shaneson partially supported by grants from DARPA. L. Maxim partially supported by a grant from the NYU Research Challenge Fund.

1. Introduction

In the mid 1950's, Chern, Hirzebruch and Serre [15] proved that if $F \hookrightarrow E \to B$ is a fiber bundle of closed, coherently oriented, topological manifolds such that the fundamental group of the base B acts trivially on the cohomology of F, then the signatures of the spaces involved are related by a simple multiplicative relation: $\sigma(E) = \sigma(F) \cdot \sigma(B)$. A decade later, Atiyah [2], and respectively Kodaira [29], observed that without the assumption on the action of the fundamental group the multiplicativity relation fails. Moreover, Atiyah showed that the deviation from multiplicativity is controlled by the cohomology of the fundamental group of B.

The main goal of this paper is to describe in a systematic way multiplicativity properties of the Hirzebruch χ_y-genus (and the associated characteristic classes) for suitable holomorphic submersions of compact complex manifolds. Indeed we study such problems for general algebraic maps, even of singular varieties, by using extensions of the Hirzebruch χ_y-genus to the singular setting; our results in this generality are expressed in terms of a stratified multiplicative property.

For example, in the classical case of fibrations of closed manifolds we extend the results of Chern, Hirzebruch and Serre in several different directions in the analytic and algebraic setting. First, we show that in the case when the Chern-Hirzebruch-Serre assumption of the triviality of the monodromy action is fulfilled, the χ_y-genus is multiplicative for a suitable holomorphic submersion $f : E \to B$ of compact complex manifolds, for which all the direct image sheaves $R^k f_* \mathbb{R}_E$ underly a polarizable variation of real Hodge structures, as studied by Griffiths [24]. In fact, the key to multiplicativity in this case is Griffiths' rigidity theorem for polarizable variations of real Hodge structures on compact complex manifolds. Hence, since by the Hodge index theorem the signature of a Kähler manifold is the value of the χ_y-genus for $y = 1$, this theorem can indeed be viewed as an Hodge-theoretic extension of the Chern-Hirzebruch-Serre result. Secondly, we consider fibrations with non-trivial monodromy action, and prove a Hodge-theoretic analogue of the Atiyah signature formula. We also derive a formula for the χ_y-genus of E in which the correction from the multiplicativity of the χ_y-genus is measured via pullbacks under the period map associated with our fibration, of certain cohomology classes of the quotient of the period domain by the action of the monodromy group. Only for some manifolds F serving as a fiber of the fibration in discussion, the quotient of the period domain is also the classifying space of the monodromy group in the topological sense. Nevertheless, when one is only interested in the value $y = 1$ yielding the signature, our correction terms coincide with those of Atiyah. In fact, the Atiyah terms are the appropriate Novikov-type higher-signatures, and our correction terms are thus Hodge-theoretic extensions of these Novikov invariants to the analytic and algebraic category.

We now present in detail the content of each section and summarize our main results.

In Section 2, we study the behavior of χ_y-genera under maps of complex algebraic varieties. We first consider a morphism $f : E \to B$ of complex algebraic varieties with B smooth and connected, which is a locally trivial topological fibration in the (strong) complex topology, and show that under certain assumptions on monodromy the χ_y-genera are multiplicative:

Lemma 1.1. *Let $f : E \to B$ be an algebraic morphism of complex algebraic varieties, with B smooth and connected. Assume that f is a locally trivial fibration in the strong (complex) topology of B with fiber F. If the action of $\pi_1(B)$ on $H^*(F, \mathbb{Q})$, respectively $H^*_c(F, \mathbb{Q})$, is trivial, then*

$$\chi_y(E) = \chi_y(B) \cdot \chi_y(F), \quad \textit{resp.} \quad \chi^c_y(E) = \chi^c_y(B) \cdot \chi^c_y(F) \,. \tag{1.1}$$

Such multiplicativity properties of genera were previously studied in certain special cases in connection with rigidity (e.g., see [25, 26, 36]). For instance, Hirzebruch's χ_y-genus is multiplicative in bundles of (stably) almost complex manifolds with structure group a compact connected Lie group (the latter condition implies trivial monodromy), and in fact it is uniquely characterized by this property. The proof of our multiplicativity result uses the fact that the Leray spectral sequences of the map f are spectral sequences in the category of mixed Hodge structures. The later claim for the case of quasi-projective varieties E and B has a nice geometric proof due to Arapura [1]. In full generality, it is a well known simple application of Saito's deep theory of algebraic mixed Hodge modules, as we explain in some detail in Section §5.2.

In Section 2.3, we consider algebraic morphisms that are allowed to have singularities, and extend the above multiplicativity property to this general stratified case. More precisely, we prove that, under the assumption of trivial monodromy along the strata of our map, the additive χ^c_y-genus that arises in the motivic context satisfies the so-called "stratified multiplicative property":

Proposition 1.2. *Let $f : X \to Y$ be an algebraic morphism of (possible singular) complex algebraic varieties. Assume that there is a (finite) decomposition of Y into locally closed and connected complex algebraic submanifolds $S \subset Y$ such that the restrictions $(R^k f_! \mathbb{Q}_X)|_S$ of all direct image sheaves to all pieces S are constant. Then*

$$\chi^c_y(X) = \sum_S \chi^c_y(S) \cdot \chi^c_y(F_s) \,, \tag{1.2}$$

were $F_s = \{f = s\}$ is a fiber over a point $s \in S$.

This property shows how to compute the invariant of the source of an algebraic morphism from its values on various varieties that arise from the singularities of the map, thus yielding powerful topological constraints on the singularities of any algebraic map. It also provides a method of inductively computing these genera of varieties. A similar result was obtained by Cappell, Maxim and Shaneson, for the behavior of intersection homology Hodge-theoretic invariants, both genera and characteristic classes (see [9], and also [8]). Such formulae were first predicted by Cappell and Shaneson in the early 1990's, see the announcements [13, 46], following their earlier work [12] on stratified multiplicative properties for signatures and associated topological characteristic classes defined using intersection homology (see also [[7], §4], and [49] for a functorial interpretation of Cappell-Shaneson's L-classes of self-dual sheaves, generalizing the Goresky-MacPherson L-classes).

In the special case of maps to a smooth curve, and under certain assumptions for the monodromy along the strata of special fibers, in §3.2 (see also §5.4) we obtain a Hodge-theoretic analogue of the Riemann-Hurwitz formula [28]. A simple but important special case is the following

Example 1.3. Let $f : X \to C$ be a proper algebraic morphism of complex algebraic manifolds, with C a curve. Assume f has only isolated singularities so that the set $Sing(f)$ of all singularities of f is finite. Assume, moreover, that the restriction $(R^k f_! \mathbb{Q}_E)|_{C\setminus D}$ of all direct image sheaves to the complement of the discriminant $D := f(Sing(f)) \subset C$ are constant. Then

$$\chi_y^c(X) = \chi_y^c(C) \cdot \chi_y^c(F_s) - \sum_{x \in Sing(f)} \chi_y([\tilde{H}^*(M_x, \mathbb{Q})]) \,, \tag{1.3}$$

were $F_s = \{f = s\}$ is a fiber over a point $s \in C\setminus D$. Here $\tilde{H}^*(M_x, \mathbb{Q})$ is the reduced cohomology of the Milnor fiber M_x of f at $x \in Sing(f)$, which carries a canonical mixed Hodge structure (e.g. by [34]).

The proof uses Hodge-theoretic aspects of the nearby and vanishing cycles in the context of one-parameter degenerations of compact algebraic manifolds.

The contribution of monodromy to χ_y-genera is studied in Section 4. This can often be applied to compute the summands arising from singularities in stratified multiplicativity formulae without monodromy assumptions. For simplicity, we first consider a suitable holomorphic submersion $f : E \to B$ of compact complex manifolds (thus a fibration in the strong topology), and compute $\chi_y(E)$ so that the (monodromy) action of $\pi_1(B)$ on the cohomology of the typical fiber is taken into account. We can prove this important result also in the analytic context:

Theorem 1.4. *Let $f : E \to B$ be a holomorphic submersion of compact complex manifolds. Assume we are in any one of the following cases:*

(1) *E is a Kähler manifold, or more generally bimeromorphic to a compact Kähler manifold.*
(2) *f is a projective morphism.*
(3) *f is an algebraic morphism of complex algebraic manifolds.*

(Note that, by Griffiths' work [24], the direct image sheaf $R^k f_* \mathbb{R}_E$ defines a polarizable variation of $\mathbb{R}$-Hodge structures of weight k, so that

$$\mathcal{H}^{p,q} := Gr^p_{\mathcal{F}}(R^{p+q} f_* \mathbb{R}_E \otimes_{\mathbb{R}} \mathcal{O}_B) \simeq R^q f_* \Omega^p_{E/B} \,.$$

In particular, all the coherent sheaves $\mathcal{H}^{p,q}$ are locally free.)

Then the χ_y-genus of E can be computed by the following formula:

$$\chi_y(E) = \int_{[B]} ch^*(\chi_y(f)) \cup \tilde{T}^*_y(TB) \,, \tag{1.4}$$

where

$$\chi_y(f) := \sum_k (-1)^k \cdot \chi_y([R^k f_* \mathbb{R}_E, \mathcal{F}^\cdot]) \in K^0(B)[y]$$

is the K-theory χ_y-characteristic of f, with

$$\chi_y([\mathcal{L}, \mathcal{F}^\cdot]) := \sum_p [Gr^p_{\mathcal{F}}(\mathcal{L} \otimes_{\mathbb{R}} \mathcal{O}_B)] \cdot (-y)^p \in K^0(B)[y, y^{-1}]$$

the K-theory χ_y-characteristic of a polarizable variation of $\mathbb{R}$-Hodge structures $\mathcal{L}$ (for $\mathcal{F}^\cdot$ the corresponding Hodge filtration of the associated flat vector bundle).

Here $K^0(B)$ is the Grothendieck group of holomorphic vector bundles, ch^* denotes the Chern character and $\tilde{T}^*_y(-)$ is the (unnormalized) Hirzebruch characteristic class [25] corresponding to the χ_y-genus. The proof of this result uses the Grothendieck-Riemann-Roch theorem (which for compact complex manifolds follows from [30]) and standard facts from classical Hodge theory. We prove in fact the following characteristic class version in the spirit of a Riemann-Roch theorem for the trivial variation of Hodge structures $[\mathbb{R}_{\mathbb{E}}, \mathcal{F}^\cdot]$ on E:

$$\begin{aligned} f_*(\ \tilde{T}^*_y(TE)\ ) &= f_*(\ ch^*(\chi_y([\mathbb{R}_{\mathbb{E}}, \mathcal{F}^\cdot])\ ) \cup \tilde{T}^*_y(TE)\ ) \\ &= ch^*(\ \chi_y(f_*[\mathbb{R}_{\mathbb{E}}, \mathcal{F}^\cdot])\ ) \cup \tilde{T}^*_y(TB) \,, \end{aligned} \tag{1.5}$$

where

$$f_*[\mathbb{R}_{\mathbb{E}}, \mathcal{F}^\cdot] := \sum_k (-1)^k \cdot [R^k f_* \mathbb{R}_E, \mathcal{F}^\cdot] \in K^0(\mathrm{VHS}^p(B))$$

in the Grothendieck group of polarizable variations of $\mathbb{R}$-Hodge structures on B.

Our formula (1.4) is a Hodge-theoretic analogue of Atiyah's formula for the signature of fiber bundles [2], and measures the deviation from multiplicativity of the χ_y-genus in the presence of monodromy. As a consequence of formula (1.4), we point out that if the action of $\pi_1(B)$ on the cohomology of the typical fiber F underlies locally constant variations of Hodge structures, then the χ_y-genus is still multiplicative, i.e., $\chi_y(E) = \chi_y(B) \cdot \chi_y(F)$.

More generally, the deviation from multiplicativity in (1.4) can be expressed in terms of higher-genera associated to cohomology classes of the quotient by the monodromy group of the corresponding period domain associated to the polarizable variation of $\mathbb{R}$-Hodge structures $[\mathcal{L}, \mathcal{F}^\cdot] = [R^k f_* \mathbb{R}_E, \mathcal{F}^\cdot]$. Let D be Griffiths' classifying space [24] for the polarizable variation of $\mathbb{R}$-Hodge structures $[\mathcal{L}, \mathcal{F}^\cdot]$. This is a homogeneous space for a suitable real Lie group G. The choice of a polarization and a base point $b \in B$ induces a group homomorphism $\pi_1(B, b) \to G$, whose image $\bar{\Gamma}$ is by definition the corresponding monodromy group. We assume that this is a discrete subgroup of G, e.g., $\bar{\Gamma}$ is finite. $\bar{\Gamma}$ is automatically discrete, if we have a rational polarization of the variation of $\mathbb{Z}$-Hodge structures $\mathcal{L} \simeq \mathcal{L}(\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R}$, with $\mathcal{L}(\mathbb{Z})$ a local system of finitely generated abelian groups. In theorem 1.4, this is for example true in the cases 2 and 3. Then a finite index subgroup $\Gamma \subset \bar{\Gamma}$ acts also freely on D, so that D/Γ is a complex manifold. Moreover, $[\mathcal{L}, \mathcal{F}^\cdot]$ is classified by a holomorphic map

$$\pi : B \to D/\Gamma \, ,$$

in such a way that the holomorphic vector bundle

$$Gr^p_{\mathcal{F}}(\mathcal{L} \otimes_{\mathbb{R}} \mathcal{O}_B) \simeq \pi^* \mathcal{H}^p$$

is the pullback under π of a corresponding (universal) vector bundle $\mathcal{H}^p \to D/\Gamma$. Therefore one obtains that

$$ch^*(\chi_y([\mathcal{L}, \mathcal{F}^\cdot])) \cup \tilde{T}^*_y(TB) = \sum_p \Big(\pi^*(ch^*(\mathcal{H}^p) \cup \tilde{T}^*_y(TB) \Big) \cdot (-y)^p \, ,$$

and formula (1.4) can easily be rewritten in terms of the higher-genera

$$\chi_y^{[\alpha_\Gamma^{p,k}]}(B) := \int_{[B]} \pi^*(ch^*(\mathcal{H}^p_k)) \cup \tilde{T}^*_y(TB) \tag{1.6}$$

corresponding to the polarized variation of Hodge structures $[R^k f_* \mathbb{R}_E, \mathcal{F}^\cdot]$ as follows:

$$\chi_y(E) = \sum_{p,k} (-1)^k \chi_y^{[\alpha_\Gamma^{p,k}]}(B) \cdot (-y)^p \, . \tag{1.7}$$

From a different perspective, we can also use

$$\chi_y([\mathcal{L}, \mathcal{F}^\cdot])(B) := \int_{[B]} ch^*(\chi_y([\mathcal{L}, \mathcal{F}^\cdot])) \cup \tilde{T}^*_y(TB) \tag{1.8}$$

as a cohomological definition of the χ_y-genus of a polarizable variation of $\mathbb{R}$-Hodge structures $[\mathcal{L}, \mathcal{F}^\cdot]$ on the compact complex manifold B. If moreover B is a compact Kähler manifold, then by a classical result of Zucker [[51], Thm 2.9], the cohomology groups $H^k(B; \mathcal{L})$ have an induced polarizable $\mathbb{R}$-Hodge structure, so that one can also define

$$\chi_y([H^*(B; \mathcal{L}), F^\cdot]) := \sum_k (-1)^k \cdot \chi_y([H^k(B; \mathcal{L}), F^\cdot]) \in \mathbb{Z}[y, y^{-1}] \, . \tag{1.9}$$

For a compact complex algebraic manifold B, the same is true for a polarizable variation of $\mathbb{Q}$-Hodge structures by Saito's theory of algebraic mixed Hodge modules [38, 39]. And here we can prove in both cases the following Hodge-theoretic analogue of Meyer's formula for twisted signatures [33]:

$$\chi_y([H^*(B; \mathcal{L}), F^\cdot]) = \int_{[B]} ch^*(\chi_y([\mathcal{L}, \mathcal{F}^\cdot])) \cup \tilde{T}^*_y(TB) = \chi_y(([\mathcal{L}, \mathcal{F}^\cdot])(B) \, . \tag{1.10}$$

At the end of Section 4.4, we present several interesting extensions of these Hodge-theoretic Atiyah-Meyer formulae to much more general situations. Firstly, we allow the fiber and the total space of the fibration to be singular. Then we also allow the base to be non-compact, in which case we need to include contributions at infinity in our formulae, see Theorem 4.11 and Corollary 4.16.

In Section 6, we extend some of the above mentioned results on χ_y-genera to Atiyah-Meyer type formulae for the corresponding Hirzebruch characteristic classes. The proofs are much more involved, and use in an essential way the construction of Hirzebruch classes via Saito's theory of algebraic mixed Hodge modules (see [7] for the construction of these classes). The key point here is the functoriality of the Hirzebruch class transformation

$$\widetilde{MHT}_y : K_0(MHM(Z)) \to H_{2*}^{BM}(Z) \otimes \mathbb{Q}[y, y^{-1}]$$

in the algebraic context. For Z a complex algebraic manifold, an admissible variation of $\mathbb{Q}$-(mixed) Hodge structures $[\mathcal{L}, \mathcal{F}^\cdot]$ (with quasi-unipotent monodromy at infinity) corresponds by Saito's theory to an algebraic mixed Hodge module $\mathcal{L}^H$ on Z (up to a shift), whose underlying rational sheaf is the local system $\mathcal{L}$. Conversely, any algebraic mixed Hodge module whose underlying rational perverse sheaf is a local system (up to shift), arises in this way from such a variation. Then our main result is the following identification (see Theorem 6.7):

$$\tilde{T}_{y*}(Z;\mathcal{L}) := \widetilde{MHT}_y(\mathcal{L}^H) = \Big(ch^*(\chi_y([\mathcal{L}, \mathcal{F}^\cdot])) \cup \tilde{T}_y^*(TZ) \Big) \cap [Z] \,. \tag{1.11}$$

We note that many of our previous results in the algebraic context can be reproved in greater generality from this identification and the functoriality of $\widetilde{MHT}_y$. We present the results for genera and variations of Hodge structures first, since they can be proven in many interesting cases by using just standard methods of classical Hodge theory. Moreover, some of these results can even be proved in a suitable context of compact complex manifolds. It is plausible that many more of our results remain valid in such an analytic context, by using Saito's theory of analytic mixed Hodge modules [38, 39, 41]. But in the analytic context one does't have the full functorial calculus on derived categories of mixed Hodge modules, e.g., one doesn't have a constant Hodge module (complex) $\mathbb{Q}_Z^H$ on a singular analytic space Z. Therefore one is forced to use the underlying complexes of filtered D-modules, a difficulty that can be avoided in the algebraic context. For simplicity we therefore present the functorial results only in this algebraic context.

We have tried to make this paper self-contained. For this reason, we first present our results, as much as possible, without using Saito's deep theory of algebraic mixed Hodge modules. Only in the end (see §5.1) we explain, as much as needed, the necessary background for this theory. We also point out how many of our previous results in the algebraic context follow quickly from this functorial theory, thus showing the reader the power of Saito's machinery. No knowledge of our pevious paper [9] is needed, were similar results are discussed for Hodge theoretical invariants related to intersection homology. In §3.1 we recall Deligne's formalism of nearby and vanishing cycles, but we assume reader's familiarity with certain aspects of Hodge theory ([17, 37]), e.g., the notion of a variation of Hodge structures.

In a future paper, we will consider extensions of our monodromy formulae to the singular setting, both for genera and characteristic classes (e.g., see [11], but see also [32] for a preliminary result). Such general results are motivated by the considerations in [9] (where the case of trivial monodromy was considered), and by an extension of the Atiyah-Meyer signature formula to the singular case, which is due to Banagl, Cappell and Shaneson [4].

Acknowledgements. *We are grateful to Jörg Schürmann for reading a first draft of this work, and for making many valuable comments and suggestions for improvement. We are indebted to the anonymous referee for carefully reading the manuscript, and for pointing out the more general context in which our results hold. We also thank Mark Andrea de Cataldo, Alexandru Dimca and Nero Budur for inspiring conversations on this subject, and Joseph Steenbrink and Shoji Yokura for valuable comments on a preliminary version of this work.*

2. Hodge genera and singularities of maps

2.1. **Hodge genera. Definitions.** In this section, we define the Hodge-theoretic invariants of complex algebraic varieties, which will be studied in the sequel. We assume reader's familiarity with Deligne's theory of mixed Hodge structures [17].

For any complex algebraic variety Z, we define its χ_y-genus in terms of the Hodge-Deligne numbers of the cohomology of Z (see [16]). More precisely,

$$\chi_y(Z) = \sum_p \left(\sum_{i,q} (-1)^{i-p} h^{p,q}(H^i(Z;\mathbb{C})) \right) \cdot y^p = \sum_{i,p\geq 0} (-1)^{i-p} \dim_{\mathbb{C}} Gr_F^p H^i(Z;\mathbb{C}) \cdot y^p,$$

where $h^{p,q}(H^i(Z;\mathbb{C})) = \dim_{\mathbb{C}} Gr_F^p(Gr^W_{p+q} H^i(Z) \otimes \mathbb{C})$, with $F^{\cdot}$ and $W_{\cdot}$ the Hodge and respectively the weight filtration of Deligne's mixed Hodge structure on $H^i(Z)$. Similarly, we define the χ_y^c-genus of Z, $\chi_y^c(Z)$, by using the Hodge-Deligne numbers of the compactly supported cohomology $H^*_c(Z;\mathbb{C})$. Of course, for a compact variety Z we have that $\chi_y(Z) = \chi_y^c(Z)$. If Z is smooth and compact, then each cohomology group $H^i_c(Z;\mathbb{C}) = H^i(Z;\mathbb{C})$ has a pure Hodge structure of weight i, and the above formulae define Hirzebruch's χ_y-genus ([25]). Note that for any complex variety Z, we have that $\chi_{-1}^c(Z) = \chi_{-1}(Z) = \chi(Z)$ is the usual Euler characteristic, where for the first equality we refer to [[21], p.141-142]. Similarly, χ_0 and χ_0^c are two possible extensions to singular varieties of the arithmetic genus.

The compactly supported χ_y-genus, χ_y^c, satisfies the so-called "scissor relations" for complex varieties, that is: $\chi_y^c(Z) = \chi_y^c(W) + \chi_y^c(Z \setminus W)$, for W a closed subvariety of Z. Therefore, χ_y^c can be defined on $K_0(\mathrm{Var}_{\mathbb{C}})$, the Grothendieck group of varieties over $\mathbb{C}$ which arises in the motivic context.

More generally, we can define χ_y-genera on the Grothendieck group of mixed Hodge structures $K_0(\mathrm{MHS}) = K_0(D^b\mathrm{MHS})$, where we denote by MHS the abelian category of (rational) mixed Hodge structures. Indeed, if $K \in \mathrm{MHS}$, define

$$\chi_y([K]) := \sum_p \dim_{\mathbb{C}} Gr_F^p(K \otimes \mathbb{C}) \cdot (-y)^p, \tag{2.1}$$

where $[K]$ is the class of K in $K_0(\mathrm{MHS})$. This is well-defined on $K_0(\mathrm{MHS})$ since the functor Gr_F^p is exact on mixed Hodge structures. For $K^\bullet$ a bounded complex of mixed Hodge structures, we define

$$[K^\bullet] := \sum_{i \in \mathbb{Z}} (-1)^i [K^i] \quad \in K_0(\mathrm{MHS})$$

and note that we have:

$$[K^\bullet] = \sum_{i \in \mathbb{Z}} (-1)^i [H^i(K^\bullet)].$$

In view of (2.1), we set

$$\chi_y([K^\bullet]) := \sum_{i \in \mathbb{Z}} (-1)^i \chi_y([K^i]). \tag{2.2}$$

Note that $\chi_y : K_0(\mathrm{MHS}) \to \mathbb{Z}[y, y^{-1}]$ is a ring homomorphism with respect to the tensor product. In this language, we have that:

$$\chi_y^c(Z) = \chi_y([H_c^*(Z; \mathbb{Q})])$$

and

$$\chi_y(Z) = \chi_y([H^*(Z; \mathbb{Q})]),$$

where $H_c^*(Z; \mathbb{Q})$ and $H^*(Z; \mathbb{Q})$ are regarded as bounded complexes of mixed Hodge structures, with all differentials equal to zero.

2.2. **Multiplicativity properties of χ_y-genera.** In this section we use Leray spectral sequences for studying simple multiplicative properties of the Hodge-theoretic genera (compare with results of [9]).

Let $f : E \to B$ be a morphism of complex algebraic varieties. Then one has the Leray spectral sequence for f, that is,

$$E_2^{p,q} = H^p(B; R^q f_* \mathbb{Q}_E) \Longrightarrow H^{p+q}(E; \mathbb{Q}), \tag{2.3}$$

and similarly, there is a compactly supported version of the Leray spectral sequence, namely

$$E_2^{p,q} = H_c^p(B; R^q f_! \mathbb{Q}_E) \Longrightarrow H_c^{p+q}(E; \mathbb{Q}). \tag{2.4}$$

We will assume for now that these are spectral sequences in the category of mixed Hodge structures. In the case of algebraic maps of quasi-projective varieties, this fact was proved by Arapura [1]. The general case will be proved later on (see §5.2), when Saito's machinery of mixed Hodge modules will be developed.

We can now prove the following:

Lemma 2.1. *Let $f : E \to B$ be an algebraic morphism of complex algebraic varieties, with B smooth and connected. Assume that f is a locally trivial fibration in the strong (complex) topology of B with fiber F. If the action of $\pi_1(B)$ on $H^*(F;\mathbb{Q})$, respectively $H^*_c(F;\mathbb{Q})$, is trivial, then*

$$\chi_y(E) = \chi_y(B) \cdot \chi_y(F), \quad \textit{resp.} \quad \chi^c_y(E) = \chi^c_y(B) \cdot \chi^c_y(F)\,. \tag{2.5}$$

Proof. Consider the Leray spectral sequences (2.3) and (2.4) for f, which by our assumption are spectral sequences in the category of mixed Hodge structures. The monodromy conditions imposed in the statement of the lemma imply that the local systems $R^qf_*\mathbb{Q}_E$, respectively $R^qf_!\mathbb{Q}_E$, are constant. It follows then by Griffiths' rigidity that the corresponding variations of mixed Hodge structures are trivial: indeed, the natural morphisms

$$H^0(B; R^qf_*\mathbb{Q}_E) \to (R^qf_*\mathbb{Q}_E)_b \to H^q(F;\mathbb{Q}), \tag{2.6}$$

respectively

$$H^0(B; R^qf_!\mathbb{Q}_E) \to (R^qf_!\mathbb{Q}_E)_b \to H^q_c(F;\mathbb{Q}), \tag{2.7}$$

are isomorphisms of mixed Hodge structures, with $F = \{f = b\}$ the fiber over $b \in B$. So the following identifications hold in the category of mixed Hodge structures:

$$H^p(B; R^qf_!\mathbb{Q}_E) = H^p(B;\mathbb{Q}) \otimes H^q(F;\mathbb{Q}),$$

respectively

$$H^p_c(B; R^qf_!\mathbb{Q}_E) = H^p_c(B;\mathbb{Q}) \otimes H^q_c(F;\mathbb{Q}).$$

This assertion follows from Saito's machinery which will be explained later on, and it will only be assumed here.

Since all differentials in the Leray spectral sequence are mixed Hodge structure morphisms, thus strict with respect to the Hodge and weight filtrations, by [[17], Lem. 1.1.11] we get a corresponding spectral sequence for the Hodge components of a given type (k,l):

$$E(k,l)^{p,q}_2 := Gr^k_F Gr^W_{k+l} E^{p,q}_2 \Longrightarrow Gr^k_F Gr^W_{k+l} H^{p+q}(E;\mathbb{Q}), \tag{2.8}$$

and similarly for the cohomology with compact support. Now let $e^{k,l}$ be the Euler characteristic of Hodge-type (k,l), i.e., for a complex algebraic variety Z we define

$$e^{k,l}(Z) = \sum_i (-1)^i h^{k,l}(H^i(Z;\mathbb{Q})).$$

By the invariance of Euler characteristics under spectral sequences, from (2.6) and (2.8) we obtain

$$e^{k,l}(E) = \sum_i (-1)^i \dim\left(\oplus_{p+q=i} E(k,l)^{p,q}_2\right) = \sum_{r+t=k, s+u=l} e^{r,s}(B) \cdot e^{t,u}(F).$$

The multiplicativity of χ_y follows now by noting that for a variety Z we have

$$\chi_y(Z) = \sum_{k,l} e^{k,l}(Z) \cdot (-y)^k.$$

Similar considerations apply to the compactly supported version of (2.8), yielding the multiplicativity of the χ_y^c-polynomial, provided $\pi_1(B)$ acts trivially on $H^*(F;\mathbb{Q})$.

Of course, if all E, B and F are smooth, the multiplicativity of χ_y and χ_y^c respectively are equivalent by Verdier duality.

□

Remark 2.2. In the proof of Lemma 2.1 we only needed the fact that all direct image sheaves (resp. with proper support) are locally constant, except for the base change isomorphism $(R^q f_* \mathbb{Q}_E)_b \simeq H^q(F;\mathbb{Q})$, where the fact that f is a locally trivial fibration is used. So Lemma 2.1 holds in fact under weaker assumptions.

Remark 2.3. The same argument can be used to show that the result of Lemma 2.1 also holds for the Hodge-Deligne polynomials (or the E-functions) defined by $E(Z;u,v) = \sum_{k,l} e^{k,l}(Z)u^k v^l$ (resp. for the E-functions $E_c(Z;u,v)$ defined by using the compactly supported cohomology).[1] In particular, the result holds for the weight polynomials $W(Z;t) := E(Z;t,t)$, resp. $W_c(Z;t) := E_c(Z;t,t)$, considered in [[19], Thm 6.1].

Example 2.4. (1) As an example, consider the case of the Hopf fibration defining $\mathbb{CP}^n$. Then $\chi_y(\mathbb{CP}^n) = \chi_y^c(\mathbb{CP}^n) = 1 + (-y) + \cdots + (-y)^n$, $\chi_y^c(\mathbb{C}^{n+1} \setminus \{0\}) = (-y)^{n+1} - 1$, $\chi_y^c(\mathbb{C}^*) = -y - 1$, and by Poincaré Duality, $\chi_y(\mathbb{C}^{n+1} \setminus \{0\}) = 1 - (-y)^{n+1}$, $\chi_y(\mathbb{C}^*) = 1 + y$. Thus the multiplicativity for both χ_y and χ_y^c holds.
(2) Let f be the Milnor fibration of a weighted homogeneous isolated hypersurface singularity at the origin in $\mathbb{C}^{n+1}$, that is, $F = \{f = 1\} \hookrightarrow E = \mathbb{C}^{n+1} \setminus \{p = 0\} \to B = \mathbb{C}^*$, for f a weighted homogeneous polynomial in $n+1$ variables, with an isolated singular point at the origin. In this case, the monodromy is an algebraic morphism of finite order equal to $\deg(f)$, and the mixed Hodge structure on $H^*(F;\mathbb{Q})$ is known by work of Steenbrink [47]. It turns out that even in this special case, the χ_y-genus is *not* multiplicative. Here is a concrete example: let $f(x,y) = x^3 - y^2$ defining the cuspidal cubic in $\mathbb{C}^2$. Then, in the notations above and by [47], the (mixed) Hodge numbers of F are $h^{0,0}(H^0(F)) = 1$, $h^{1,0}(H^1(F)) = 1$, $h^{0,1}(H^1(F)) = 1$ and $h^{1,1}(H^1(F)) = 0$ (note that $H^2(F) = 0$ since F is affine of complex dimension 1). Therefore, we obtain that $\chi_y(F) = y$, so by Poincaré Duality [2] it follows that $\chi_y^c(F) = (-y)\cdot\chi_{y^{-1}}(F) = -1$. It also follows easily that $\chi_y^c(E) = y^2 + y$, $\chi_y^c(B) = -y - 1$, and $\chi_y(E) = 1 + y$, $\chi_y(B) = 1 + y$.

Remark 2.5. The assumption of trivial monodromy is closely related to, but different from the situation of "algebraic piecewise trivial" maps coming up in the motivic context (e.g., see [7]). For example, the multiplicativity of the χ_y^c-polynomial holds (without any assumption on monodromy) for a Zariski locally trivial fibration of possibly singular complex algebraic varieties (see [[16], Cor. 1.9], or [[7], Ex. 3.3]). Note that if the base space B is smooth and connected, then a Zariski locally trivial fibration is also a locally trivial fibration in the

[1] Note that $\chi_y(Z) = E(Z;-y,1)$, and similarly, $\chi_y^c(Z) = E_c(Z;-y,1)$.

[2] If Z is a complex algebraic manifold of dimension n, then $\chi_y^c(Z) = (-y)^n \chi_{y^{-1}}(Z)$. Indeed, the Poincaré duality isomorphism takes classes of type (p,q) in $H_c^j(Z;\mathbb{Q})$ to classes of type $(n-p, n-q)$ in $H^{2n-j}(Z;\mathbb{Q})$.

complex topology, with trivial monodromy action since $\pi_1(U) \to \pi_1(B)$ is surjective for a Zariski open subset U of B.

2.3. **χ_y^c-genera and singularities of maps.** In this section, by analogy with the results of [8, 9], we discuss the behavior of the χ_y^c-genus under morphisms of algebraic varieties, and show that χ_y^c satisfies the *stratified multiplicative property* in the sense of [13]. The result will be further refined in §3.2 and §5.4, in the case of maps onto curves.

Let $f : X \to Y$ be a morphism of complex algebraic varieties. Since f can be extended to a proper algebraic map, it can be stratified with subvarieties as strata, i.e., there exist finite algebraic Whitney stratifications of X and Y respectively, such that for any component S of a stratum of Y, $f^{-1}(S)$ is a union of connected components of strata in X, each of which is mapping submersively to S. This implies that $f_{|f^{-1}(S)} : f^{-1}(S) \to S$ is a locally trivial map of Whitney stratified spaces (see [[23], §I.1.6]). In particular, all direct image sheaves $R^k f_! \mathbb{Q}_X$ are constructible, i.e., each restriction $(R^k f_! \mathbb{Q}_X)|_S$ is locally constant on a stratum S of such a stratification on Y.

The following easy consequence of Lemma 2.1 shows the deviation from multiplicativity of the χ_y^c-genus in the case of a stratified map (compare with results in [8, 9]):

Proposition 2.6. *Let $f : X \to Y$ be an algebraic morphism of (possible singular) complex algebraic varieties. Assume that there is a (finite) decomposition of Y into locally closed and connected complex algebraic submanifolds $S \subset Y$ such that the restrictions $(R^k f_! \mathbb{Q}_X)|_S$ of all direct image sheaves to all pieces S are constant. Then*

$$\chi_y^c(X) = \sum_S \chi_y^c(S) \cdot \chi_y^c(F_s) \,, \tag{2.9}$$

were $F_s = \{f = s\}$ is a fiber over a point $s \in S$.

Proof. Without loss of generality, we can assume (after refining the stratification) that for each stratum S, the set $\mathrm{cl}(S) \setminus S$ is a union of other pieces (strata) S' of the decomposition of Y. Then by the additivity of the χ_y^c-genus, and by the multiplicativity result of Lemma 2.1, it follows that:

$$\chi_y^c(X) = \sum_S \chi_y^c(f^{-1}(S)) = \sum_S \chi_y^c(S) \cdot \chi_y^c(F_s).$$

□

Remark 2.7. With no assumption on the monodromy along the strata of f, each summand $\chi_y^c(f^{-1}(S))$ of equation (2.9) can be calculated by means of Atiyah-Meyer type formulae as in Section 4.2.

Remark 2.8. More generally, by Remark 2.3 and additivity, formula (2.9) above is also satisfied by the Hodge-Deligne polynomial $E_c(-; u, v)$ defined by means of compactly supported cohomology. In other words, the polynomial $E_c(-; u, v)$ satisfies the stratified multiplicative property.

Example 2.9. *Smooth blow-up*
Let Y be a smooth subvariety of codimension $r+1$ in a smooth variety X. Let $\pi : \tilde{X} \to X$ be the blow-up of X along Y. Then π is an isomorphism over $X \setminus Y$ and a projective bundle (Zariski locally trivial) over Y, coresponding to the normal bundle of Y in X of rank $r+1$. The result of Proposition 2.6 then yields

$$\chi_y^c(\tilde{X}) = \chi_y^c(X) + \chi_y^c(Y) \cdot (-y + \cdots + (-y)^r) . \tag{2.10}$$

In fact this formula holds without any assumption on monodromy, by using instead Remark 2.5 and the fact that $\pi^{-1}(Y)$ is a Zariski locally trivial fibration over Y with fiber $\mathbb{CP}^r$ (see [[16], §1.10]).

3. A Hodge-theoretic analogue of the Riemann-Hurwitz formula

In this section, we specialize the results of §2.3 to the case of morphisms onto a curve and obtain a Hodge-theoretic analogue of the Riemann-Hurwitz formula. We first recall the definition of nearby and vanishing cycle functors.

3.1. Vanishing and nearby cycles.

Definition 3.1. Let $f : X \to \Delta$ be a holomorphic map from a reduced complex space X to a disc. Denote by $X_0 = f^{-1}(0)$ the fiber over the center of the disc, with $i_0 : X_0 \hookrightarrow X$ the inclusion map. The canonical fiber X_∞ of f is defined by

$$X_\infty := X \times_{\Delta^*} \hbar,$$

where $\hbar$ is the complex upper-half plane (i.e., the universal cover of the punctured disc via the map $z \mapsto \exp(2\pi i z)$). Let $k : X_\infty \to X$ be the induced map. Then the *nearby cycle complex* is defined by

$$\psi_f(\mathbb{Q}_X) := i_0^* R k_* k^* \mathbb{Q}_X. \tag{3.1}$$

As it turns out, $\psi_f(\mathbb{Q}_X)$ is in fact a constructible complex, i.e., $\psi_f(\mathbb{Q}_X) \in D_c^b(X_0)$ (e.g., see [[43], Thm 4.0.2, Lem. 4.2.1]). The *vanishing cycle complex* $\phi_f(\mathbb{Q}_X) \in D_c^b(X_0)$ is the cone on the comparison morphism $\mathbb{Q}_{X_0} = i_0^*\mathbb{Q}_X \to \psi_f(\mathbb{Q}_X)$, that is, there exists a canonical morphism $can : \psi_f(\mathbb{Q}_X) \to \phi_f(\mathbb{Q}_X)$ such that

$$i_0^*\mathbb{Q}_X \to \psi_f(\mathbb{Q}_X) \overset{can}{\to} \phi_f(\mathbb{Q}_X) \overset{[1]}{\to} \tag{3.2}$$

is a distinguished triangle in $D_c^b(X_0)$.

It follows directly from the definition that for $x \in X_0$,

$$H^j(M_x;\mathbb{Q}) = \mathcal{H}^j(\psi_f\mathbb{Q}_X)_x \quad \text{and} \quad \tilde{H}^j(M_x;\mathbb{Q}) = \mathcal{H}^j(\phi_f\mathbb{Q}_X)_x, \tag{3.3}$$

where M_x denotes the Milnor fiber of f at x. This identification can be used as in [34, 35] to put canonical mixed Hodge structures on the (reduced) cohomology of the Milnor fiber (even in the analytic context for non-isolated singularities). If X is smooth, the identification in (3.3) can also be used to show that $\mathrm{Supp}(\phi_f\mathbb{Q}_X) \subset \mathrm{Sing}(X_0)$, e.g., see [[20], Ex. 4.2.6, Prop.4.2.8].

In fact, by replacing $\mathbb{Q}_X$ by any complex in $D^b_c(X)$, we obtain in this way functors $\psi_f, \phi_f : D^b_c(X) \to D^b_c(X_0)$. It is well-known that if X is a pure $(n+1)$-dimensional locally complete intersection (e.g., X is smooth), then $\psi_f \mathbb{Q}_X[n]$ and $\phi_f \mathbb{Q}_X[n]$ are perverse complexes. This is just a particular case of the fact that the shifted functors ${}^p\psi_f := \psi_f[-1]$ and ${}^p\phi_f := \phi_f[-1]$ take perverse sheaves on X into perverse sheaves on the central fiber X_0 (e.g., see [[43], Thm 6.0.2]).

The above construction of the vanishing and nearby cycles comes up in the following global context (for details, see [[20], §4.2]). Let X be a complex algebraic (resp. analytic) variety, and $f : X \to \mathbb{C}$ a non-constant regular (resp. analytic) function. Then for any $t \in \mathbb{C}$, one has functors

$$\mathcal{K}^\cdot \in D^b_c(X) \mapsto \psi_{f-t}(\mathcal{K}^\cdot), \phi_{f-t}(\mathcal{K}^\cdot) \in D^b_c(X_t)$$

where $X_t = f^{-1}(t)$ is assumed to be a non-empty hypersurface, by simply repeating the above considerations for the function $f - t$ restricted to a tube $T(X_t) := f^{-1}(\Delta)$ around the fiber X_t (here Δ is a small disc centered at t). By stratification theory, in the algebraic context (or in the analytic context for a proper holomorphic map f) one can find a small disc $\triangle$ centered at $t \in \mathbb{C}$ such that $f : f^{-1}(\triangle^*) \to \triangle^*$ is a (stratified) locally trivial fibration, with $\Delta^* := \Delta \setminus \{t\}$.

3.2. **A Riemann-Hurwitz formula for χ_y-genera.** Let $f : X \to C$ be a proper algebraic morphisms from a smooth $(n+1)$-dimensional complex algebraic variety onto a smooth algebraic curve. Let $\Sigma(f) \subset C$ be the critical locus of f. Then f is a submersion over $C^* := C \setminus \Sigma(f)$, hence locally differentiably trivial (by Ehresmann's fibration theorem). For a point $c \in C$ we let X_c denote the fiber $f^{-1}(c)$.

We want to relate the χ_y^c-genera of X and respectively C via the singularities of f, and to obtain a stronger version of Proposition 2.6 in our setting. The outcome is a Hodge-theoretic version of a formula of Iversen, or of the Riemann-Hurwitz formula for the Euler characteristic (e.g., see [[20], Cor. 6.2.5, Rem. 6.2.6], or [[28], (III, 32)]), see Theorem 3.2 and Example 3.4 below, as well as the more detailed results of §5.4. The proof uses the additivity of the χ_y^c-genus, together with the study of genera of singular fibers of f by means of vanishing cycles at a critical value.

Theorem 3.2. *Let $f : X \to C$ be a proper algebraic morphism from a smooth $(n+1)$-dimensional complex algebraic variety onto a non-singular algebraic curve C. Let $\Sigma(f) \subset C$ be the set of critical values of f, and set $C^* = C \setminus \Sigma(f)$. If the action of $\pi_1(C^*)$ on the cohomology of the generic fibers X_t of f is trivial, then*

$$\chi_y^c(X) = \chi_y^c(C) \cdot \chi_y^c(X_t) - \sum_{c \in \Sigma(f)} \chi_y([\mathbb{H}^*(X_c; \phi_{f-c}\mathbb{Q}_X)]) \tag{3.4}$$

where on $\mathbb{H}^*(X_c; \phi_{f-c}\mathbb{Q}_X)$ we have the mixed Hodge structure constructed in [34, 35].

Proof. Under our assumptions, the fibers of f are compact complex algebraic varieties, and fibers over points in C^* are smooth. By additivity, we can write:

$$\chi_y^c(X) = \chi_y^c(X^*) + \sum_{c \in \Sigma(f)} \chi_y(X_c),$$

where $X^* := f^{-1}(C^*)$. Then by Lemma 2.1 (or by [[9], Prop.3.1]), we have that

$$\chi_y^c(X^*) = \chi_y^c(C^*) \cdot \chi_y(X_t), \tag{3.5}$$

where X_t is the smooth (generic) fiber of f.

Now let $c \in \Sigma(f)$ be a critical value of f and restrict the morphism to a tube $T(X_c) := f^{-1}(\Delta_c)$ around the singular fiber X_c, where Δ_c denotes a small disc in C centered at c. By our assumptions, $f : T(X_c) \to \Delta_c$ is a proper holomorphic function, smooth over Δ_c^*, with compact complex algebraic fibers, that is, a one-parameter degeneration of compact complex algebraic manifolds. Then there is a long exact sequence of mixed Hodge structures (e.g., see [34, 35]):

$$\cdots \to H^j(X_c; \mathbb{Q}) \to \mathbb{H}^j(X_c; \psi_{f-c}\mathbb{Q}_X) \to \mathbb{H}^j(X_c; \phi_{f-c}\mathbb{Q}_X) \to \cdots, \tag{3.6}$$

where $\mathbb{H}^j(X_c; \psi_{f-c}\mathbb{Q}_X)$ carries the "limit mixed Hodge structure" defined on the cohomology of the canonical fiber X_∞ of the one-parameter degeneration $f : T(X_c) \to \Delta_c$ (e.g., see [[37], §11.2]). However, a consequence of the definition of the limit mixed Hodge structure is that (see [[37], Cor. 11.25])

$$\dim_{\mathbb{C}} F^p H^j(X_\infty; \mathbb{C}) = \dim_{\mathbb{C}} F^p H^j(X_t; \mathbb{C}),$$

for X_t the generic fiber of the family (and of f). Therefore,

$$\chi_y(X_\infty) := \chi_y([\mathbb{H}^*(X_c; \psi_{f-c}\mathbb{Q}_X)]) = \chi_y(X_t).$$

With this observation, from (3.6) we obtain that for a critical value c of f the following holds:

$$\begin{aligned} \chi_y(X_c) &= \chi_y([\mathbb{H}^*(X_c; \psi_{f-c}\mathbb{Q}_X)]) - \chi_y([\mathbb{H}^*(X_c; \phi_{f-c}\mathbb{Q}_X)]) \\ &= \chi_y(X_t) - \chi_y([\mathbb{H}^*(X_c; \phi_{f-c}\mathbb{Q}_X)]). \end{aligned}$$

The formula in (3.4) follows now from (3.5) and additivity. □

Remark 3.3. The key point in the proof of the above theorem was to observe that in a one-parameter degeneration of compact complex algebraic manifolds the χ_y-genus of the canonical fiber coincides with the χ_y-genus of the generic fiber of the family. This fact is not true for the corresponding E-polynomials, since, while the Hodge structure on the cohomology of the generic fiber is pure, the limit mixed Hodge structure on the cohomology of the canonical fiber carries the monodromy weight filtration.

Example 3.4. If X is smooth and f has only isolated singularities, then

$$\chi_y^c(X) = \chi_y^c(C) \cdot \chi_y^c(X_t) + (-1)^{n+1} \sum_{x \in \mathrm{Sing}(f)} \chi_y([\tilde{H}^n(M_x; \mathbb{Q})]), \tag{3.7}$$

where M_x is the Milnor fiber of f at x. Indeed, if f has only isolated singular points, then each critical fiber X_c has only isolated singularities and the corresponding vanishing cycles $\phi_{f-c}\mathbb{Q}_X$ are supported only at these points. Each Milnor fiber M_x at one of these singularities is $(n-1)$-connected. Moreover, for each integer j, the isomorphism

$$\mathbb{H}^j(X_c; \phi_{f-c}\mathbb{Q}_X) \simeq \sum_{x \in Sing(X_c)} \tilde{H}^j(M_x; \mathbb{Q})$$

is compatible with the mixed Hodge structures.

Remark 3.5. In the special case of the Euler characteristic $\chi = \chi_{-1}$, the formulae in Theorem 3.2 and in the example above remain true for any proper algebraic morphism onto a curve, without any assumption on the monodromy (see [[20], Cor. 6.2.5]). This follows from the multiplicativity of the Euler characteristic χ under fibrations, the additivity of compactly supported Euler characteristic χ_c, and from the fact that in the category of complex algebraic varieties we have the equality $\chi = \chi_c$ (see [[21], p.141-142]).

4. The presence of monodromy. Atiyah-Meyer formulae for the χ_y-genus.

In this section we prove Hodge-theoretic analogues of Atiyah's formula for the signature of a fibre bundles in the presence of monodromy [2], and of Meyer's twisted signature formula [33]. As already mentioned in the introduction, in some important cases we can prove these results even in the complex analytic context.

4.1. **Hirzebruch classes of complex manifolds and the Hirzebruch-Riemann-Roch theorem.** Recall that if X is a complex manifold, its Hirzebruch class $\widetilde{T}^*_y(TX)$ corresponds to the (un-normalized) power series

$$\widetilde{Q}_y(\alpha) := \frac{\alpha(1+ye^{-\alpha})}{1-e^{-\alpha}} \in \mathbb{Q}[y][[\alpha]], \quad \widetilde{Q}_y(0) = 1+y. \tag{4.1}$$

In fact,

$$\widetilde{T}^*_y(TX) := Td^*(TX) \cup ch^*(\lambda_y(T^*X)), \tag{4.2}$$

where $Td^*(X)$ is the total Todd class of X, ch^* is the Chern character, and

$$\lambda_y(T^*X) := \sum_p \Lambda^p T^*X \cdot y^p \tag{4.3}$$

is the total λ-class of (the cotangent bundle of) X. Hirzebruch's class appears in the *generalized Hirzebruch-Riemann-Roch theorem*, (g-HRR) for short, which asserts that if $\mathcal{E}$ is a holomorphic vector bundle on a compact complex manifold X then the χ_y-characteristic of $\mathcal{E}$, i.e.,

$$\chi_y(X,\mathcal{E}) := \sum_{p\geq 0} \chi(X, \mathcal{E}\otimes\Lambda^p T^*X)\cdot y^p = \sum_{p\geq 0}\left(\sum_{i\geq 0}(-1)^i \mathrm{dim} H^i(X, \Omega(\mathcal{E})\otimes\Lambda^p T^*X)\right)\cdot y^p,$$

with T^*X the cotangent bundle of X and $\Omega(\mathcal{E})$ the coherent sheaf of germs of sections of $\mathcal{E}$ [3], can in fact be expressed in terms of the Chern classes of $\mathcal{E}$ and the tangent bundle of X, or more precisely,

$$\chi_y(X,\mathcal{E}) = \int_{[X]} ch^*(\mathcal{E}) \cup \widetilde{T}_y^*(TX). \tag{4.4}$$

In particular, if $\mathcal{E} = \mathcal{O}_X$ we have that

$$\chi_y(X) = \int_{[X]} \widetilde{T}_y^*(TX). \tag{4.5}$$

Also note that the value $y = 0$ in (4.4) yields the classical Hirzebruch-Riemann-Roch theorem (in short, HRR) for the holomorphic Euler characteristic of $\mathcal{E}$, that is,

$$\chi(X,\mathcal{E}) = \int_{[X]} ch^*(\mathcal{E}) \cup Td^*(TX). \tag{4.6}$$

We should point out that while Hirzebruch's original proof of the (g-HRR) theorem was given in the context of complex projective manifolds (see [[25], §21.3]), the result remains valid more generally for compact complex manifolds by the Atiyah-Singer Index theorem [3]. In this context, it is also a special case of the Grothendieck-Riemann-Roch theorem (GRR) for compact complex manifolds (which will be explained in Theorem 4.1 of the next section).

Important special cases of the χ_y-genus of a compact complex manifold include the Euler characteristic (for $y = -1$), the arithmetic genus (for $y = 0$), and the signature (for $y = 1$).

4.2. **χ_y-genera of smooth families.** Let $f : E \to B$ be a holomorphic submersion of compact complex manifolds, with B connected. Then we have a short exact sequence of holomorphic vector bundles

$$0 \to T_f \to TE \to f^*TB \to 0\,, \tag{4.7}$$

with T_f the tangent bundle to the fibers, so that

$$\tilde{T}_y^*(TE) = \tilde{T}_y^*(T_f) \cup f^*(\tilde{T}_y^*(TB))\,. \tag{4.8}$$

By using Poincaré duality and the projection formula we obtain

$$f_*(\tilde{T}_y^*(TE)) = f_*(\tilde{T}_y^*(T_f)) \cup \tilde{T}_y^*(TB), \tag{4.9}$$

so that

$$\chi_y(E) = \int_{[B]} f_*(\tilde{T}_y^*(T_f)) \cup \tilde{T}_y^*(TB)\,. \tag{4.10}$$

For the degree-zero component

$$f_*(\tilde{T}_y^*(T_f))^0 \in H^0(B;\mathbb{Q}[y])$$

[3] For X smooth and projective, $\chi_y(X,\mathcal{O}_X)$ agrees with the Hodge-theoretic χ_y-genus defined in the first part of this paper (indeed, by Hodge theory, $h^{p,q} = \dim_{\mathbb{C}} H^q(X,\Lambda^p T^*X)$), and this is in fact a special case of formula (1.10) from the introduction.

of $f_*(\tilde{T}^*_y(T_f))$ we have by restriction to any fiber $F_b = \{f = b\}$ (with $T_f|_{F_b} = TF_b$) that:

$$\chi_y(F_b) = f_*(\tilde{T}^*_y(T_f))^0 \,. \tag{4.11}$$

So the χ_y-genus of all fibers is the same, and the multiplicativity relation

$$\chi_y(E) = \chi_y(F_b) \cdot \chi_y(B)$$

would follow from the equality

$$f_*(\tilde{T}^*_y(T_f)) = f_*(\tilde{T}^*_y(T_f))^0 \in H^0(B;\mathbb{Q}[y]) \subset H^{2*}(B;\mathbb{Q}[y]) \,,$$

which classically is called "strict multiplicativity" (compare [[26], p.47]).

For a better understanding of $f_*(\tilde{T}^*_y(T_f))$ we can use a relative version of the (HRR), i.e., the following Grothendieck-Riemann-Roch theorem for compact complex analytic manifolds:

Theorem 4.1. (GRR). *For a compact complex analytic manifold M there is a (unique) Chern character homomorphism $ch^* : G_0(M) \to H^{2*}(M;\mathbb{Q})$ from the Grothendieck group of coherent sheaves to cohomology, with the following two properties:*

1. *For the (class of a) coherent sheaf $\mathcal{V}$ of sections of a holomorphic vector bundle $V \to M$ this is the usual Chern character of V, i.e.,*
$$ch^*([\mathcal{V}]) = ch^*(V) \in H^{2*}(M;\mathbb{Q}) \,.$$
2. *For a holomorphic map $f : M \to N$ of compact complex manifolds the following diagram commutes:*
$$\begin{array}{ccc} G_0(M) & \xrightarrow{ch^*(-)\cup Td^*(TM)} & H^{2*}(M;\mathbb{Q}) \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle f_*} \\ G_0(N) & \xrightarrow{ch^*(-)\cup Td^*(TN)} & H^{2*}(N;\mathbb{Q}) \,. \end{array}$$
Here the left pushforward is defined taking the alternating sum of the (classes of the) higher direct image sheaves $R^i f_*$ (which are coherent by Grauert's theorem). □

Note that for an algebraic manifold the Grothendieck groups of algebraic vector bundles and coherent algebraic sheaves are the same $K^0(M) \simeq G_0(M)$, because any coherent algebraic sheaf has a finite resolution by algebraic vector bundles. So in the algebraic context this is the usual (GRR) (compare e.g. [[22], Ch. 15]). But for complex analytic manifolds this needs not to be the case, and for defining ch^* we have to compose the usual topological Chern character $ch^* : K^0_{top}(M) \to H^{2*}(M;\mathbb{Q})$ with the K-theoretic Riemann-Roch transformation $\alpha : G_0(M) \to K_0^{top}(M) \simeq K^0_{top}(M)$ constructed in [30] (even for compact complex spaces). The uniqueness of the Chern character follows from properties 1. and 2., since $G_0(N)$ is generated by classes $f_*([\mathcal{V}])$ with $f : M \to N$ a holomorphic map of compact manifolds and $V \to M$ a holomorphic vector bundle.

For our holomorphic submersion $f : E \to B$ of compact complex manifolds the above commutative diagram can be rewritten as:

$$\begin{array}{ccc} G_0(E) & \xrightarrow{ch^*(-)\cup Td^*(T_f)} & H^{2*}(E;\mathbb{Q}) \\ f_* \downarrow & & \downarrow f_* \\ G_0(B) & \xrightarrow{ch^*(-)} & H^{2*}(B;\mathbb{Q}) . \end{array}$$

So for the class $\tilde{T}^*_y(T_f) = ch^*(\Lambda_y T^*_f) \cup Td^*(T_f)$ we get from (GRR):

$$f_*(\tilde{T}^*_y(T_f)) = ch^*(f_*[\Lambda_y T^*_f]) , \tag{4.12}$$

with $\Lambda_y T^*_f = \sum_p \Omega^p_{E/B} \cdot y^p$. Therefore, we need to understand the higher direct image sheaves $R^q f_* \Omega^p_{E/B}$, which a priori are only coherent sheaves. But by the (relative) Hodge to de Rham spectral sequence (compare [[37], Prop.10.29]):

$$'E_1^{p,q} = R^q f_* \Omega^p_{E/B} \Rightarrow R^{p+q} f_* \Omega^{\bullet}_{E/B} = H^{p+q}_{DR}(E/B) \simeq R^{p+q} f_* \mathbb{C}_E \otimes_{\mathbb{C}} \mathcal{O}_B , \tag{4.13}$$

these coherent sheaves are related to the locally constant sheaves $R^k f_* \mathbb{C}_E$. Note that this spectral sequence is induced by the "trivial filtration" of the relative de Rham complex $\Omega^{\bullet}_{E/B}$. By analytic restriction to a fiber of f one has the (absolute) Hodge to de Rham spectral sequence of the fiber F_b:

$$'E_1^{p,q} = H^q(F_b; \Omega^p_{F_b}) \Rightarrow \mathbb{H}^{p+q}(F_b; \Omega^{\bullet}_{F_b}) = H^{p+q}_{DR}(F_b) \simeq H^{p+q}(F_b;\mathbb{C}) . \tag{4.14}$$

And under suitable assumptions on F_b, this spectral sequence degenerates at E_1. A sufficient condition for this to happen is, for example, that F_b is bimeromorphic to a compact Kähler manifold (compare [[37], Cor. 2.30]), e.g. F_b is itself a Kähler manifold or an algebraic manifold. But (4.14) also degenerates at E_1 for any 2-dimensional compact surface. Finally, if the Hodge to de Rham spectral sequence (4.14) degenerates at E_1 for all fibers F_b of f, then (by Grauert's base change theorem) the relative Hodge to de Rham spectral sequence (4.13) also degenerates at E_1, and all coherent direct image sheaves $R^q f_* \Omega^p_{E/B}$ are locally free (compare [[37], p.251]). All the above considerations imply the following result:

Theorem 4.2. *Let $f : E \to B$ be a holomorphic submersion of compact complex manifolds such that for all fibers F_b of f the Hodge to de Rham spectral sequence (4.14) degenerates at E_1. Then all coherent direct image sheaves $R^q f_* \Omega^p_{E/B}$ are locally free, and the χ_y-genus of E can be computed by the formula:*

$$\chi_y(E) = \int_{[B]} ch^*(\chi_y(f)) \cup \tilde{T}^*_y(TB) , \tag{4.15}$$

where

$$\chi_y(f) := \sum_{p,q} (-1)^q [R^q f_* \Omega^p_{E/B}] \cdot y^p \in K^0(B)[y]$$

is the K-theory χ_y-characteristic of f. □

We shall now explain how one obtains from this a proof of Theorem 1.4 and of formula (1.5) from the introduction. If a compact complex manifold M is bimeromorphic to a compact Kähler manifold, then not only the Hodge to de Rham spectral sequence degenerates at E_1, but the induced decreasing filtration $F^{\cdot}$ on

$$\mathbb{H}^k(M;\Omega^{\cdot}_M) = H^k_{DR}(M) \simeq H^k(M;\mathbb{C}) =: V$$

defines a pure Hodge structure of weight k (compare [[37], Cor. 2.30]), i.e.,

$$F^p(V) \cap \overline{F^{k-p+1}(V)} = \{0\}\,.$$

Here the conjugation $^-$ on V comes from the real (or rational) structure

$$H^k(M;\mathbb{C}) \simeq H^k(M;\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C} \simeq H^k(M;\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C}\,.$$

Moreover, this pure Hodge structure is polarizable over $\mathbb{R}$ (or over $\mathbb{Q}$ for M algebraic) (compare [17]), i.e., there is a $(-1)^k$-symmetric pairing

$$\mathcal{Q} : V \otimes_{\mathbb{R}} V \to \mathbb{R} \quad (\text{or} \quad \mathcal{Q} : V \otimes_{\mathbb{Q}} V \to \mathbb{Q})\ ,$$

satisfying suitable properties (compare [[37], p.38-39]).

By the arguments of Griffiths [24], these considerations when applied to the fibers of a holomorphic submersion $f : E \to B$ of compact complex manifolds, yield a polarizable variation of $\mathbb{R}$-Hodge structures of weight k on the direct image sheaves $R^k f_* \mathbb{R}_E$ with

$$\mathcal{H}^{p,q} := Gr^p_{\mathcal{F}}(R^{p+q} f_* \mathbb{C}_E \otimes_{\mathbb{C}} \mathcal{O}_B) \simeq R^q f_* \Omega^p_{E/B}, \tag{4.16}$$

provided we are in any one of the following cases (compare [[37], Cor. 10.32] , [17], [24]):

(1) E is a Kähler manifold, or more generally bimeromorphic to a compact Kähler manifold.
(2) f is a projective morphism.
(3) f is an algebraic morphism of complex algebraic manifolds.

So we get a $(-1)^k$-symmetric pairing $\mathcal{Q} : \mathcal{L} \otimes_{\mathbb{R}} \mathcal{L} \to \mathbb{R}_B$ for the local system $\mathcal{L} = R^k f_* \mathbb{R}_E$, together with a decreasing filtration $\mathcal{F}^{\cdot}$ of $(\mathcal{L} \otimes_{\mathbb{R}} \mathbb{C}_B) \otimes_{\mathbb{C}} \mathcal{O}_B =: \mathcal{V}$ by holomorphic subbundles, which on each fiber define a polarized Hodge structure of weight k. Finally, the induced flat connection $\nabla : \mathcal{V} \to \mathcal{V} \otimes \Omega^1_B$ satisfies the Griffiths' transversality condition

$$\nabla(\mathcal{F}^p \mathcal{V}) \subset \mathcal{F}^{p-1} \mathcal{V} \otimes \Omega^1_B$$

for all p. Note that the structure of such a geometric variation of Hodge structures is a condition on the cohomology of the fibers of f, whereas the existence of a polarization is a global property, were we use one of our assumptions 1, 2 or 3. In the last two cases, the polarization is even defined over $\mathbb{Q}$. This finishes the proof of Theorem 1.4 which we recall below for the convenience of the reader, and it also proves formula (1.5) from the introduction:

Theorem 4.3. *Let $f : E \to B$ be a holomorphic submersion of compact complex manifolds. Assume we are in any one of the following cases:*

(1) *E is a Kähler manifold, or more generally bimeromorphic to a compact Kähler manifold.*
(2) *f is a projective morphism.*
(3) *f is an algebraic morphism of complex algebraic manifolds.*

Then the χ_y-genus of E can be computed by the following formula:

$$\chi_y(E) = \int_{[B]} ch^*(\chi_y(f)) \cup \tilde{T}_y^*(TB) \, , \tag{4.17}$$

where

$$\chi_y(f) := \sum_k (-1)^k \cdot \chi_y([R^k f_* \mathbb{R}_E, \mathcal{F}^{\cdot}]) \in K^0(B)[y]$$

is the K-theory χ_y-characteristic of f, for

$$\chi_y([\mathcal{L}, \mathcal{F}^{\cdot}]) := \sum_p [Gr^p_{\mathcal{F}}(\mathcal{L} \otimes_{\mathbb{R}} \mathcal{O}_B)] \cdot (-y)^p \in K^0(B)[y, y^{-1}]$$

the K-theory χ_y-characteristic of a polarizable variation of $\mathbb{R}$-Hodge structures $\mathcal{L}$ (with $\mathcal{F}^{\cdot}$ the Hodge filtration on the associated flat vector bundle).

Remark 4.4. (1.) Note that if $\pi_1(B)$ acts trivially on the real cohomology of the fibers of a holomorphic submersion $f : E \to B$ as in the statement of Theorem 4.3, then by Griffiths' "rigidity" theorem [24] all polarizable variations of Hodge structures $R^k f_* \mathbb{R}_E$ are constant, since the underlying local systems are constant. We get in this case the "multiplicativity" of the χ_y-genus for holomorphic submersions as in one of the cases 1, 2, or 3 above. We should also point out that by formula (4.17) the multiplicativity property holds provided only that the locally-free sheaves $\mathcal{H}^{p,q}$ are all flat (since by [27] rational Chern classes of flat bundles are trivial in positive degrees). This is the case if the above variations are what is called "local systems of Hodge structures" (compare [[37], Ex. 10.7]).

(2.) Formula (4.17) shows the deviation from multiplicativity of the χ_y-genus of fiber bundles in the presence of monodromy. The right-hand side of (4.17) is a sum of polynomials, one of the summands being $\chi_y(B) \cdot \chi_y(F)$, for F the generic fiber. Indeed, as already pointed out, the zero-dimensional piece of $ch^*(\chi_y(f))$ is $\chi_y(F)$.

(3.) Formula (4.17) is a Hodge-theoretic analogue of Atiyah's signature formula [[2], (4.3)] in the complex analytic/algebraic setting. Indeed, if $y = 1$, then by [[50], Rem. 3], and in the notation of [[2] §4],

$$\widetilde{T}_1^*(T_B) = \prod_{i=1}^{dimB} \frac{\alpha_i}{\tanh(\frac{1}{2}\alpha_i)} =: \widetilde{\mathcal{L}}(B), \tag{4.18}$$

where α_i are the Chern numbers of the tangent bundle of B. Moreover, it is known that $\chi_1(E) = \sigma(E)$ is the usual signature (see [25]), and in a similar fashion one can show that

$(-1)^q\mathcal{H}^{p,q}$ is the K-theory signature $\mathrm{Sign}(f)$ from [2]. In other words, the value at $y = 1$ of (4.17) yields Atiyah's formula

$$\sigma(E) = \int_{[B]} ch^*(\mathrm{Sign}(f)) \cup \widetilde{\mathcal{L}}(B). \tag{4.19}$$

(4.) In [2], Atiyah pointed out that his examples showing the non-multiplicativity for the signature of holomorphic fibrations $Z \to C$ with $\dim_{\mathbb{C}} Z = 2$, $\dim_{\mathbb{C}} C = 1$ and with non-trivial monodromy action on the cohomology of the fiber G, also show the non-multiplicativity of the Todd genus $\mathrm{Td}(-)$. Examples of non-multiplicativity in higher dimensions can be obtained as follows. Let $D \to C$ be an arbitrary holomorphic fiber bundle with fiber F and having a trivial monodromy. Then the Todd (and hence χ_y-) genus is non-multiplicative for the fibration $Z \times_C D \to D$, since $Z \times_C D$ also fibers over Z with fiber F and trivial monodromy, and

$$\mathrm{Td}(Z \times_C D) = \mathrm{Td}(F) \cdot \mathrm{Td}(Z) \neq \mathrm{Td}(F) \cdot \mathrm{Td}(C) \cdot \mathrm{Td}(G) = \mathrm{Td}(D) \cdot \mathrm{Td}(G). \tag{4.20}$$

More such examples can be obtained via standard constructions, e.g., fiber or direct products of Atiyah's examples, or higher dimensional examples as above.

(5.) Theorem 4.3 can be extended so that we allow E and F to be singular. We can also discard the compactness assumption on the base B, but in this case we need to allow contributions "at infinity" in our formula; see the discussion at the end of §4.4 for precise formulations of these general results.

4.3. **Higher χ_y-genera and period domains.** In this section, we express the deviation from multiplicativity of the χ_y-genus in (1.4) in terms of higher-genera associated to cohomology classes of the quotient by the monodromy group of the corresponding period domain associated to the polarizable variation of $\mathbb{R}$-Hodge structures $[R^k f_* \mathbb{R}_E, \mathcal{F}^\cdot]$ ($k \in \mathbb{Z}$). These higher-genera are analogous to the previously considered Novikov-type invariants corresponding to the cohomology classes of the fundamental group (e.g., see [6]), and in some cases they coincide with the latter.

Let $[\mathcal{L}, \mathcal{F}^\cdot]$ be a polarizable variation of $\mathbb{R}$-Hodge structures of weight k on a compact complex manifold B. And let D be Griffiths' classifying space [24] for $[\mathcal{L}, \mathcal{F}^\cdot]$. More precisely, if $(L, F^\cdot)$ is the stalk of $\mathcal{L}$ at a point $b \in B$ together with its Hodge filtration, and if $\epsilon = (-1)^k$ is the type of a polarization $\mathcal{Q}$ on L with η the Hodge partition given by $\dim(L \otimes \mathbb{C}) = \sum_p h^{p,k-p}$ (for $h^{p,k-p} := \dim F^p/F^{p+1}$), then $D := D_{\epsilon,\eta}$ is the classifying space of weight k pure Hodge structures of type (ϵ, η). This space is a subset in the flag manifold consisting of flags in L which satisfy the Riemann bilinear relations. In particular, D is the base of the universal flag bundle $\mathcal{F}_\eta := (\cdots \subset \mathcal{F}^{p+1} \subset \mathcal{F}^p \subset \cdots)$ with rank $\mathcal{F}^p = \dim F^p$, which has the flags as its fiber, and it is also the base for the bundles $\mathcal{H}^p := \mathcal{F}^p/\mathcal{F}^{p+1}$.

Let $\bar{\Gamma}$ be the monodromy group corresponding to $\mathcal{L}$. Then $\bar{\Gamma}$ is a subgroup in the subgroup of $\mathrm{GL}(\dim(L), \mathbb{R})$ consisting of transformations preserving the biliniar form $\mathcal{Q}$. We assume $\bar{\Gamma}$ is discrete, e.g. $\bar{\Gamma}$ is finite. Note that $\bar{\Gamma}$ is automatically discrete if we have a rational polarization of the variation of $\mathbb{Z}$-Hodge structures $\mathcal{L} \simeq \mathcal{L}(\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R}$, with $\mathcal{L}(\mathbb{Z})$ a local system of finitely generated abelian groups. In Theorem 1.4 (or 4.3) this is for example true in the cases 2 and 3. Then a finite index subgroup $\Gamma \subset \bar{\Gamma}$ acts also freely on D so that D/Γ is a complex manifold. Moreover, $[\mathcal{L}, \mathcal{F}^{\cdot}]$ is classified by a holomorphic map

$$\pi : B \to D/\Gamma ,$$

in such a way that the holomorphic vector bundle

$$Gr^p_{\mathcal{F}}(\mathcal{L} \otimes_{\mathbb{R}} \mathcal{O}_B) \simeq \pi^* \mathcal{H}^p$$

is the pullback under π of a corresponding (universal) vector bundle $\mathcal{H}^p \to D/\Gamma$. Indeed, the action of the group Γ on L and D induces an action on the total space of $\mathcal{F}_\eta$ so that the projection $\mathcal{F}_\eta \to D$ is Γ-equivariant. The latter map induces the locally trivial fibration $\mathcal{F}_\eta/\Gamma \to D/\Gamma$ and, moreover, for any $h^{p,k-p} \in \eta$ the bundle $\mathcal{H}^p$ over D descends to a universal bundle (also denoted by $\mathcal{H}^p$) over the quotient D/Γ. Therefore, we can write:

$$ch^*(\chi_y([\mathcal{L}, \mathcal{F}^{\cdot}])) \cup \tilde{T}^*_y(TB) = \sum_p \left(\pi^*(ch^*(\mathcal{H}^p) \cup \tilde{T}^*_y(TB) \right) \cdot (-y)^p . \tag{4.21}$$

We next make the following definition:

Definition 4.5. *Let $\alpha \in H^*(D/\Gamma)$. The higher genus $\chi_y^{[\alpha]}$ is defined by:*

$$\chi_y^{[\alpha]} = \int_{[B]} \pi^*(\alpha) \cup \tilde{T}_y(TB) .$$

Among polarizable variations of Hodge structures one can single out those for which, if $\epsilon = -1$ there are at most two non-vanishing Hodge numbers, and if $\epsilon = +1$ and $p \neq q$ then all $h^{p,q} = 0$ except for at most two of them for which one has $h^{p,q} = 1$. In this case, the period domain is simply-connected, since it is the Siegel upper-half plane for $\epsilon = -1$, and it is $SO(2, h^{p,p})/U(1) \times SO(h^{p,p})$, i.e., the quotient by the maximal compact subgroup, for $\epsilon = +1$ (see [[14], p.145]). It follows that the period map factors as $B \to B\pi_1(B) \to D/\Gamma = B\Gamma$ (where $B\Gamma$ is the classifying space of Γ), and $\chi_y^{[\alpha]}$ coincides in this case with the higher χ_y-genus considered in [6]. We shall refer to such variations as "topological variations" of Hodge structures.

Let us now consider the polarized variation of Hodge structures $[R^k f_* \mathbb{R}_E, \mathcal{F}^{\cdot}]$ coming from a map $f : E \to B$ as in the statement of Theorem 1.4. If we denote by $\mathcal{H}^p_k$ the corresponding universal bundle on the quotient of the associated classifying space by the monodromy group, then by (4.21) our formula (1.4) can easily be rewritten in terms of the higher genera

$$\chi_y^{[\alpha_\Gamma^{p,k}]}(B) := \int_{[B]} \pi^*(ch^*(\mathcal{H}^p_k)) \cup \tilde{T}^*_y(TB) \tag{4.22}$$

corresponding to the polarized variation of Hodge structures $[R^k f_* \mathbb{R}_E, \mathcal{F}^{\cdot}]$ as follows:

Theorem 4.6.

$$\chi_y(E) = \sum_{p,k} (-1)^k \chi_y^{[\alpha_\Gamma^{p,k}]}(B) \cdot (-y)^p . \tag{4.23}$$

Remark 4.7. If $\Gamma = 1$ (i.e., the monodromy group $\bar{\Gamma}$ is trivial or finite), we obtain the multiplicativity of the χ_y-genus. More generally, if the pieces $H^{p,q}$ of the Hodge decomposition on the cohomology of the fiber are monodromy invariant then the period map is homotopic to the map to a point, and again one has multiplicativity.

Remark 4.8. Fibrations for which the fibers are curves or K3 surfaces induce topological variations of Hodge structures, hence the χ_y-genus of the total space can be expressed in terms of Novikov-type higher χ_y-genera. On the other hand, for fibrations with fibers of higher dimensions one needs the generalization of the higher χ_y-genus as defined in this section (except for very special cases).

4.4. **A Hodge-theoretic analogue of Meyer's twisted signature formula.** Let $[\mathcal{L}, \mathcal{F}^\cdot]$ be a polarizable variation of $\mathbb{R}$-Hodge structures on a compact Kähler manifold B (with $\mathcal{F}^\cdot$ the corresponding Hodge filtration on the associated flat vector bundle). Then by a classical result of Zucker [[51], Thm 2.9, Lem. 2.11], the cohomology $H^*(B;\mathcal{L})$ gets an induced polarizable $\mathbb{R}$-Hodge structure with the Hodge filtration induced from the filtered (by Griffiths' transversality) de Rham complex $(\Omega^\cdot_B(\mathcal{V}), F^\cdot)$ of the associated flat vector bundle $\mathcal{V} = \mathcal{L} \otimes_{\mathbb{R}} \mathcal{O}_B$, where $\Omega^\cdot_B(\mathcal{V}) := \Omega^\cdot_B \otimes_{\mathcal{O}_B} \mathcal{V}$.

For a compact complex algebraic manifold B, the same is true for a polarizable variation of $\mathbb{Q}$-Hodge structures by Saito's theory of algebraic mixed Hodge modules [38, 39].

The key point used in the results of this section is the fact that the spectral sequence induced by the above filtration $F^\cdot$ of the twisted de Rham complex, i.e.,

$$'E_1^{p,q} = \mathbb{H}^{p+q}(B; Gr^p_F(\Omega^\cdot_B(\mathcal{V}))) \Rightarrow \mathbb{H}^{p+q}(B; \Omega^\cdot_B(\mathcal{V})) \simeq H^{p+q}(B; \mathcal{L} \otimes \mathbb{C}) \tag{4.24}$$

degenerates at E_1. In the Kähler case this is the result from [[51], Lem. 2.11], whereas a corresponding relative version for a projective morphism is the main result of [[38], Thm 1, Thm 5.3.1]. Finally, a compact algebraic manifold B is bimeromorphic to a projective algebraic manifold M, i.e., there is a projective birational morphism $\pi : M \to B$. Then the spectral sequence for $(\Omega^\cdot_B(\mathcal{V}), F^\cdot)$ is a direct summand of the spectral sequence for $(\Omega^\cdot_M(\pi^*\mathcal{V}), F^\cdot)$, which implies the degeneration claim (compare also with [[41], Thm 2.2] for a much more general relative version in a suitable Kähler context). This will be used in the proof of the following Hodge-theoretic version of Meyer's twisted signature formula [33]:

Theorem 4.9. *Let $[\mathcal{L}, \mathcal{F}^\cdot]$ be a (rational) polarizable variation of Hodge structures on a compact Kähler (or compact algebraic) manifold B so that $H^*(B;\mathcal{L})$ gets an induced (rational) polarizable Hodge structure with $F^\cdot$ the associated Hodge filtration. Then*

$$\chi_y([H^*(B;\mathcal{L}), F^\cdot]) = \int_{[B]} ch^*(\chi_y([\mathcal{L}, \mathcal{F}^\cdot])) \cup \tilde{T}^*_y(TB) \, , \tag{4.25}$$

where

$$\chi_y([\mathcal{L}, \mathcal{F}^\cdot]) := \sum_p \, [Gr^p_{\mathcal{F}}(\mathcal{L} \otimes_{\mathbb{R}} \mathcal{O}_B)] \cdot (-y)^p \in K^0(B)[y, y^{-1}]$$

is the K-theory χ_y-characteristic of $[\mathcal{L}, \mathcal{F}^\cdot]$.

Proof. By definition,

$$\chi_y([H^*(B;\mathcal{L}),F^{\cdot}])=\sum_{i,p}(-1)^i\dim_{\mathbb{C}}Gr^p_F H^i(B;\mathcal{L}\otimes\mathbb{C})\cdot(-y)^p=\sum_p \chi^p(B;\mathcal{L})\cdot(-y)^p,$$

where $\chi^p(B;\mathcal{L}):=\sum_i(-1)^i\dim_{\mathbb{C}}Gr^p_F H^i(B;\mathcal{L}\otimes\mathbb{C})$ is the Euler characteristic associated to the exact functor Gr^p_F.

If $\mathcal{V}=\mathcal{L}\otimes_{\mathbb{R}}\mathcal{O}_B$ is the flat bundle associated to $\mathcal{L}$ with decreasing Hodge filtration $\mathcal{F}^{\cdot}$, then the Hodge filtration on $H^j(B;\mathcal{L})$ is induced via the isomorphism

$$H^j(B;\mathcal{L}\otimes\mathbb{C})\simeq\mathbb{H}^j(B;\Omega^{\cdot}_B\otimes_{\mathcal{O}_B}\mathcal{V}),$$

from the filtration $F^{\cdot}$ defined by Griffiths' transversality on the twisted de Rham complex $\Omega^{\cdot}_B(\mathcal{V}):=\Omega^{\cdot}_B\otimes_{\mathcal{O}_B}\mathcal{V}$:

$$F^p(\Omega^{\cdot}_B\otimes_{\mathcal{O}_B}\mathcal{V}):=\left[\mathcal{F}^p\mathcal{V}\xrightarrow{\bigtriangledown}\Omega^1_B\otimes\mathcal{F}^{p-1}\mathcal{V}\xrightarrow{\bigtriangledown}\cdots\xrightarrow{\bigtriangledown}\Omega^i_B\otimes\mathcal{F}^{p-i}\mathcal{V}\xrightarrow{\bigtriangledown}\cdots\right]$$

The associated graded is the complex

$$Gr^p_F(\Omega^{\cdot}_B\otimes_{\mathcal{O}_B}\mathcal{V})=\left(\Omega^{\cdot}_B\otimes_{\mathcal{O}_B}Gr^{p-\cdot}_{\mathcal{F}}\mathcal{V},\ Gr_F\bigtriangledown\right)$$

with the induced differential.

Then

$$\begin{aligned}\chi^p(B;\mathcal{L}) &= \sum_k(-1)^k\dim_{\mathbb{C}}Gr^p_F H^k(B;\mathcal{L}\otimes\mathbb{C})\\ &= \sum_k(-1)^k\dim_{\mathbb{C}}Gr^p_F\mathbb{H}^k(B;\Omega^{\cdot}_B\otimes_{\mathcal{O}_B}\mathcal{V})\\ &\overset{(*)}{=} \sum_k(-1)^k\dim_{\mathbb{C}}\mathbb{H}^k(B;Gr^p_F(\Omega^{\cdot}_B\otimes_{\mathcal{O}_B}\mathcal{V}))\\ &= \chi(B,\Omega^{\cdot}_B\otimes_{\mathcal{O}_B}Gr^{p-\cdot}_{\mathcal{F}}\mathcal{V}),\end{aligned}$$

where $(*)$ follows from the E_1-degeneration of the spectral sequence (4.24).

The last term in the above equality can be computed by using the invariance of the Euler characteristic under spectral sequences. In general, if $\mathcal{K}^{\cdot}$ is a complex of sheaves on a topological space B, then there is the following spectral sequence calculating its hypercohomology (e.g., see [[20], §2.1]):

$$E_1^{i,j}=H^j(B;\mathcal{K}^i)\Longrightarrow\mathbb{H}^{i+j}(B;\mathcal{K}^{\cdot}).$$

Assuming all $\chi(B;\mathcal{K}^i)$ are defined, then $\chi(B;\mathcal{K}^{\cdot})$ is also defined, and it can be computed from the E_1-term as

$$\chi(B;\mathcal{K}^{\cdot})=\sum_{i,j}(-1)^{i+j}\dim H^j(B;\mathcal{K}^i)=\sum_i(-1)^i\chi(B;\mathcal{K}^i)$$

Therefore:

$$\chi_y([H^*(B;\mathcal{L}),F^{\cdot}]) \quad=\quad \sum_p\chi^p(B;\mathcal{L})\cdot(-y)^p$$

$$= \sum_p \chi(B; \Omega^{\cdot}_B \otimes_{\mathcal{O}_B} Gr^{p-\cdot}_{\mathcal{F}} \mathcal{V}) \cdot (-y)^p$$

$$= \sum_{i,p} (-1)^i \chi(B; \Omega^i_B \otimes Gr^{p-i}_{\mathcal{F}} \mathcal{H}_l) \cdot (-y)^p$$

$$\overset{(HRR)}{=} \int \Big(\sum_{i,p} (-1)^i \big(ch^*(\Omega^i_B \otimes Gr^{p-i}_{\mathcal{F}} \mathcal{V}) \cup Td^*(TB) \cap [B] \big) \cdot (-y)^p \Big) .$$

Finally, the characteristic class under the integral sign can be computed as

$$\sum_{i,p} (-1)^i \Big(ch^*(\Omega^i_B \otimes Gr^{p-i}_{\mathcal{F}} \mathcal{V}) \cup Td^*(TB) \cap [B] \Big) \cdot (-y)^p =$$

$$= \Big(\sum_{i,p} ch^*(Gr^{p-i}_{\mathcal{F}} \mathcal{V}) \cdot (-y)^{p-i} \Big) \cdot \Big(\sum_i ch^*(\Omega^i_B) \cdot y^i \Big) \cup Td^*(TB) \cap [B]$$

$$= ch^*(\chi_y([\mathcal{L}, \mathcal{F}^{\cdot}])) \cup \tilde{T}^*_y(TB) \cap [B] .$$

which finishes the proof of the theorem.

□

Remark 4.10. The result and the proof of Theorem 4.9 remain the same for $\mathcal{L}$ a graded polarizable variation of mixed Hodge structures. In this case $H^*(B; \mathcal{L})$ gets an induced graded polarizable mixed Hodge structure, in such a way that the spectral sequence coming from the Hodge filtration $F^{\cdot}$ on the twisted de Rham complex degenerates as before at E_1 (whereas the weight filtration $W_{\cdot}$ is not used); compare [[39], Prop.2.15] or [[41], Prop.1.8].

We can now prove the following generalization of Theorem 1.4 in the algebraic context:

Theorem 4.11. *Let $f : E \to B$ be a morphism of complex algebraic varieties, with B smooth, connected and compact. Assume that all direct image sheaves $R^s f_* \mathbb{Q}_E$ (respectively $R^s f_! \mathbb{Q}_E$) are locally constant on B ($s \in \mathbb{Z}$). Then the χ_y- (resp. χ^c_y-) genus of E can be computed by the following formula:*

$$\chi_y(E) = \int_{[B]} ch^*(\chi_y(f)) \cup \tilde{T}^*_y(TB) \; ; \tag{4.26}$$

respectively,

$$\chi^c_y(E) = \int_{[B]} ch^*(\chi^c_y(f)) \cup \tilde{T}^*_y(TB) \, , \tag{4.27}$$

where $\chi_y(f)$ (resp. $\chi^c_y(f)$) is the K-theoretic χ_y- (resp. χ^c_y-) characteristic of f, i.e.,

$$\chi_y(f) = \sum_i (-1)^i \chi_y([R^i f_* \mathbb{Q}_E, \mathcal{F}^{\cdot}]), \quad \textit{resp.} \quad \chi^c_y(f) = \sum_i (-1)^i \chi_y([R^i f_! \mathbb{Q}_E, \mathcal{F}^{\cdot}]) \, .$$

Proof. First note that since the direct image sheaves $R^s f_*\mathbb{Q}_E$ (resp. $R^s f_!\mathbb{Q}_E$) are locally constant, they underly in the algebraic context admissible variations of mixed Hodge structures. In our case B is also compact, so admissible here just means graded-polarizable. (For more details and classical references for the fact that such "geometric variations" of mixed Hodge structures are admissible, compare with [[37], Thm 14.51] for the case when f quasi-projective. In the next section this will be explained by using Saito's theory of algebraic mixed Hodge modules.)

Since the Leray spectral sequences (2.3) and resp. (2.4) are compatible with the mixed Hodge structures, we easily obtain that

$$\chi_y(E) = \sum_s (-1)^s \cdot \chi_y([H^*(B; R^s f_*\mathbb{Q}_E), F^\cdot]), \tag{4.28}$$

and respectively

$$\chi_y^c(E) = \sum_s (-1)^s \cdot \chi_y([H^*(B; R^s f_!\mathbb{Q}_E), F^\cdot]). \tag{4.29}$$

The result follows now from the above remark and formula (4.25) of Theorem 4.9. □

Remark 4.12. As stated in [33], Meyer's formula for the signature $\sigma(B;\mathcal{L})$ of a Poincaré local system $\mathcal{L}$ (that is, a local system equipped with a nondegenerate bilinear pairing $\mathcal{L}\otimes\mathcal{L}\to\mathbb{R}_B$) on a closed, oriented, even-dimensional smooth manifold B involves a twisted Chern character and the total L-polynomial of B (as opposed to Atiyah's formula [2], where an un-normalized version of the L-polynomial is used). More precisely ([33]),

$$\sigma(Z;\mathcal{L}) = \int_{[B]} \widetilde{ch}^*([\mathcal{L}]_K) \cup L^*(B), \tag{4.30}$$

where $[\mathcal{L}]_K$ is the K-theory signature of $\mathcal{L}$, $L^*(B)$ is the total Hirzebruch L-polynomial of B, and $\widetilde{ch}^* := ch^* \circ \psi^2$ is a modified Chern character obtained by composition with the second Adams operation. Similarly, following [[26], p.61–62] (see also [[44], §6]), we can reformulate our Hodge-theoretic Atiyah-Meyer formulae in terms of the normalized Hirzebruch classes $T_y^*(TB)$ corresponding to the power series [4]

$$Q_y(\alpha) := \widetilde{Q}_y(\alpha(1+y)) \cdot (1+y)^{-1} = \frac{\alpha(1+y)}{1-e^{-\alpha(1+y)}} - \alpha y \in \mathbb{Q}[y][[\alpha]], \tag{4.31}$$

by using instead a modified Chern character, $ch^*_{(1+y)}$, whose value on a complex vector bundle $\mathcal{E}$ is

$$ch^*_{(1+y)}(\mathcal{E}) = \sum_{j=1}^{rk\xi} e^{\beta_j(1+y)}, \tag{4.32}$$

[4]For B smooth and projective, the total Hirzebruch class $T_y^*(TB)$ unifies the total Chern class, Todd class and respectively L-class of B; in fact, $T_{-1}^* = c^*$, $T_0^* = td^*$ and $T_1^* = L^*$.

for β_j the Chern roots of $\mathcal{E}$. (In this notation, Meyer's modified Chern character is simply $ch^*_{(2)}$.) For example, in the notations of Theorem 4.9, formula (4.25) is equivalent to [5]

$$\chi_y([H^*(B;\mathcal{L}),F^\cdot]) = \int_{[B]} ch^*_{(1+y)}(\chi_y([\mathcal{L},\mathcal{F}^\cdot])) \cup T^*_y(TB). \tag{4.33}$$

Similar methods can be used to compute $\chi_y([H^*(U;\mathcal{L}),F^\cdot])$ [6], the twisted χ_y-polynomial associated to the canonical mixed Hodge structure on $H^*(U;\mathcal{L})$, for U any (not necessarily compact) complex algebraic manifold and $\mathcal{L}$ an admissible variation of mixed Hodge structure with quasi-unipotent monodromy at infinity on U. (The existence of such mixed Hodge structures follows for example from Saito's theory, see also [[37], Thm 14.52] and the references therein.) In this case, the Hodge filtration on $H^*(U;\mathcal{L}\otimes\mathbb{C})$ is induced by the filtered logarithmic de Rham complex associated to the Deligne extension of $\mathcal{L}$ on a good compactification of U. More precisely, let $(\mathcal{V},\bigtriangledown)$ be the corresponding vector bundle on U with its flat connection and Hodge filtration $\mathcal{F}^\cdot$. Then we can choose a smooth compactification $j: U \hookrightarrow Z$ such that $D = Z \setminus U$ is a divisor with normal crossings, and for each half-open interval of length one there is a unique extension of $(\mathcal{V},\bigtriangledown)$ to a vector bundle $(\bar{\mathcal{V}}^I, \bar{\bigtriangledown}^I)$ with a logarithmic connection on Z such that the eigenvalues of the residues lie in I ([18]). If we set $\bar{\mathcal{V}} := \bar{\mathcal{V}}^{[0,1)}$, then the twisted logarithmic de Rham complex $\Omega^\cdot_Z(\log D)\otimes\bar{\mathcal{V}}$ is quasi-isomorphic (on Z) to $Rj_*\mathcal{L}\otimes\mathbb{C}$, and the filtration $\mathcal{F}^\cdot$ on $\mathcal{V}$ extends to a filtration $\bar{\mathcal{F}}^\cdot \subset \bar{\mathcal{V}}$ since the variation of mixed Hodge structures was assumed to be admissible. As before, by Griffiths' transversality, we can filter the twisted logarithmic de Rham complex, and this filtration becomes part of a cohomological mixed Hodge complex that calculates $H^*(U;\mathcal{L})$. The spectral sequence analogous to (4.24) for the corresponding filtered logarithmic de Rham complex $(\Omega^\cdot_Z(\log D)\otimes\bar{\mathcal{V}}, F^\cdot)$ also degenerates at the E_1-tem (e.g., see [[37], Thm 3.18 (II.iv)]). So by repeating the arguments in the proof of Theorem 4.9, we obtain the following result involving contributions "at infinity" (i.e., forms on Z with logarithmic poles along D):

Theorem 4.13. *Let U be a smooth (not necessarily compact) complex algebraic variety and $\mathcal{L}$ an admissible variation of mixed Hodge structures on U with quasi-unipotent monodromy at infinity. Then in the above notations, we have that*

$$\chi_y([H^*(U;\mathcal{L}),F^\cdot]) = \int_{[Z]} ch^*(\sum_p [Gr^p_{\bar{\mathcal{F}}}\bar{\mathcal{V}}]\cdot(-y)^p) \cup ch^*\left(\lambda_y(\Omega^1_Z(logD))\right) \cup Td^*(TZ), \tag{4.34}$$

where $\lambda_y\left(\Omega^1_Z(logD)\right) := \sum_i \Omega^i_Z(logD)\cdot y^i$.

Remark 4.14. By Poincaré duality, it follows that $\chi^c_y(U;\mathcal{L}) = (-y)^n\chi_{y^{-1}}(U;\check{\mathcal{L}})$, where $\check{\mathcal{L}}$ is the dual (admissible) variation of mixed Hodge structures on the n-dimensional complex algebraic manifold U.

[5] This is in fact the formulation of our result in [10].

[6] We don't mention the weight filtration in the notation since it is not used at all in our arguments.

Remark 4.15. If we let $\mathcal{L} = \mathbb{Q}_U$ be the trivial variation on U, then formula (4.34) yields a calculation of $\chi_y(U)$ in terms of logarithmic forms on a good compactification (Z, D), i.e.,

$$\chi_y(U) = \int_{[Z]} ch^* \left(\lambda_y(\Omega^1_Z(\log D)) \right) \cup Td^*(TZ). \tag{4.35}$$

In view of formula (4.34) and by using again the Leray spectral sequences, we can obtain an even more general Atiyah-type result for an algebraic map f as in Theorem 4.11 by dropping the compactness assumption on its target B:

Corollary 4.16. *Let $f : E \to B$ be a morphism of complex algebraic varieties, with B smooth and connected. Assume that all direct image sheaves $R^s f_* \mathbb{Q}_E$ (resp. $R^s f_! \mathbb{Q}_E$) are locally constant on B ($s \in \mathbb{Z}$). Then the χ_y- (resp. χ^c_y-) genus of E can be computed by the following formula:*

$$\chi_y(E) = \int_{[Z]} ch^* (\sum_{i,p} (-1)^i [Gr^p_{\bar{\mathcal{F}}}(\bar{\mathcal{V}}_i)] \cdot (-y)^p) \cup ch^*(\lambda_y \Omega^1_Z(log\ D)) \cup Td^*(TZ)\ ;$$

respectively,

$$\chi^c_y(E) = (-1)^n \cdot \int_{[Z]} ch^* (\sum_{i,p} (-1)^i [Gr^p_{\bar{\mathcal{F}}}(\bar{\mathcal{V}}_i)] \cdot (-y)^p) \cup ch^*(\lambda_y \Omega^1_Z(log\ D)) \cup Td^*(TZ)\ ,$$

where n is the complex dimension of B and (Z, D) is a good compactification of B with D a normal crossing divisor. Finally $\bar{\mathcal{V}}_i$ denotes the unique extension of $(R^i f_* \mathbb{Q}_E) \otimes \mathcal{O}_B$ (resp. $(R^i f_! \mathbb{Q}_E)\check{} \otimes \mathcal{O}_B$) to a vector bundle with logarithmic connection on Z such that the eigenvalues of the residues lie in $[0, 1)$ (for $i \in \mathbb{Z}$).

5. The calculus of mixed Hodge modules and applications

In this section we begin with a brief overview of Saito's theory of algebraic mixed Hodge modules. Then we present some quick applications of this theory, by showing for example that the Leray spectral sequences are compatible with mixed Hodge theory. We also obtain in this section a more explicit version of our Hodge-theoretic Riemann-Hurwicz formula.

5.1. **Basics of Saito's theory of mixed Hodge modules.** Even though the theory of mixed Hodge modules is very involved, in this section we give a brief overview adapted to our needs. (See also [40] and [[37], Ch. 14] for a quick introduction to this theory.)

We recall that for any complex algebraic variety Z, the derived category of bounded cohomologically constructible complexes of sheaves of $\mathbb{Q}$-vector spaces on Z is denoted by $D^b_c(Z)$, and it contains as a full subcategory the category $\mathrm{Perv}_{\mathbb{Q}}(Z)$ of perverse $\mathbb{Q}$-complexes. The Verdier duality operator $\mathbb{D}_Z$ is an involution on $D^b_c(Z)$ preserving $\mathrm{Perv}_{\mathbb{Q}}(Z)$. Associated to a morphism $f : X \to Y$ of complex algebraic varieties, there are pairs of adjoint functors (f^*, Rf_*) and $(Rf_!, f^!)$ between the respective categories of cohomologically constructible complexes, which are interchanged by Verdier duality. For details, see the books [20, 43].

M. Saito associated to a complex algebraic variety Z an abelian category $\mathrm{MHM}(Z)$, the category of *algebraic mixed Hodge modules* on Z, together with a forgetful functor

$$rat : D^b\mathrm{MHM}(Z) \to D^b_c(Z)$$

such that $rat(\mathrm{MHM}(Z)) \subset \mathrm{Perv}_{\mathbb{Q}}(Z)$ is faithful. For $\mathcal{M}^\cdot \in D^b\mathrm{MHM}(Z)$, $rat(\mathcal{M}^\cdot)$ is called the underlying rational complex of $\mathcal{M}^\cdot$.

Since $\mathrm{MHM}(Z)$ is an abelian category, the cohomology groups of any complex $\mathcal{M}^\cdot \in D^b\mathrm{MHM}(Z)$ are mixed Hodge modules. The underlying rational complexes of the cohomology groups of a complex of mixed Hodge modules are the perverse cohomologies of the underlying rational complex, that is, $rat(H^j(\mathcal{M}^\cdot)) = {}^p\mathcal{H}^j(rat(\mathcal{M}^\cdot))$.

The Verdier duality functor $\mathbb{D}_Z$ lifts to $\mathrm{MHM}(Z)$ as an involution, in the sense that it commutes with the forgetful functor: $rat \circ \mathbb{D}_Z = \mathbb{D}_Z \circ rat$.

For a morphism $f : X \to Y$ of complex algebraic varieties, there are induced functors $f_*, f_! : D^b\mathrm{MHM}(X) \to D^b\mathrm{MHM}(Y)$ and $f^*, f^! : D^b\mathrm{MHM}(Y) \to D^b\mathrm{MHM}(X)$, exchanged under the Verdier duality functor, and which lift the analogous functors on the level of constructible complexes. Moreover, if f is proper, then $f_! = f_*$.

Let us give a rough picture of what the objects in Saito's category of mixed Hodge modules look like. For Z *smooth*, $\mathrm{MHM}(Z)$ is a full subcategory of the category of objects $((\mathcal{M}, F_\cdot), \mathcal{K}^\cdot, W_\cdot)$ such that:

1. $(\mathcal{M}, F_\cdot)$ is an algebraic holonomic filtered D-module $\mathcal{M}$ on Z, with an increasing "Hodge" filtration $F_\cdot$ by coherent algebraic $\mathcal{O}_Z$-modules;
2. $\mathcal{K}^\cdot \in \mathrm{Perv}_{\mathbb{Q}}(Z)$ is the underlying rational sheaf complex, and there is a quasi-isomorphism $\alpha : DR(\mathcal{M}) \simeq \mathbb{C} \otimes \mathcal{K}^\cdot$ in $\mathrm{Perv}_{\mathbb{C}}(Z)$, where DR is the de Rham functor shifted by the dimension of Z;
3. $W_\cdot$ is a pair of (weight) filtrations on $\mathcal{M}$ and $\mathcal{K}^\cdot$ compatible with α.

For a singular variety Z, one works with local embeddings into manifolds and corresponding filtered D-modules with support on Z. In addition, these objects have to satisfy a long list of very complicated properties, but the details of the full construction are not needed here. Instead, we will only use certain formal properties that will be explained below. In this notation, the functor rat is defined by $rat((\mathcal{M}, F_\cdot), \mathcal{K}^\cdot, W_\cdot) = \mathcal{K}^\cdot$.

It follows from the definition of mixed Hodge modules that every $\mathcal{M} \in \mathrm{MHM}(Z)$ has a functorial increasing filtration $W_\cdot$ in $\mathrm{MHM}(Z)$, called the *weight filtration* of $\mathcal{M}$, so that the functor $\mathcal{M} \to Gr^W_k\mathcal{M}$ is exact. We say that $\mathcal{M} \in \mathrm{MHM}(Z)$ is *pure of weight* k if $Gr^W_i\mathcal{M} = 0$ for all $i \neq k$. If $\mathcal{M} \in \mathrm{MHM}(X)$ is pure of weight k and $f : X \to Y$ is proper, then $H^i(f_*\mathcal{M})$ is pure of weight $i + k$.

We say that $\mathcal{M} \in \mathrm{MHM}(Z)$ is supported on S if and only if $rat(\mathcal{M})$ is supported on S. Saito showed that the category of mixed Hodge modules supported on a point, $\mathrm{MHM}(pt)$, coincides with the category MHS^p of (graded) polarizable rational mixed Hodge structures. Here one has to switch the increasing D-module filtration $F_\cdot$ of the mixed Hodge module to the decreasing Hodge filtration of the mixed Hodge structure by $F^\cdot := F_{-\cdot}$, so that $gr^p_F \simeq gr^F_{-p}$. In this case, the functor rat associates to a mixed Hodge structure the underlying rational vector space. Following [39], there exists a unique object $\mathbb{Q}^H_{pt} \in \mathrm{MHM}(pt)$ such that

$rat(\mathbb{Q}^H_{pt}) = \mathbb{Q}$ and $\mathbb{Q}^H_{pt}$ is of type $(0,0)$. In fact, $\mathbb{Q}^H_{pt} = ((\mathbb{C}, F_{\bullet}), \mathbb{Q}, W_{\bullet})$, with $gr^F_i = 0 = gr^W_i$ for all $i \neq 0$, and $\alpha : \mathbb{C} \to \mathbb{C} \otimes \mathbb{Q}$ the obvious isomorphism. For a complex variety Z, define $\mathbb{Q}^H_Z := k^*\mathbb{Q}^H_{pt} \in D^b\mathrm{MHM}(Z)$, with $rat(\mathbb{Q}^H_Z) = \mathbb{Q}_Z$, for $k : Z \to pt$ the map to a point. If Z is *smooth* of dimension n, then $\mathbb{Q}_Z[n] \in \mathrm{Perv}_{\mathbb{Q}}(Z)$ and $\mathbb{Q}^H_Z[n] \in \mathrm{MHM}(Z)$ is a single mixed Hodge module (in degree 0), explicitly described by $\mathbb{Q}^H_Z[n] = ((\mathcal{O}_Z, F_{\bullet}), \mathbb{Q}_Z[n], W_{\bullet})$, with $gr^F_i = 0 = gr^W_{i+n}$ for all $i \neq 0$. So if Z is smooth of dimension n, then $\mathbb{Q}^H_Z[n]$ is a pure mixed Hodge module of weight n.

Note that a graded-polarizable mixed Hodge structure on the (compactly supported) rational cohomology of a complex algebraic variety Z can be obtained by noting that, for $k : Z \to pt$ the constant map, we have that

$$H^i(Z; \mathbb{Q}) = rat(H^i(k_*k^*\mathbb{Q}^H_{pt})) \quad \text{and} \quad H^i_c(Z; \mathbb{Q}) = rat(H^i(k_!k^*\mathbb{Q}^H_{pt})) \, .$$

Moreover, by a deep result of Saito [42], these structures coincide with the classical mixed Hodge structures constructed by Deligne.

If Z is smooth of dimension n, an object $\mathcal{M} \in \mathrm{MHM}(Z)$ is called *smooth* if and only if $rat(\mathcal{M})[-n]$ is a local system on Z. Smooth mixed Hodge modules are (up to a shift) admissible variations of mixed Hodge structures with quasi-unipotent monodromy at infinity. Conversely, such an admissible variation $\mathcal{L}$ on a smooth variety Z of pure dimension n gives rise to a smooth mixed Hodge module (see [39]), i.e., to an element $\mathcal{L}^H[n] \in \mathrm{MHM}(Z)$ with $rat(\mathcal{L}^H[n]) = \mathcal{L}[n]$. A pure polarizable variation of weight k with quasi-unipotent monodromy at infinity yields a pure (polarizable) Hodge module of weight $k+n$ on Z. An easy consequence of this is the following:

Corollary 5.1. *Let Z be a complex algebraic manifold and $\mathcal{L}$ be an admissible variation of mixed Hodge structures on Z with quasi-unipotent monodromy at infinity. Then the groups $H^j(Z; \mathcal{L})$ and $H^j_c(Z; \mathcal{L})$ get induced (graded polarizable) mixed Hodge structures. Moreover, these structures are pure if Z is compact and $\mathcal{L}$ is a variation of pure Hodge structures.*

5.2. **Spectral sequences of mixed Hodge modules.** In this section, we explain how the Leray-type spectral sequences (e.g., the Leray spectral sequence of an algebraic morphism, or the hypercohomology spectral sequence) are in fact spectral sequences of mixed Hodge structures.

From the general theory of spectral sequences, since the category of mixed Hodge modules is abelian, the canonical filtration τ on $D^b\mathrm{MHM}(Z)$ preserves complexes of mixed Hodge modules. Therefore, the second fundamental spectral sequence (e.g., see [[37], §A.3.4]) for any (left exact) functor F sending mixed Hodge modules to mixed Hodge modules, that is, the spectral sequence

$$E_2^{p,q} = H^pF(H^q(\mathcal{M}^{\bullet})) \Longrightarrow H^{p+q}F(\mathcal{M}^{\bullet}), \tag{5.1}$$

is a spectral sequences of mixed Hodge modules.

Note that the canonical t-structure τ on $D^b\mathrm{MHM}(Z)$ corresponds to the perverse truncation ${}^p\tau$ on $D^b_c(Z)$. However, Saito [[39], Rem.4.6(2)] constructed another t-structure ${}'\tau$ on $D^b\mathrm{MHM}(Z)$ that corresponds to the classical t-structure on $D^b_c(Z)$. By using the t-structure ${}'\tau$ in the construction of the second fundamental spectral sequence above, one can show that the classical Leray spectral sequences are, in fact, spectral sequences of mixed Hodge structures.

Example 5.2. (1) *Hypercohomology spectral sequences.*
Let Z be a complex algebraic variety. Then for $\mathcal{K}^\cdot$ a bounded complex of sheaves with constructible cohomology on Z, we have a spectral sequence with the E_2-term given by

$$E_2^{p,q} = H^p(Z; \mathcal{H}^q(\mathcal{K}^\cdot)) \Longrightarrow \mathbb{H}^{p+q}(Z; \mathcal{K}^\cdot), \tag{5.2}$$

which is induced by the natural filtration on the complex $\mathcal{K}^\cdot$. If $\mathcal{K}^\cdot$ underlies a complex of mixed Hodge modules, then the spectral sequence is compatible with mixed Hodge structures. Indeed, this follows by using the t-structure $'\tau$, and the fundamental spectral sequence (5.1) for $F = \Gamma(Z, \cdot) = k_*$, together with the fact that mixed Hodge modules over a point are (graded polarizable) mixed Hodge structures. As usual, $k : Z \to pt$ is the constant map to a point.

Under the previous assumptions on the complex $\mathcal{K}^\cdot$, by taking $F = \Gamma_c(Z, \cdot) = k_!$ together with the t-structure $'\tau$ above, we get that the compactly supported hypercohomology spectral sequence

$$E_2^{p,q} = H_c^p(Z; \mathcal{H}^q(\mathcal{K}^\cdot)) \Longrightarrow \mathbb{H}_c^{p+q}(Z; \mathcal{K}^\cdot), \tag{5.3}$$

is a spectral sequence in the category of mixed Hodge structures, provided $\mathcal{K}^\cdot$ underlies a bounded complex of mixed Hodge modules.

(2) *Leray spectral sequences.*
Let $f : E \to B$ be a morphism of complex algebraic varieties. The Leray spectral sequence for f, that is,

$$E_2^{p,q} = H^p(B; R^q f_* \mathbb{Q}_E) \Longrightarrow H^{p+q}(E; \mathbb{Q}) \tag{5.4}$$

is a special case of (5.2) where $\mathcal{K}^\cdot = Rf_*\mathbb{Q}_E$. Since $\mathcal{K}^\cdot$ underlies $\mathcal{M}^\cdot = f_*\mathbb{Q}_E^H \in D^b\mathrm{MHM}(B)$, it follows that (5.4) is compatible with Hodge structures.

Finally, the compactly supported Leray spectral sequence, namely

$$E_2^{p,q} = H_c^p(B; R^q f_! \mathbb{Q}_E), \Longrightarrow H_c^{p+q}(E; \mathbb{Q}). \tag{5.5}$$

is obtained from (5.3) for $\mathcal{K}^\cdot = Rf_!\mathbb{Q}_E$, which is the rational complex for $\mathcal{M}^\cdot = f_!\mathbb{Q}_E^H \in D^b\mathrm{MHM}(B)$, whence the compatibility with the mixed Hodge structures.

Note that for B an algebraic manifold and $\mathcal{M}$ a smooth mixed Hodge module, the two Leray spectral sequences for f coincide (up to a shift), so in this case the use of the t-structure $'\tau$ can be avoided.

5.3. **χ_y-polynomials of mixed Hodge modules.** In this section we use Saito's theory of algebraic mixed Hodge modules to derive some easy additivity properties of χ_y-genera of complexes of mixed Hodge structures. We begin with a consequence of the fact that mixed Hodge modules over a point are just (graded-polarizable) mixed Hodge structures:

Lemma 5.3. *Let Z be a complex algebraic variety, and $k : Z \to pt$ be the constant map to the point. For any bounded complex $\mathcal{M}^\bullet$ of mixed Hodge modules on Z with underlying rational complex $\mathcal{K}^\bullet$, the vector spaces*

$$\mathbb{H}^p(Z; \mathcal{K}^\bullet) = rat(H^p(k_*\mathcal{M}^\bullet)) \quad \textit{and} \quad \mathbb{H}^p_c(Z; \mathcal{K}^\bullet) = rat(H^p(k_!\mathcal{M}^\bullet))$$

get rational (graded) polarizable mixed Hodge structures.

As a corollary, we obtain some very useful facts for the global-to-local study of χ_y-genera. If Z is a complex algebraic variety, and $i : Y \hookrightarrow Z$ is a closed immersion, with $j : U \hookrightarrow Z$ the inclusion of the open complement, then there is a functorial distinguished triangle for $\mathcal{M}^\bullet \in D^b\mathrm{MHM}(Z)$, lifting the corresponding one from $D^b_c(Z)$ (see [[39], p.321]):

$$j_!j^*\mathcal{M}^\bullet \to \mathcal{M}^\bullet \to i_*i^*\mathcal{M}^\bullet \overset{[1]}{\to} \tag{5.6}$$

In particular, by taking hypercohomology with compact supports in (5.6) and together with Lemma 5.3, we obtain the following long exact sequence in the category of mixed Hodge structures:

$$\cdots \to \mathbb{H}^p_c(U; j^*\mathcal{M}^\bullet) \to \mathbb{H}^p_c(Z; \mathcal{M}^\bullet) \to \mathbb{H}^p_c(Y; i^*\mathcal{M}^\bullet) \to \cdots \tag{5.7}$$

Therefore,

$$\chi_y([\mathbb{H}^*_c(Z; \mathcal{M}^\bullet)]) = \chi_y([\mathbb{H}^*_c(Y; \mathcal{M}^\bullet)]) + \chi_y([\mathbb{H}^*_c(U; \mathcal{M}^\bullet)]). \tag{5.8}$$

As a corollary of (5.8), by induction on strata we obtain the following additivity property:

Corollary 5.4. *Let $\mathcal{S}$ be the set of components of strata of an algebraic Whitney stratification of the complex algebraic variety Z. Then for any $\mathcal{M}^\bullet \in D^bMHM(Z)$ so that $rat(\mathcal{M}^\bullet)$ is constructible with respect to the stratification,*

$$\chi_y([\mathbb{H}^*_c(Z; \mathcal{M}^\bullet)]) = \sum_{S \in \mathcal{S}} \chi_y([\mathbb{H}^*_c(S; \mathcal{M}^\bullet)]). \tag{5.9}$$

Remark 5.5. By taking $\mathcal{M}^\bullet = \mathbb{Q}^H_Z$ in (5.9), we obtain the usual additivity of the χ^c_y-genus. This is a consequence of the fact that Deligne's and Saito's mixed Hodge structures on cohomology (with compact support) coincide, where the latter assertion can be seen by construction if the variety can be embedded into a manifold, but in general it is a very deep result of Saito, see [42]. However, the use of [42] can be avoided here by noting that both χ^c_y-genera of complex algebraic varieties, in the sense of Saito and Deligne respectively, are additive, so they are the same since they coincide in the smooth case.

Each of the terms in the sum of the right-hand side of equation (5.9) can be further computed by means of the Leray spectral sequence for hypercohomology (see §5.2). Indeed, by using the fact that the spectral sequence (5.3) calculating $\mathbb{H}^*_c(S; \mathcal{M}^\bullet)$ is a spectral sequence of mixed Hodge structures, we obtain that for each stratum $S \in \mathcal{S}$,

$$\chi_y([\mathbb{H}^*_c(S; \mathcal{M}^\bullet)]) = \sum_q (-1)^q \cdot \chi^c_y(S; \mathcal{H}^q(rat(\mathcal{M}^\bullet))), \tag{5.10}$$

where $\chi_y^c(S;\mathcal{H}^q(rat(\mathcal{M}^\cdot))) := \chi_y([H_c^*(S;\mathcal{H}^q(rat(\mathcal{M}^\cdot))]])$, with the Hodge structures induced by the (admissible) variation of Hodge structures $\mathcal{H}^q(rat(\mathcal{M}^\cdot))|_S$ on the stratum S. Each of the twisted χ_y^c-polynomials in formula (5.10) can be calculated by means of Atiyah-Meyer type formulae as in §4.4. However, we first discuss the simple case, where we assume that the monodromy along each stratum is trivial (e.g., all strata are simply-connected).

Assume as before that $\mathcal{S}$ is an algebraic Whitney stratification with respect to which $\mathcal{K}^\cdot := rat(\mathcal{M}^\cdot) \in D_c^b(Z)$ has constructible cohomology. Then each stratum $S \in \mathcal{S}$ is a smooth, connected complex algebraic variety, and each cohomology sheaf $\mathcal{H}^q(\mathcal{K}^\cdot)|_S$ is a local system underlying an admissible variation of mixed Hodge structures. We have the following extension of Lemma 2.1:

Proposition 5.6. *Assume the local systems $\mathcal{H}^j(\mathcal{K}^\cdot)|_S$ are constant on S for each $j \in \mathbb{Z}$, e.g. $\pi_1(S) = 0$. Then*

$$\chi_y([\mathbb{H}^*(S;\mathcal{K}^\cdot)]) = \chi_y(S) \cdot \chi_y([\mathcal{K}_s^\cdot]) \tag{5.11}$$

and

$$\chi_y([\mathbb{H}_c^*(S;\mathcal{K}^\cdot)]) = \chi_y^c(S) \cdot \chi_y([\mathcal{K}_s^\cdot]), \tag{5.12}$$

where $[\mathcal{K}_s^\cdot] := [i_s^*\mathcal{K}^\cdot] = [\mathcal{H}^*(\mathcal{K}^\cdot)_s] \in K_0(MHS)$ is the complex of mixed Hodge structures induced by the pullback of the complex of mixed Hodge modules $\mathcal{M}^\cdot$ over the point $s \in S$ under the inclusion $i_s : \{s\} \hookrightarrow S$.

Proof. Since the Leray spectral sequence (5.2) is a spectral sequence of mixed Hodge structures, it follows that

$$\chi_y([\mathbb{H}^*(S;\mathcal{K}^\cdot)]) = \sum_q (-1)^q \cdot \chi_y(S;\mathcal{H}^q(\mathcal{K}^\cdot)|_S), \tag{5.13}$$

where $\chi_y(S;\mathcal{H}^q(\mathcal{K}^\cdot)|_S) := \chi_y([H^*(S;\mathcal{H}^q(\mathcal{K}^\cdot)|_S)]) = \sum_p (-1)^p \cdot \chi_y([H^p(S;\mathcal{H}^q(\mathcal{K}^\cdot)|_S)])$.

By our assumption, the variation of mixed Hodge structures $\mathcal{H}^q(\mathcal{K}^\cdot)|_S$ on S is trivial by "rigidity", fact which admits the following simple proof in the language of mixed Hodge modules: if $\mathcal{L}$ is an admissible (at infinity) variation of mixed Hodge structures with constant underlying local system, then the restriction map $H^0(S;\mathcal{L}) \to \mathcal{L}_s$ underlies the adjunction map $H^0(k_*\mathcal{L}^H) \to H^0(k_*i_{s*}i_s^*\mathcal{L}^H)$ in the category of (graded-polarizable) mixed Hodge structures (corresponding to MHM(pt)), so it is an isomorphism since rat is faithfull on MHM. Here k denotes as usual the constant map to a point $k : S \to s$, and $i_s : s \hookrightarrow S$ is the inclusion of a point.

Hence by (5.2) and as in the proof of Lemma 2.1, there are mixed Hodge structure isomorphisms

$$E_2^{p,q} = H^p(S;\mathcal{H}^q(\mathcal{K}^\cdot)|_S) = H^p(S) \otimes V^q, \tag{5.14}$$

where $V^q := \mathcal{H}^q(\mathcal{K}^\cdot)_s \cong H^0(S;\mathcal{H}^q(\mathcal{K}^\cdot)|_S)$, for any $s \in S$. Therefore, since χ_y is a ring homomorphism, we get that $\chi_y([H^p(S;\mathcal{H}^q(\mathcal{K}^\cdot)|_S)]) = \chi_y([H^p(S)]) \cdot \chi_y([V^q])$. Formula (5.11) follows now from the identification $[\mathcal{K}_s^\cdot] = [\mathcal{H}^*(\mathcal{K}^\cdot)_s]$. The proof of formula (5.12) is similar, but one has to work instead with the spectral sequence (5.3).

□

Altogether, Corollary 5.4 and Proposition 5.6 yield the following global-to-local formula:

Theorem 5.7. *Let $\mathcal{S}$ be the set of components of strata of an algebraic Whitney stratification of the complex algebraic variety Z. Assume that for $\mathcal{M}^\cdot \in D^bMHM(Z)$ the underlying complex $\mathcal{K}^\cdot = rat(\mathcal{M}^\cdot) \in D^b_c(Z)$ is constructible with respect to the given stratification and, moreover, the local systems $\mathcal{H}^j(\mathcal{K}^\cdot)|_S$ are constant on each pure stratum $S \in \mathcal{S}$ for each $j \in \mathbb{Z}$, e.g. $\pi_1(S) = 0$ for all $S \in \mathcal{S}$. Then*

$$\chi_y([\mathbb{H}^*_c(Z;\mathcal{K}^\cdot)]) = \sum_{S\in\mathcal{S}}\sum_{q\in\mathbb{Z}}(-1)^q \cdot \chi^c_y(S;\mathcal{H}^q(\mathcal{K}^\cdot)|_S) = \sum_{S\in\mathcal{S}} \chi^c_y(S)\cdot\chi_y([\mathcal{K}^\cdot_s]), \tag{5.15}$$

for some points $s \in S$.

Remark 5.8. For an algebraic morphism $f : X \to Z$ and $\mathcal{M}^\cdot = f_!\mathbb{Q}^H_X \in D^b\mathrm{MHM}(Z)$, Theorem 5.7 specializes to our earlier result from Proposition 1.2.

5.4. **Nearby and vanishing cycles as mixed Hodge modules.** Of particular importance is the fact that the nearby and vanishing functors can be defined at the level of Saito's mixed Hodge modules [38, 39]. More precisely, if f is a non-constant regular (resp. holomorphic) function on the complex algebraic (resp. analytic) space X and $X_c = f^{-1}(c)$ is the fiber over c, then one has functors $\psi^H_{f-c}, \phi^H_{f-c} : \mathrm{MHM}(X) \to \mathrm{MHM}(X_c)$ compatible with the corresponding perverse cohomological functors on the underlying perverse sheaves by the forgetful functor

$$rat : \mathrm{MHM}(X) \to \mathrm{Perv}_{\mathbb{Q}}(X)$$

which assigns to a mixed Hodge module the underlying $\mathbb{Q}$-perverse sheaf. In other words, $rat \circ \psi^H_{f-c} = {}^p\psi_{f-c} \circ rat$, and similarly for ϕ^H_{f-c}. As a consequence, for each $x \in X_c$ we get canonical mixed Hodge structures on the groups

$$H^j(M_x;\mathbb{Q}) = rat\left(H^j(i^*_x\psi^H_{f-c}\mathbb{Q}^H_X[1])\right), \quad \tilde{H}^j(M_x;\mathbb{Q}) = rat\left(H^j(i^*_x\phi^H_{f-c}\mathbb{Q}^H_X[1])\right), \tag{5.16}$$

where M_x denotes the Milnor fiber of f at $x \in X_c$, and $i_x : \{x\} \hookrightarrow X_c$ is the inclusion of the point. And similarly, we obtain in this way the "limit mixed Hodge structure" on

$$\mathbb{H}^j(X_c;\psi_{f-c}\mathbb{Q}_X) = rat\left(H^j(k_*\psi^H_{f-c}\mathbb{Q}^H_X[1])\right)$$

with $k : X_c \to \{c\}$ the constant map.

5.4.1. *The Hodge-theoretic Riemann-Hurwicz formula revisited.* We will now extend the formula of Example 3.4 to the case of general singularities. By (3.4), it suffices to restrict f over a small disc Δ_c centered at a critical value $c \in \Sigma(f)$ and to study the polynomial $\chi_y([\mathbb{H}^*(X_c;\phi_{f-c}\mathbb{Q}_X)])$. Recall that the fibers of f are compact complex algebraic varieties, which are smooth over points in $\Delta_c \setminus \{c\}$. Fix an algebraic Whitney stratification of X_c with respect to which $\phi_{f-c}\mathbb{Q}_X$ is constructible. For each $q \in \mathbb{Z}$ and each pure stratum $S \subset \mathrm{Sing}(X_c)$, $\mathcal{H}^q(\phi_{f-c}\mathbb{Q}_X)|_S$ is a local coefficient system on S (underlying an admissible variation of mixed Hodge structures) with stalk $\tilde{H}^q(M_s;\mathbb{Q})$, where M_s is the local Milnor fibre at some point in $s \in S$. Then, according to Theorem 5.7, for $\mathcal{M}^{\bullet} = \phi^H_{f-c}(\mathbb{Q}^H_X[n])$ and by assuming trivial monodromy along all strata $S \subset \mathrm{Sing}(X_c)$, we obtain

$$\chi_y([\mathbb{H}^*(X_c; \phi_{f-c}\mathbb{Q}_X)]) = \sum_{S \subset \mathrm{Sing}(X_c)} \chi^c_y(S) \cdot \chi_y([\tilde{H}^*(M_s; \mathbb{Q})]).$$

Under no monodromy assumptions, the left hand side of the above equality can be computed as an alternating sum of twisted χ^c_y-polynomials as in (5.10).

All these facts yield the following general Hodge-theoretic version of the Riemann-Hurwicz formula:

Corollary 5.9. *Let $f : X \to C$ be a proper algebraic morphism from a smooth $(n+1)$-dimensional complex algebraic variety onto a non-singular algebraic curve C. Let $\Sigma(f) \subset C$ be the set of critical values of f, and set $C^* = C \setminus \Sigma(f)$. Assume that each special fiber X_c has an algebraic stratification with respect to which the corresponding vanishing cycle complex is constructible, and moreover the monodromy along each pure stratum is trivial. If the action of $\pi_1(C^*)$ on the cohomology of the generic fibers X_t of f is trivial, then*

$$\chi^c_y(X) = \chi^c_y(C) \cdot \chi^c_y(X_t) - \sum_{c \in \Sigma(f)} \sum_{S \subset Sing(X_c)} \chi^c_y(S) \cdot \chi_y([\tilde{H}^*(M_s; \mathbb{Q})]), \tag{5.17}$$

where M_s is the local Milnor fibre at some point in $s \in S$.

Remark 5.10. In view of (5.10) and Theorem 5.7, and by using our Atiyah-Meyer type formulae for twisted χ^c_y-polynomials, one can formulate a very general Hodge-theoretic Riemann-Hurwitz formula without any assumptions on monodromy. We leave the details as an exercise for the interested reader.

6. Atiyah-Meyer type characteristic class formulae.

In this section, we present characteristic class versions of our Atiyah-Meyer formulae for the χ_y-genus. The proofs of these characteristic class formulae are much more involved, and make use of Saito's theory of algebraic mixed Hodge modules and the construction of the motivic Hirzebruch classes ([7]), which we recall here.

Let Z be a complex algebraic variety. Then for any $p \in \mathbb{Z}$ one has a functor of triangulated categories

$$gr^F_p DR : D^b\mathrm{MHM}(Z) \to D^b_{coh}(Z) \tag{6.1}$$

commuting with proper push-down, with $gr^F_p(DR(\mathcal{M}^{\bullet})) \simeq 0$ for almost all p and $\mathcal{M}^{\bullet} \in D^b\mathrm{MHM}(Z)$ fixed, where $D^b_{coh}(Z)$ is the bounded derived category of sheaves of $\mathcal{O}_Z$-modules with algebraic coherent cohomology sheaves. If $\mathbb{Q}^H_Z \in D^b\mathrm{MHM}(Z)$ denotes the constant Hodge module on Z, and if Z is smooth and pure dimensional, then

$$gr^F_{-p} DR(\mathbb{Q}^H_Z) \simeq \Omega^p_Z[-p] \in D^b_{coh}(Z).$$

The transformations $gr^F_p DR(-)$ are functors of triangulated categories, so they induce functors on the level of Grothendieck groups. Thus, if $G_0(Z) \simeq K_0(D^b_{coh}(Z))$ denotes the

Grothendieck group of algebraic coherent sheaves on Z, we obtain the following group homomorphism commuting with proper push-down:

$$gr^F_{-*}DR : K_0(\mathrm{MHM}(Z)) \to G_0(Z) \otimes \mathbb{Z}[y, y^{-1}], \tag{6.2}$$
$$[\mathcal{M}] \mapsto \sum_p \left(\sum_i (-1)^i [\mathcal{H}^i(gr^F_{-p}DR(\mathcal{M}))] \right) \cdot (-y)^p.$$

We can now make the following definitions (see [7, 9]).

Definition 6.1. The transformation $\widetilde{MHT_y}$ is defined as the composition [7]:

$$\widetilde{MHT_y} := td_* \circ gr^F_{-*}DR : K_0(\mathrm{MHM}(Z)) \to H^{BM}_{2*}(Z) \otimes \mathbb{Q}[y, y^{-1}], \tag{6.3}$$

where td_* is the Baum-Fulton-MacPherson Todd class transformation [5], which is linearly extended over $\mathbb{Z}[y, y^{-1}]$. Note that $\widetilde{MHT_y}$ commutes with proper push-forward.

Remark 6.2. Let $K^0(Z)$ be the Grothendieck group of complex algebraic vector bundles on Z. If Z an algebraic manifold, the canonical map $K^0(Z) \to G_0(Z)$ induced by taking the sheaf of sections is an isomorphism, and the Todd class transformation of the classical Grothendieck-Riemann-Roch theorem is explicitly described by

$$td_*(-) = ch^*(-)Td^*(TZ) \cap [Z]. \tag{6.4}$$

The transformation α of [30] which is used in the definition of the analytic (GRR) for compact complex spaces is a K-theoretic counterpart of the Todd transformation td_* from [5], and Theorem 4.1 is the analytic version of the fact that for an algebraic manifold Z $td_*(-) = ch^*(-)Td^*(TZ) \cap [Z]$ commutes with proper push-down.

Definition 6.3. The *(homology) Hirzebruch class* of an n-dimensional complex algebraic variety Z is defined by the formula

$$\widetilde{T}_{y*}(Z) := \widetilde{MHT_y}([\mathbb{Q}^H_Z]). \tag{6.5}$$

Similarly, if Z is an n-dimensional complex algebraic manifold, and $\mathcal{L}$ an admissible variation of mixed Hodge structures with quasi-unipotent monodromy at infinity on Z, we define *twisted Hirzebruch characteristic classes* in homology by

$$\widetilde{T}_{y*}(Z; \mathcal{L}) = \widetilde{MHT_y}([\mathcal{L}^H]), \tag{6.6}$$

where $\mathcal{L}^H[n]$ is the smooth mixed Hodge module on Z corresponding to $\mathcal{L}$.

By [[7], Lem.3.1, Thm 3.1], the following normalization holds: if Z is smooth and pure dimensional, then $\widetilde{T}_{y*}(Z) = \widetilde{T}^*_y(TZ) \cap [Z]$, thus $\widetilde{T}_{y*}(Z)$ is an extension to the singular setting of (the Poincaré dual of) the un-normalized Hirzebruch class.

The precise relationship between Hirzebruch characteristic classes and χ_y-genera is given by the following

[7] The special case of the transformation $\widetilde{MHT_y}$ at $y = 1$ was previously used by Totaro [48] for finding numerical invariants of singular varieties, more precisely Chern numbers that are invariant under small resolutions.

Proposition 6.4. *Let Z be a compact (possibly singular) complex algebraic variety, and $k : Z \to pt$ the constant map to a point. Then*

$$\chi_y(Z) = k_* \widetilde{T}_{y*}(Z). \tag{6.7}$$

Proof. Since k is proper, by the definition of the Hirzebruch class we obtain

$$k_* \widetilde{T}_{y*}(Z) = k_* \widetilde{MHT}_y([\mathbb{Q}^H_Z]) = \widetilde{MHT}_y([k_* \mathbb{Q}^H_Z]) = \sum_i (-1)^i \widetilde{MHT}_y([H^i(k_* \mathbb{Q}^H_Z)]).$$

Note that each $H^i(k_* \mathbb{Q}^H_Z)$ is a mixed Hodge module over a point, thus a (graded) polarizable mixed $\mathbb{Q}$-Hodge structure. (Here one has to switch the increasing D-module filtration of the mixed Hodge module to the decreasing Hodge filtration of the mixed Hodge structure, so that $gr^F_{-p} \simeq gr^p_F$.) So, by the definition of the transformation $\widetilde{MHT}_y$, for each $i \in \mathbb{Z}$,

$$\begin{aligned} \widetilde{MHT}_y([H^i(k_* \mathbb{Q}^H_Z)]) &= \widetilde{MHT}_y([H^i(Z; \mathbb{Q})]) \\ &= \sum_p td_0([Gr^p_F H^i(Z; \mathbb{C})]) \cdot (-y)^p \\ &= \sum_p \dim_{\mathbb{C}}(Gr^p_F H^i(Z; \mathbb{C})) \cdot (-y)^p. \end{aligned}$$

Putting together the two equalities above, we obtain

$$k_* \widetilde{T}_{y*}(Z) = \sum_{i,p} (-1)^i \dim_{\mathbb{C}}(Gr^p_F H^i(Z; \mathbb{C})) \cdot (-y)^p = \chi_y(Z).$$

□

Remark 6.5. If Z is smooth and compact, and $\mathcal{L}$ is an admissible variation of mixed Hodge structures on Z, then by replacing $\mathbb{Q}^H_Z$ with $\mathcal{L}^H$ in the above arguments, we obtain a similar relationship between twisted characteristic classes and twisted χ_y-genera.

Remark 6.6. *Normalized Hirzebruch classes in the singular setting.*
One can also construct a "normalized" characteristic class transformation MHT_y, whose value at the Hodge sheaf $\mathbb{Q}^H_Z$ yields an extension to the singular setting of the (Poincaré dual of the) normalized Hirzebruch class $T^*_y(TZ)$ which was considered in Remark 4.12. The transformation MHT_y is defined as the composition (see [7]):

$$MHT_y := td_{(1+y)} \circ gr^F_{-*} DR : K_0(\mathrm{MHM}(Z)) \to H^{BM}_{2*}(Z) \otimes \mathbb{Q}[y, y^{-1}, (1+y)^{-1}], \tag{6.8}$$

where $td_{(1+y)}$ is the natural transformation

$$td_{(1+y)} : G_0(Z) \otimes \mathbb{Z}[y, y^{-1}] \to H^{BM}_{2*}(Z) \otimes \mathbb{Q}[y, y^{-1}, (1+y)^{-1}], \tag{6.9}$$

$$[\mathcal{G}] \mapsto \sum_{k \geq 0} td_k([\mathcal{G}]) \cdot (1+y)^{-k},$$

with td_k the degree k component of the Todd class transformation td_* of Baum-Fulton-MacPherson [5], which is linearly extended over $\mathbb{Z}[y, y^{-1}]$. The new transformation MHT_y also commutes with proper push-forward, and we set $T_{y*}(Z) := MHT_y(\mathbb{Q}^H_Z)$ (and similarly for the twisted case). Then, if Z is smooth and pure dimensional, it follows from [7] that $T_{y*}(Z) = T^*_y(TZ) \cap [Z]$. Moreover, exactly as in Proposition 6.4, one can show that if Z is a compact (possibly singular) variety and $k : Z \to pt$ is the constant map to a point, then

$$\chi_y(Z) = k_* T_{y*}(Z).$$

The importance of the normalized characteristic class transformation MHT_y comes from the fact that its lifting to the relative Grothendieck group $K_0(\mathrm{Var}/Z)$ of varieties over Z "unifies" (in the sense of [7]) the (rationalized) Chern-MacPherson transformation $c_* \otimes \mathbb{Q}$ ([31]), the Todd class transformation td_* of Baum-Fulton-MacPherson ([5]) and the L-class transformation of Cappell-Shaneson ([12, 49]). At the level of (Borel-Moore) homology characteristic classes, it can be shown that if Z is a complex algebraic variety then $T_{-1*}(Z) = c_*(Z) \otimes \mathbb{Q}$, but the other special cases (i.e., for $y = 0$ and resp. $y = 1$) fail in general to satisfy such a precise relationship. For complete details, see [7].

The first important result of this section is the following Meyer-type formula for the twisted Hirzebruch characteristic classes:

Theorem 6.7. *Let Z be a complex algebraic manifold of pure dimension n, and $\mathcal{L}$ an admissible variation of mixed Hodge structures on Z with quasi-unipotent monodromy at infinity, with associated flat bundle with "Hodge" filtration $(\mathcal{V}, \mathcal{F}^{\cdot})$. Then*

$$\widetilde{T}_{y*}(Z;\mathcal{L}) = \Big(ch^*(\chi_y([\mathcal{L},\mathcal{F}^{\cdot}])) \cup \widetilde{T}_y^{\,*}(TZ)\Big) \cap [Z] = ch^*(\chi_y([\mathcal{L},\mathcal{F}^{\cdot}])) \cap \widetilde{T}_{y*}(Z), \tag{6.10}$$

where $\chi_y([\mathcal{L},\mathcal{F}^{\cdot}])$ is as before the K-theory χ_y-characteristic of $[\mathcal{L},\mathcal{F}^{\cdot}]$.

Proof. Let

$$\mathcal{L}^H[n] = ((\mathcal{V}, \mathcal{F}_{-\cdot}), W_{\cdot}, \mathcal{L}[n])$$

be the smooth mixed Hodge module on Z corresponding to $\mathcal{L}$, with $\mathcal{F}_{-p} := \mathcal{F}^p$ the increasing filtration on the D-module $\mathcal{V}$, and with $W_k\mathcal{V} := W_k\mathcal{L} \otimes_{\mathbb{Q}} \mathcal{O}_Z$ (e.g., see [[37], Def. 14.53]). It follows from Saito's work that there is a filtered quasi-isomorphism between $(DR(\mathcal{L}^H), F_{-\cdot})$ and the usual filtered de Rham complex $(\Omega^{\cdot}_Z(\mathcal{V}), F^{\cdot})$ with the filtration induced by the Griffiths' transversality condition on $\bigtriangledown$, that is:

$$F^p\Omega^{\cdot}_Z(\mathcal{V}) : \left[\mathcal{F}^p\mathcal{V} \overset{\bigtriangledown}{\to} \Omega^1_Z \otimes \mathcal{F}^{p-1}\mathcal{V} \overset{\bigtriangledown}{\to} \cdots \overset{\bigtriangledown}{\to} \Omega^i_Z \otimes \mathcal{F}^{p-i}\mathcal{V} \overset{\bigtriangledown}{\to} \cdots\right].$$

Note that since $\mathcal{L}$ is an admissible variation of mixed Hodge structures with quasi-unipotent monodromy at infinity, the associated filtered de Rham complex extends as before to a filtered logarithmic de Rham complex on a compact algebraic manifold, so that by GAGA all sheaves can be regarded as algebraic sheaves. Therefore,

$$\begin{aligned}
\widetilde{T}_{y*}(Z;\mathcal{L}) &= td_*\left(\sum_p \left(\sum_i (-1)^i[\mathcal{H}^i(gr^F_{-p}DR(\mathcal{L}^H))]\right)\cdot(-y)^p\right) \\
&= td_*\left(\sum_p \left(\sum_i (-1)^i[\mathcal{H}^i(gr^p_F\Omega^{\cdot}_Z(\mathcal{V}))]\right)\cdot(-y)^p\right)
\end{aligned}$$

$$\begin{aligned}
&= td_*\left(\sum_p \left(\sum_i (-1)^i [\Omega_Z^i \otimes Gr_{\mathcal{F}}^{p-i}\mathcal{V}]\right)\cdot(-y)^p\right)\\
&= \sum_p \left(\sum_i (-1)^i \; td_*([\Omega_Z^i \otimes Gr_{\mathcal{F}}^{p-i}\mathcal{V}])\right)\cdot(-y)^p\\
&= \sum_p \left(\sum_i (-1)^i \; ch^*(\Omega_Z^i \otimes Gr_{\mathcal{F}}^{p-i}\mathcal{V}) \cup Td^*(TZ) \cap [Z]\right)\cdot(-y)^p\\
&= ch^*(\chi_y([\mathcal{L},\mathcal{F}^\bullet])) \cup ch^*(\lambda_y(T^*Z)) \cup Td^*(TZ) \cap [Z]\\
&= ch^*(\chi_y([\mathcal{L},\mathcal{F}^\bullet])) \cup \tilde{T_y}^*(TZ) \cap [Z].
\end{aligned}$$

□

Remark 6.8. If Z is compact, Theorem 4.9 in the algebraic context can be obtained from Theorem 6.7 by pushing down to a point via the constant map $Z \to pt$.

Jörg Schürmann [45] communicated to us that for the special case of a proper submersion the following Atiyah-type result can be obtained as a direct application of the Verdier-Riemann-Roch formula for a smooth proper morphism (see [[7], Cor. 3.1(3)]), if one makes the identification $\mathcal{H}^{p,q} \simeq R^q f_*(\Lambda^p T_f^*)$, with T_f^* the dual of the tangent bundle T_f to the fibers of f (cf. [[37], Prop.10.29]). In fact, this argument extends to the case of smooth proper maps between singular varieties. However, the proof we give here is based only on the definition of the Hirzebruch classes and on Theorem 6.7 in the context of geometric variations of Hodge structures.

Theorem 6.9. *Let $f : E \to B$ be a proper morphism of complex algebraic varieties, with B smooth and connected, such that the sheaves $R^s f_*\mathbb{Q}_E$, $s \in \mathbb{Z}$ are locally constant on B, e.g., f is a locally trivial topological fibration. Then*

$$f_*\widetilde{T}_{y*}(E) = ch^*\left(\chi_y(f)\right) \cap \widetilde{T}_{y*}(B), \tag{6.11}$$

where $\chi_y(f) := \sum_{i,p}(-1)^i[Gr_{\mathcal{F}}^p\mathcal{V}_i]\cdot(-y)^p \in K^0(B)[y]$ is the K-theory χ_y-characteristic of f, for $\mathcal{V}_i$ the flat bundle associated to the local system $R^i f_*\mathbb{Q}_E$.

Proof. Since f is proper and the transformation $\widetilde{MHT_y}$ commutes with proper pushdowns, we first obtain the following:

$$f_*\widetilde{T}_{y*}(E) = f_*(\widetilde{MHT_y}([\mathbb{Q}_E^H])) = \widetilde{MHT_y}([f_*\mathbb{Q}_E^H]). \tag{6.12}$$

Now let $\tau_\leq$ be the natural truncation on $D^b\mathrm{MHM}(B)$ with associated cohomology H^*. Then for any complex $\mathcal{M}^\bullet \in D^b\mathrm{MHM}(B)$ we have the identification (e.g., see [[20], p.95-96], [[43], Lem.3.3.1])

$$[\mathcal{M}^\bullet] = \sum_{i\in\mathbb{Z}}(-1)^i[H^i(\mathcal{M}^\bullet)] \in K_0(D^b\mathrm{MHM}(B)) \cong K_0(\mathrm{MHM}(B)). \tag{6.13}$$

In particular, if for any $k \in \mathbb{Z}$ we regard $H^{i+k}(\mathcal{M}^\bullet)[-k]$ as a complex concentrated in degree k, then

$$\big[H^{i+k}(\mathcal{M}^\bullet)[-k]\big] = (-1)^k[H^{i+k}(\mathcal{M}^\bullet)] \in K_0(\mathrm{MHM}(B)). \tag{6.14}$$

Therefore, if we let $\mathcal{M}^\bullet = f_*\mathbb{Q}^H_E$, we have that

$$f_*\widetilde{T}_{y*}(E) = \sum_{i\in\mathbb{Z}}(-1)^i\widetilde{MHT}_y([H^i(f_*\mathbb{Q}^H_E)]) = \sum_{i\in\mathbb{Z}}(-1)^i\widetilde{MHT}_y(\big[H^{i+\dim B}(f_*\mathbb{Q}^H_E)[-\dim B]\big]). \tag{6.15}$$

Note that $H^i(f_*\mathbb{Q}^H_E) \in \mathrm{MHM}(B)$ is the smooth mixed Hodge module on B whose underlying rational complex is (recall that B is smooth)

$$\mathrm{rat}(H^i(f_*\mathbb{Q}^H_E)) = {}^p\mathcal{H}^i(Rf_*\mathbb{Q}_E) = (R^{i-\dim B}f_*\mathbb{Q}_E)[\dim B], \tag{6.16}$$

where ${}^p\mathcal{H}$ denotes the perverse cohomology functor. In this case, each of the local systems $\mathcal{L}_s := R^s f_*\mathbb{Q}_E$ underlies a geometric variation of Hodge structures.

Altogether, (6.15) becomes

$$f_*\widetilde{T}_{y*}(E) = \sum_{i\in\mathbb{Z}}(-1)^i\widetilde{T}_{y*}(B;\mathcal{L}_i), \tag{6.17}$$

where $\mathcal{L}^H_i[\dim B] := H^{i+\dim B}(f_*\mathbb{Q}^H_E)$ is the smooth mixed Hodge module whose underlying perverse sheaf is $\mathcal{L}_i[\dim B]$. Our formula (6.11) follows now from Theorem 6.7.

□

Remark 6.10. If B is compact, then by pushing (6.11) down to a point, we get back our earlier formula (4.26). Also, if f is a proper submersion, the above result reproves (1.5) from the introduction.

Remark 6.11. By analogy with Remark 4.12, we can reformulate our Atiyah-Meyer type characteristic class formulae in this section in terms of the normalized Hirzebruch classes $T_{y*}(\cdot)$, by using instead the twisted Chern character $ch^*_{(1+y)}$ (see the announcement [10]).

References

[1] Arapura, D., *The Leray spectral sequence is motivic*, Invent. Math. 160 (2005), 567–589.

[2] Atiyah, M. F., *The signature of fiber bundles*, in *Global Analysis (Papers in Honor of K. Kodaira)*, 73–84, Univ. Tokyo Press, Tokyo, 1969.

[3] Atiyah, M. F., Singer, I.M., *The index of elliptic operators on compact manifolds*, Bull. Amer. Math. Soc. 69 (1963), 422-433.

[4] Banagl, M., Cappell, S. E., Shaneson, J. L., *Computing twisted signatures and L-classes of stratified spaces*, Math. Ann. 326 (2003), 589-623.

[5] Baum, P., Fulton, W., MacPherson, M., *Riemann-Roch for singular varieties*, Publ. Math. I.H.E.S. 45, 101-145 (1975).

[6] Borisov, L., Libgober, *Higher elliptic genera*, Math. Res. Lett. 15 (2008), no. 3, 511–520.

[7] Brasselet, P., Schürmann, J., Yokura, S., *Hirzebruch classes and motivic Chern classes of singular spaces*, math.AG/0503492.

[8] Cappell, S. E., Maxim, L. G., Shaneson, J. L., *Euler characteristics of algebraic varieties*, Comm. Pure Appl. Math. 61 (2008), no. 3, 409-421.

[9] Cappell, S. E., Maxim, L. G., Shaneson, J. L., *Hodge genera of algebraic varieties, I*, Comm. Pure Appl. Math. 61 (2008), no. 3, 422-449.

[10] Cappell, S. E., Libgober, A., Maxim, L. G., Shaneson, J. L., *Hodge genera and characteristic classes of complex algebraic varieties*, Electron. Res. Announc. Math. Sci. 15 (2008), 1-7.

[11] Cappell, S.E., Maxim, L., Schürmann, J., Shaneson, J.L., *Atiyah-Meyer formulae for intersection homology genera*, work in progress.

[12] Cappell, S. E., Shaneson, J. L. *Stratifiable maps and topological invariants*, J. Amer. Math. Soc. 4 (1991), no. 3, 521–551

[13] Cappell, S. E., Shaneson, J. L., *Genera of algebraic varieties and counting of lattice points*, Bull. Amer. Math. Soc. (N.S.) 30 (1994), no. 1, 62–69.

[14] Carlson, J., Muller-Stach, S., Peters, C., *Period mappings and Period Domains*, Cambridge studies in advanced mathematics 85, 2003.

[15] Chern, S. S., Hirzebruch, F., Serre, J.-P., *On the index of a fibered manifold*, Proc. Amer. Math. Soc. 8 (1957), 587–596.

[16] Danilov, V. I., Khovanskii, A. G., *Newton polyhedra and an algorithm for computing Hodge-Deligne numbers*, Bull. Amer. Math. Soc. 30 (1994), no 1, 62–69.

[17] Deligne, P., *Théorie de Hodge, II, III*, Publ. Math. IHES 40, 44 (1972, 1974).

[18] Deligne, P., *Equation différentielles a point singular régulier*, Springer 1969.

[19] Dimca, A.; Lehrer, G. I., *Purity and equivariant weight polynomials*, Algebraic groups and Lie groups, 161–181, Austral. Math. Soc. Lect. Ser., 9, Cambridge Univ. Press, Cambridge, 1997.

[20] Dimca, A., *Sheaves in Topology*, Universitext, Springer-Verlag, 2004.

[21] Fulton, W., *Introduction to toric varieties*, Annals of Math. Studies 131, Princeton University Press, 1993.

[22] Fulton, W., *Intersection theory*, Springer Verlag, 1984.

[23] Goresky, M., MacPherson, R., *Stratified Morse theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], 14. Springer-Verlag, Berlin, 1988.

[24] Griffiths, Phillip A., *Periods of integrals on algebraic manifolds, III*, Publ. Math. I.H.E.S 38 (1970), 125–180.

[25] Hirzebruch, F., *Topological methods in algebraic geometry*, Springer, 1966.

[26] Hirzebruch, F., Berger, T., Jung, R., *Manifolds and modular forms*, Aspects of Math., vol. E20, Vieweg, 1992.

[27] Kamber, F., Tondeur, Ph., *Flat manifolds*, LNM no. 67, Springer, 1968.

[28] Kleiman, S., *The enumerative theory of singularities*, Real and Complex Singularities (P. Holm, ed.), Sijthoff and Noordhoff, 1976, pp. 298–384.

[29] Kodaira, K., *A certain type of irregular algebraic surfaces*, J. Analyse Math. 19 1967 207–215.

[30] Levy, R., *Riemann-Roch theorem for complex spaces*, Acta Math. 158 (1987), 149–188.

[31] MacPherson, R., *Chern classes for singular algebraic varieties*, Ann. of Math. 100 (1974), 423–432.

[32] Maxim, L. G., Schürmann, J., *Hodge-theoretic Atiyah-Meyer formulae and the stratified multiplicative property*, in "Singularities I: Algebraic and Analytic Aspects", Contemp. Math. 474 (2008), pp. 145-167.

[33] Meyer, W., *Die Signatur von lokalen Koeffizientensystemen und Faserbündeln*, Bonner Mathematische Schriften 53 (Universität Bonn), 1972.

[34] Navarro Aznar, V., *Sur les structures de Hodge mixtes associées aux cycles évanescents,* Hodge Theory (Sant Cugat, 1985), 143–153, Lecture Notes in Math., 1246, Springer, Berlin, 1987.

[35] Navarro Aznar, V., *Sur la théorie de Hodge-Deligne*, Invent. Math. 90 (1987), 11–76.

[36] Ochanine, S., *Genres elliptiques équivariants*, Elliptic curves and modular forms in algebraic topology (Princeton, NJ, 1986), 107–122, Lecture Notes in Math., 1326, Springer, Berlin, 1988.

[37] Peters, C., Steenbrink, J., *Mixed Hodge structures*, A Series of Modern Surveys in Mathematics, vol. 52, Springer 2008.

[38] Saito, M., *Modules de Hodge polarisables*, Publ. RIMS 24 (1988), 849–995.
[39] Saito, M., *Mixed Hodge Modules*, Publ. RIMS 26 (1990), 221–333.
[40] Saito, M., *Introduction to mixed Hodge modules*, Actes du Colloque de Théorie de Hodge (Luminy, 1987), Astérisque No. 179–180 (1989), 10, 145–162.
[41] Saito, M., *Decomposition theorem for proper Kähler morphisms* , Tohoku Math. J. 42 (1990), 127–148.
[42] Saito, M., *Mixed Hodge complexes on algebraic varieties*, Math. Ann. 316, 283–331 (2000).
[43] Schürmann, J., *Topology of singular spaces and constructible sheaves*, Monografie Matematyczne, 63. Birkhäuser Verlag, Basel, 2003.
[44] Schürmann, J., Yokura, S., *A survey of characteristic classes of singular spaces*, in Singularity theory, 865–952, World Sci. Publ., Hackensack, NJ, 2007.
[45] Schürmann, J., *private communication.*
[46] Shaneson, S., *Characteristic classes, lattice points and Euler-MacLaurin formulae*, Proceedings ICM, Zurich, Switzerland 1994.
[47] Steenbrink, J., *Intersection forms for quasihomogeneous singularities*, Compositio Math. 34 (1977), 211–223.
[48] Totaro, B., *Chern numbers for singular varieties and elliptic homology*, Ann. Math. 151 (2000), 757-792.
[49] Yokura, S., *On Cappell-Shaneson's homology L-classes of singular algebraic varieties*, Trans. Amer. Math. Soc. 347 (1995), 1005–1012.
[50] Yokura, S., *A singular Riemann-Roch for Hirzebruch characteristics*, Banach Center Publications, Vol. 44, 257–268, Warsaw 1998.
[51] Zucker, S., *Hodge theory with degenerating coefficients:* L_2*-cohomology in the Poincaré-metric* , Ann. Math. 109 (1979), 415–476.

S. E. Cappell: Courant Institute, New York University, 251 Mercer Street, New York, NY 10012
E-mail address: cappell@cims.nyu.edu

A. Libgober : Department of Mathematics, University of Illinois at Chicago, 851 S Morgan Street, Chicago, IL 60607
E-mail address: libgober@math.uic.edu

L. Maxim : Courant Institute, New York University, 251 Mercer Street, New York, NY 10012
E-mail address: maxim@cims.nyu.edu

J. L. Shaneson: Department of Mathematics, University of Pennsylvania, 209 S 33rd St., Philadelphia, PA 19104
E-mail address: shaneson@sas.upenn.edu