PRACTICE FINAL EXAM

Do all eight of the following problems. Show all your work, and write neatly.

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Problem I (25 points)
Find the equation of the tangent line to the curve $x^3 + y^3 = 9xy$ at the point $(x, y) = (2, 4)$.

\[x^3 + y^3 = 9xy\]
\[3x^2 dx + 3y^2 dy = 9y dx + 9x dy,\]
\[(3y^2 - 9x) dy = (9y - 3x^2) dx\]
\[
\frac{dy}{dx} = \frac{9y - 3x^2}{3y^2 - 9x}.
\]
At $(x, y) = (2, 4)$, \[
\frac{dy}{dx} \bigg|_{(x,y)=2,4} = \frac{9 \times 4 - 3 \times 2^2}{3 \times 4^2 - 9 \times 2}.
\]
\[
= \frac{36 - 12}{48 - 18} = \frac{24}{30} = \frac{4}{5}.
\]

:. The line is
\[(y - 4) = \frac{4}{5} (x - 2)\]
Problem II (25 points)

Use logarithmic differentiation to calculate the derivative \( \frac{dy}{dx} \) of

\[
y = (\ln x)^{\ln x}.
\]

\[
\ln y = \ln \left((\ln x)^{\ln x}\right)
\]

\[
\ln y = (\ln x) \ln (\ln x)
\]

\[
\frac{1}{y} y' = \frac{1}{x} \ln (\ln x) + \ln x \frac{1}{\ln x} \frac{1}{x}
\]

\[
y' = y \left(\frac{\ln x}{x} + \frac{1}{x}\right)
\]

\[
y' = (\ln x)^{\ln x} \left(\frac{\ln x}{x} + \frac{1}{x}\right)
\]
Problem III (25 points) Evaluate the following limits:

a) \[ \lim_{x \to \infty} (\ln x)^{1/x} \]
   \[= \lim_{x \to \infty} e^{\frac{1}{x} \ln(\ln x)} \]
   \[= e^{\lim_{x \to \infty} \frac{\ln(\ln x)}{x}} \]
   \[= e^0 = 1 \]

b) \[ \lim_{x \to 0} \frac{10^x - 1}{x} \]
   \[= \lim_{x \to 0} \frac{10^x \ln 10}{1} \]
   \[= \ln 10 \]

By "L'H" I mean using L'Hôpital's rule. Be sure check whether it's \( \frac{0}{0} \) or \( \frac{\infty}{\infty} \) before using it.

c) \[ \lim_{x \to 0} \frac{4 - 4e^x}{xe^x} \]
   \[= \lim_{x \to 0} \frac{-4e^x}{e^x + xe^x} \]
   \[= \frac{-4}{1} = -4 \]
Problem IV (25 points)
Find the volume of the solid obtained by revolving the region bounded by \( y = \sqrt{x}, y = 2, x = 0 \) about the line \( x = 4 \).

Washer:

\[
\begin{align*}
\int_0^2 \pi \left( 4^2 - (4-y^2)^2 \right) \, dy \\
= \int_0^2 \pi \left( 16 - (16 - 8y^2 + y^4) \right) \, dy \\
= \int_0^2 \pi \left( 8y^2 - y^4 \right) \, dy \\
= 8\pi \left[ \frac{1}{3} y^3 \right]_0^2 - \pi \left[ \frac{1}{5} y^5 \right]_0^2 \\
= \frac{64\pi}{3} - \frac{32\pi}{5} \\
= \frac{320 - 96}{15} \pi = \frac{224}{15} \pi .
\end{align*}
\]

Shell:

\[
\begin{align*}
\int_0^4 2\pi (4-x)(2-\sqrt{x}) \, dx \\
= \int_0^4 2\pi \left( 8-2x-\frac{32-x}{5} \right) \, dx \\
= 2\pi \left( 8x - x^2 - \frac{8}{3} x^2 + \frac{2}{5} x^2 \right) \bigg|_0^4 \\
= 2\pi \left( \frac{32}{3} - 8 + \frac{2}{5} \right) \\
= 2\pi \left( 16 - 16x^\frac{4}{3} + 6x^\frac{6}{5} \right) \\
= 32\pi \left( 1 - \frac{4}{3} + \frac{6}{5} \right) \\
= \frac{32\pi}{15} \left( 15 - 20 + 12 \right) \\
= \frac{32\pi}{15} \left( \frac{7}{5} \right) = \frac{224}{15} \pi .
\end{align*}
\]
(Make sure your result is positive)
Problem V (25 points)

Find the total area of the region between the curves \( y = \cos x \) and \( y = \sin x \) for \( 0 \leq x \leq 3\pi/2 \).

To find the intersection points:

\( \sin(x) = \cos(x) \),

\( \tan x = 1 \),

so \( x = \frac{\pi}{4}, \frac{5\pi}{4} \) if \( 0 \leq x \leq \frac{3\pi}{2} \).

\[
\int_0^{\pi/4} (\cos x - \sin x) \, dx + \int_{\pi/4}^{3\pi/4} (\sin x - \cos x) \, dx + \int_{3\pi/4}^{5\pi/4} (\cos x - \sin x) \, dx
\]

\[
= \left[ \sin x + \cos x \right]_0^{\pi/4} + \left[ -\cos x - \sin x \right]_{\pi/4}^{3\pi/4} + \left[ \sin x + \cos x \right]_{3\pi/4}^{5\pi/4}
\]

\[
= \left( \frac{\sqrt{2}}{2} + 1 \right) + \left( -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} \right) + \left( 0 - 1 - \left( -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} \right) \right)
\]

\[
= \sqrt{2} - 1 + 2\sqrt{2} + \sqrt{2} - 1
\]

\[
= 4\sqrt{2} - 2
\]
Problem VI (25 points) Evaluate the following integrals:

a) \( \int e^{x} \sin(e^{x}) \, dx \)

\[
= \int \sin u \, du \\
= -\cos u + C \\
= -\cos e^{x} + C
\]

\( u = e^{x} \)
\( \frac{du}{dx} = e^{x} \)
\( du = e^{x} \, dx \)

b) \( \int_{1}^{8} \left( \frac{2}{3x} - \frac{8}{x^3} \right) \, dx \)

\[
= \left[ \left( \frac{2}{3} \ln x + \frac{8}{x} \right) \right]_{1}^{8} \\
= \left( \frac{2}{3} \ln 8 + \frac{8}{8} \right) - \left( \frac{2}{3} \left( \ln 1 + \frac{8}{1} \right) \right) \\
= \frac{2}{3} \ln 8 + 1 - 8 \cdot \left( = 2 \ln 2 \right. \\
\text{This guy is zero!} \\
\left. \text{if you simplify it.} \right)
\]

c) \( \int_{2}^{4} (1 + \ln t) t \ln t \, dt \)

\[
= \int_{2}^{4} u \, du \\
= \left[ \frac{u^2}{2} \right]_{2}^{4} \\
= \frac{1}{2} \left[ (4h4)^2 - (2ln2)^2 \right] \\
= \frac{1}{2} \left[ (4ln2)^2 - (2ln2)^2 \right] \\
= 30 (ln 2)^2 \text{ (if you simplify it)}.
\]
Problem VII (25 points)
The sum of two nonnegative numbers is 36. Find the numbers if the difference of their square roots is to be as large as possible.

Suppose the two numbers are \( a \) and \( b \).
\[ a + b = 36 \]
\[ a > 0, \ b > 0 \]
Maximize: \( \sqrt{a} - \sqrt{b} \).

Notice: \( b = 36 - a \)

\[ \therefore \text{Max: } \sqrt{a} - \sqrt{36 - a} \]
\[ f(a) = \sqrt{a} - \sqrt{36 - a} \]
\[ f'(a) = \frac{1}{2 \sqrt{a}} + \frac{1}{2 \sqrt{36-a}} = 0 \]

Notice \( f'(a) > 0 \) for all \( a \) on \( 0 < a < 36 \).
So there is no solution to \( f'(a) = 0 \) on \( 0 < a < 36 \).
However, we need to check end points.
\[ f(a=0) = \sqrt{0} - \sqrt{36} = -6 \]
\[ f(a=36) = \sqrt{36} - \sqrt{0} = 6 \]

\[ \therefore a = 36 \] maximizes the objective function.

In that case, \( b = 36 - a = 0 \) and the maximum is 6.
Problem VIII (25 points)

Find the center of mass of a thin plate of constant density \( \delta \) covering the region bounded by the parabola \( y = 25 - x^2 \) and the \( x \)-axis.

There are two ways: integrate in \( x \) or in \( y \). You just use one way.

In \( x \),

\[
\begin{align*}
M &= \int_{-5}^{5} \delta (25-x^2) \, dx \\
M_y &= \int_{-5}^{5} \delta x (25-x^2) \, dx \\
M_x &= \int_{-5}^{5} \frac{\delta (25-x^2)}{2} (25-x^2) \, dx
\end{align*}
\]

\[
M = 500 \delta (25x - \frac{1}{3} x^3) \bigg|_{-5}^{5} = \frac{5000}{3} \delta
\]

\( M_y = 0 \), because of the symmetry or that the integrand is an odd function.

\[
M_x = \frac{\delta}{2} \int_{-5}^{5} (625 - 50x^2 + x^4) \, dx
= \frac{\delta}{2} \left( 6250 - \frac{50}{3} \cdot 250 + \frac{25}{5} \right) = \frac{5000}{3} \delta
\]

\[
\overline{x} = \frac{M_y}{M} = \frac{0}{\frac{5000}{3} \delta} = 0,
\]

\[
\overline{y} = \frac{M_x}{M} = \frac{\frac{5000}{3} \delta}{\frac{5000}{3} \delta} = 10.
\]

\[\therefore \text{The center of mass is (0, 10).}\]