1 Cohomology Theory (Continued)

1.1 More Applications of Poincaré Duality

\textbf{Proposition 1.1.} Any homotopy equivalence $\mathbb{CP}^2n \xrightarrow{f} \mathbb{CP}^2n$ preserves orientation ($n \geq 1$).

In other words,

\[ f^* \colon H^4n(\mathbb{CP}^2n) \to H^4n(\mathbb{CP}^2n) \]

\[ [\mathbb{CP}^2n] \mapsto [\mathbb{CP}^2n] \]

\textit{Proof.} Say $\alpha$ generates $H^2(\mathbb{CP}^2n)$. Then, as $f$ induces an isomorphism on $H^2$, we have:

\[ f^*(\alpha) = \pm \alpha \]

We also know that $\alpha^2n$ generates $H^4n(\mathbb{CP}^2n)$, i.e.

\[ \langle \alpha^2n, [\mathbb{CP}^2n] \rangle = 1 \]

or

\[ H^4n(\mathbb{CP}^2n) \cap [\mathbb{CP}^2n] \to H^0(\mathbb{CP}^2n) \]

\[ \alpha^2n \mapsto 1 \]

If $f^*([\mathbb{CP}^2n]) = -[\mathbb{CP}^2n]$, then

\[ 1 = \langle \alpha^2n, [\mathbb{CP}^2n] \rangle = \langle \alpha^2n, -f^*([\mathbb{CP}^2n]) \rangle \]

\[ = -\langle f^*(\alpha^2n), [\mathbb{CP}^2n] \rangle = -\langle (f^*(\alpha))^2n, [\mathbb{CP}^2n] \rangle \]

\[ \overset{(1)}{=} -\langle (\pm \alpha)^2n, [\mathbb{CP}^2n] \rangle = -\langle \alpha^2n, [\mathbb{CP}^2n] \rangle = -1, \]

which is a contradiction! \hfill \Box

\textbf{Proposition 1.2.} Let $M^n$ be a closed, connected, oriented $n$-manifold and let $f : S^n \to M$ be a continuous map of non-zero degree, i.e., the morphism

\[ f_* : H_n(S^n; \mathbb{Z}) \to H_n(M; \mathbb{Z}) \]

is non-trivial. Show that $H_*(M; \mathbb{Q}) \cong H_*(S^n; \mathbb{Q})$.

\textit{Proof.} Assume that $\exists 1 \leq i \leq n - 1$ such that $H_i(M; \mathbb{Q}) = 0$. Then by Universal Coefficient Theorem,

\[ H^i(M; \mathbb{Q}) \cong \text{Hom}_\mathbb{Q}(H_i(M; \mathbb{Q}), \mathbb{Q}) \neq 0 \]

If $0 \neq \alpha \in H^i(M; \mathbb{Q})$ is the generator, then by Poincaré Duality, $\exists \beta \in H^{n-i}(M; \mathbb{Q})$ such that $\alpha \cup \beta$ generates $H^n(M; \mathbb{Q}) = \mathbb{Q}$. Especially, $\alpha \cup \beta \neq 0$.

On one hand, $f^* : H^i(M; \mathbb{Q}) \to H^i(S^n; \mathbb{Q}) = 0$ is a zero map. Consequently,

\[ f^*(\alpha \cup \beta) = f^*(\alpha) \cup f^*(\beta) = 0 \cup 0 = 0. \]

On the other hand, $f^*(\alpha \cup \beta) = (\deg f) \cdot \text{generator of } H^n(S^n; \mathbb{Q}) \neq 0$, a contradiction. \hfill \Box
**Definition 1.3** (Manifold with Boundary). $M$ is a $n$-manifold with boundary if any $x \in M$ has a neighbourhood $U_x$ homeomorphic to $\mathbb{R}^n$ or $\mathbb{R}^n_+(x_n \geq 0)$.

- if $U_x \cong \mathbb{R}^n$, $H_n(M, M - x) = H_n(U_x, U_x - x) \cong \mathbb{Z}$
- if $U_x \cong \mathbb{R}^n_+$,

$$H_n(M, M - x) = H_n(U_x, U_x - x) = H_n(\mathbb{R}^n_+, \mathbb{R}^n_+ - \{0\}) = 0$$

And the boundary of $M$ is defined to be

$$\partial M = \{ x \in M \mid H_n(M, M - x) = 0 \}.$$

**Example 1.4.** $\partial(D^n) = S^{n-1}$, $\partial(\mathbb{R}^n_+) = \mathbb{R}^{n-1}$.

**Remark 1.** $\partial M$ is a manifold of dimension $n - 1$ with no boundary.

**Definition 1.5.** $(M, \partial M)$ is orientable, if $M \setminus \partial M$ is orientable as a manifold with no boundary.

**Proposition 1.6.** If $(M, \partial M)$ is compact, orientable $n$-manifold with boundary, then $\exists! \mu_M \in H_n(M, \partial M)$ inducing local orientations $\mu_x \in H_n(M, M - x)$ at all $x \in M \setminus \partial M$.

**Note 1.** In the long exact sequence for the pair $(M, \partial M)$, we have

$$H_n(M, \partial M) \xrightarrow{\partial} H_{n-1}(\partial M)$$

$$[M] = \mu_M \mapsto [\partial M]$$

if $\partial M$ is oriented.

**Theorem 1.7** (Poincaré Duality). If $(M, \partial M)$ is a connected, oriented $n$-manifold with boundary, then

$$H^i_c(M, \partial M) \xrightarrow{\cap\mu_M} H_{n-i}(M, \partial M) \quad \text{or} \quad H^i_c(M, \partial M) \xrightarrow{\cap\mu_M} H_{n-i}(M).$$

where $H^i_c(M, \partial M) \overset{\text{def}}{=} \lim_{K \subset M \setminus \partial M} H^i(M, (M \setminus K) \cup \partial M)$ is the cohomology with compact support.

**Proposition 1.8.** If $M^n = \partial V^{n+1}$ is a connected manifold with $V$ compact, then the Euler characteristic $\chi(M)$ is even.

An immediate corollary is

**Corollary 1.9.** $\mathbb{RP}^{2n}$, $\mathbb{CP}^{2n}$, $\mathbb{HP}^{2n}$ cannot be boundaries of compact manifolds.

In order to prove Proposition 1.8, we need another proposition:
**Proposition 1.10.** Assume $V^{2n+1}$ is an oriented, compact manifold with connected boundary $\partial V = M^{2n}$. If $R$ is a field (especially $\mathbb{Z}/2\mathbb{Z}$ if $M$ is non-orientable), then $\dim_R H^n(M; R)$ is even.

**Proof of Proposition 1.10.** Let’s consider the long exact sequence for the pair $(V, M)$:

$$
\begin{align*}
H^n(V; R) & \xrightarrow{i^*} H^n(M; R) \xrightarrow{\delta} H^{n+1}(V, M; R) \\
\cong & \cap [M] \cong \cap [V] \\
H_n(M; R) & \xrightarrow{i_*} H_n(V; R)
\end{align*}
$$

where $i^*, i_*$ are induced by the inclusion $i : M = \partial V \hookrightarrow V$.

By exactness, $\text{Im } i^* \cong \text{Ker } \delta \cong \text{Ker } i_*$, so

$$\dim \text{Im } i^* = \dim \text{Ker } i_* = \dim H_n(M; R) - \dim \text{Im } i_*.$$ 

Since $i^*, i_*$ are Hom-dual, $\dim \text{Im } i^* = \dim \text{Im } i_*$.

Therefore, $\dim H^n(M; R) = \dim H_n(M; R) = 2 \dim \text{Im } i_*$ is even. \hfill \square

**Proof of Proposition 1.8.** If $n = \dim M$ is odd, then $\chi(M) = 0$ is even.

If $n = 2m$ is even, then work with $\mathbb{Z}/2\mathbb{Z}$-coefficients,

$$
\chi(M) = \sum_{i=0}^{2m} (-1)^i \dim_{\mathbb{Z}/2\mathbb{Z}} H_i(M; \mathbb{Z}/2\mathbb{Z})
\begin{equation}
\begin{aligned}
&\overset{(1)}{=} 2 \sum_{i=0}^{m-1} (-1)^i \dim_{\mathbb{Z}/2\mathbb{Z}} H_i(M; \mathbb{Z}/2\mathbb{Z}) + (-1)^m \dim_{\mathbb{Z}/2\mathbb{Z}} H_m(M; \mathbb{Z}/2\mathbb{Z}) \\
&\overset{(2)}{=} 0 \mod 2
\end{aligned}
\end{equation}
$$

Equation (1) is due to Poincaré Duality, $\dim_{\mathbb{Z}/2\mathbb{Z}} H_{n-i}(M; \mathbb{Z}/2\mathbb{Z}) = \dim_{\mathbb{Z}/2\mathbb{Z}} H_i(M; \mathbb{Z}/2\mathbb{Z})$, $0 \leq i \leq m - 1$. Congruence (2) is by Proposition 1.10. \hfill \square

**Note 2.**

- $\text{Im } i^* \subset H^n(M^{2n})$ is self-annihilating with respect to cup product $\cup$, i.e. if $\alpha, \beta \in \text{Im } i^*$, then $\alpha \cup \beta = 0$.

- by Proposition 1.10, $\dim \text{Im } i^* = \frac{1}{2} \dim H^n(M^{2n})$.

**Proof.** For any $\alpha = i^*(\overline{\alpha}), \beta = i^*(\overline{\beta}) \in B$, where $\overline{\alpha}, \overline{\beta} \in H^{2n}(V)$, we have

$$
\delta(\alpha \cup \beta) = \delta(i^*(\overline{\alpha}) \cup i^*(\overline{\beta})) = \delta i^*(\overline{\alpha} \cup \overline{\beta}) = 0
$$
Hence, $\alpha \cup \beta \in \text{Ker} \ (\delta : H^{4n}(M) \to H^{4n+1}(V)) = 0$ by the following commutative diagram

$$
H^{4n}(M) \xrightarrow{\delta} H^{4n+1}(V) \\
\cong \text{P.D.} \quad \cong \text{P.D.} \\
H_0(M) \longrightarrow H_0(V)
$$

with the bottom arrow an injection. \qed

1.2 Signature

Let $M^n$ be a closed, oriented, $n$-manifold.

- $\sigma(M) = 0$ if $4 \not| \dim M = n$.
- if $\dim M = 4k$, then $\sigma(M)$ is defined to be the signature of the non-degenerate cup product pairing

$$
(,): H^{2k}(M; \mathbb{R}) \times H^{2k}(M; \mathbb{R}) \to \mathbb{R} \\
(\alpha, \beta) \mapsto (\alpha, \beta)[M]
$$

$\sigma(M):=\text{(the number of positive eigenvalues)-\text{(the number of negative eigenvalues)}}$.

Example 1.11. $\sigma(S^2 \times S^2) = \sigma\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = 0$, $\sigma(\mathbb{C}P^{2n}) = 1$, $\sigma(\mathbb{C}P^2 \# \mathbb{C}P^2) = 2$.

Remark 2. Signature $\sigma$ is a cobordism invariant, i.e. if $\partial W = M \sqcup -N$, then $\sigma(M) = \sigma(N)$.

\begin{center}
\begin{tikzpicture}
\node (W) at (0,0) {W};
\node (M) at (-1,1) {M};
\node (N) at (1,1) {N};
\draw [dotted] (W) to (M);
\draw (W) to (N);
\end{tikzpicture}
\end{center}

Theorem 1.12. If in the above notations $M^{4k} = \partial V^{4k+1}$ is connected with $V$ compact and orientable, then $\sigma(M) = 0$.

Remark 3.  
- $\sigma(\mathbb{C}P^2 \# - \mathbb{C}P^2) = 0$. In fact, $\mathbb{C}P^2 \# - \mathbb{C}P^2$ is the boundary of a connected, oriented 5-manifold;
- $\sigma(\mathbb{C}P^2 \# \mathbb{C}P^2) = 2 \neq 0$, so $\mathbb{C}P^2 \# \mathbb{C}P^2$ is NOT the boundary of a connected, oriented 5-manifold, but as we will see later, it IS the boundary of a non-orientable 5-manifold.
Proof. Let $A = H^{2n}(M)$ and we'll always work over $\mathbb{R}$-coefficients. We have a non-degenerate and symmetric pairing $\varphi : A \times A \to \mathbb{R}$.

Let $A_+$ be the subspace on which the pairing is positive-definite,

$A_-$ be the subspace on which the pairing is negative-definite.

Let $r = \dim A_+, 2l = \dim A$ (by Proposition 1.10). Then, automatically, $\dim A_- = 2l - r$ and $\sigma(M) = r - (2l - r) = 2r - 2l$.

In order to prove that $\sigma(M) = 0$, we want to show that $r = l$!

Let $B \subset A$ be the self-annihilating $l$-dimensional subspace given by Proposition 1.8 and Note 2.

Therefore, $A_+ \cap B = \{0\}, A_- \cap B = \{0\}$. Hence,

\[
\begin{align*}
\dim A_+ + \dim B & \leq \dim A = 2l, \text{ i.e. } r + l \leq 2l \text{ i.e. } r \leq l \\
\dim A_- + \dim B & \leq \dim A = 2l, \text{ i.e. } 2l - r + l \leq 2l \text{ i.e. } r \geq l
\end{align*}
\]

In conclusion, $r = l$ and $\sigma(M) = 0$. 

\[\square\]

1.3 Connected Sums

**Definition 1.13** (Connected Sum). Let $M^n, N^n$ be closed, connected, oriented $n$-manifolds, then their connected sum

\[
M \# N := (M \setminus D_1^n) \cup_f (N \setminus D_2^n)
\]

where $f : \partial D_1^n = S^{n-1} \to \partial D_2^n = S^{n-1}$ is an orientation-reversing homeomorphism.

**Remark 4.** $M \# N$ is still a closed, connected, oriented $n$-manifold, $H^0(M \# N) = \mathbb{Z}$, $H^n(M \# N) = \mathbb{Z}$ and $H^k(M \# N) \cong H^k(M) \oplus H^k(N)$, $0 < k < n$.

**Remark 5.** Cup product $\alpha \cup \beta = 0$ for any $\alpha \in H^k(M)$ and $\beta \in H^l(N)$ with $k, l > 0$.

**Example 1.14.** $S^2 \times S^2$ and $\mathbb{CP}^2 \# \mathbb{CP}^2$ have the same cohomology groups,

\[
H^0 = \mathbb{Z}, \ H^2 = \mathbb{Z} \oplus \mathbb{Z} = \mathbb{Z} \alpha \oplus \mathbb{Z} \beta, \ H^4 = \mathbb{Z},
\]

but different cohomology rings, since

\[
\alpha \cup \beta \neq 0 \text{ in } S^2 \times S^2, \text{ while } \alpha \cup \beta = 0 \text{ in } \mathbb{CP}^2 \# \mathbb{CP}^2.
\]
Example 1.15. \(\sigma(\mathbb{C}P^2 \# \mathbb{C}P^2) = 2\), so \(\mathbb{C}P^2 \# \mathbb{C}P^2\) cannot be the boundary of a compact oriented 5-manifold. However, \(\mathbb{C}P^2 \# \mathbb{C}P^2 = \partial W^5\), where \(W^5\) is a compact non-orientable 5-manifold. \(W\) can be constructed as follows:

1. Start with \((\mathbb{C}P^2 \times I) \# (\mathbb{R}P^2 \times S^3)\).

2. Run an orientation reversing path \(\gamma\) from one \(\mathbb{C}P^2\) to the other, by traveling along an orientation reversing path in \(\mathbb{R}P^2\).

3. Enlarge the path to a tube and remove its interior. What is left is a 5-dimensional non-orientable manifold with \(\partial W = \mathbb{C}P^2 \# \mathbb{C}P^2\).

2 Homotopy Theory

Let \(X\) be a topological space, \(x_0 \in X\). Earlier we defined the fundamental group

\[
\pi_1(X, x_0) = \{f : (I, \partial I) \to (X, x_0)\} / \sim \\
= \{f : (S^1, s_0) \to (X, x_0)\} / \sim
\]

where \(\sim\) refers to the homotopy of maps.

And also some properties of \(\pi_1\):

- functorial, i.e. \(f : X \to Y\) induces \(f_* : \pi_1(X, x_0) \to \pi_1(Y, f(x_0))\);
- homotopy invariant;
- group structure;
- related to \(H_1\).

2.1 Higher Homotopy Groups

Definition 2.1 (Higher homotopy groups).

\[
\pi_n(X, x_0) = \{f : (I^n, \partial I^n) \to (X, x_0)\} / \sim \\
= \{f : (S^n, s_0) \to (X, x_0)\} / \sim
\]
where $I^n = [0,1]^n$, $\partial I^n = \{(x_1, \ldots, x_n) \in I^n \mid \exists \ i, \ \text{s.t.} \ x_i \in \{0,1\}\}$ and $\sim$ refers to homotopy of maps.

When $n = 0$, $I^0 = \text{point}$, $\partial I^0 = \emptyset$. So

$$\pi_0 = \text{set of connected components of } X$$

Define for $n \geq 1$, $f, g \in \pi_n(X, x_0)$,

$$(f + g)(s_1, \ldots, s_n) = \begin{cases} f(2s_1, s_2, \ldots, s_n) & 0 \leq s_1 \leq \frac{1}{2} \\ g(2s_1 - 1, s_2, \ldots, s_n) & \frac{1}{2} \leq s_1 \leq 1 \end{cases}$$

**Lemma 2.2.** If $n \geq 2$, $\pi_n$ is an abelian group.

**Goal:** Prove Whitehead’s Theorem.

**Theorem 2.3** (Whitehead’s Theorem). If a map $f : X \to Y$ between connected CW complexes induces isomorphisms on $\pi_n$ for all $n$, then $f$ is a homotopy equivalence. If moreover, $f$ is the inclusion of a subcomplex $X \hookrightarrow Y$, then $X$ is a deformation retract of $Y$. 