Monodromy representations of conformal field theory

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The space of conformal blocks
Plan

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- Quantum representations of mapping class groups

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- Images of quantum representations of mapping class groups
Conformal Field Theory

\((\Sigma, p_1, \cdots, p_n)\): Riemann surface with marked points
\(\lambda_1, \cdots, \lambda_n\): level \(K\) highest weights
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\(\mathcal{H}_\Sigma(p, \lambda)\) : space of conformal blocks
vector space spanned by holomorphic parts of the WZW partition function.
Wess-Zumino-Witten model

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vector space spanned by holomorphic parts of the WZW partition function.

Geometry : vector bundle over the moduli space of Riemann surfaces with \(n\) marked points with projectively flat connection.
$g = sl_2(\mathbb{C})$ has a basis

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

$\lambda$ : non-negative integer

$V_\lambda$ : irreducible highest weight representation of $sl_2(\mathbb{C})$ with highest weight vector $v$ such that

$$Hv = \lambda v, \ E v = 0$$
Representations of an affine Lie algebra

\( \hat{g} = g \otimes C((\xi)) \oplus Cc : \text{affine Lie algebra} \) with commutation relation

\[
[X \otimes f, Y \otimes g] = [X, Y] \otimes fg + \text{Res}_{\xi=0} df g \langle X, Y \rangle c
\]

\( K \) a positive integer (level)
\( \hat{g} = \mathcal{N}_+ \oplus \mathcal{N}_0 \oplus \mathcal{N}_- \)
\( c \) acts as \( K \cdot \text{id} \).
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\( \lambda : \text{an integer with } 0 \leq \lambda \leq K \)
\( \mathcal{H}_\lambda : \text{irreducible quotient of } \mathcal{M}_\lambda \) called the integrable highest weight modules.
Geometric background

$G$ : the Lie group $SL(2, \mathbb{C})$

$LG = \text{Map}(S^1, G)$ : loop group

$\mathcal{L} \rightarrow LG$ : complex line bundle with $c_1(\mathcal{L}) = K$
$G$ : the Lie group $SL(2, \mathbb{C})$

$LG = \text{Map}(S^1, G)$ : loop group

$L \rightarrow LG$ : complex line bundle with $c_1(L) = K$

The affine Lie algebra $\hat{\mathfrak{g}}$ acts on the space of sections $\Gamma(L)$.
The integrable highest weight modules $\mathcal{H}_\lambda$, $0 \leq \lambda \leq K$, appears as sub representations.
As the infinitesimal version of the action of the central extension of $\text{Diff}(S^1)$ the Virasoro Lie algebra acts on $\mathcal{H}_\lambda$. 
Suppose $0 \leq \lambda_1, \cdots, \lambda_n \leq K$.
\[ p_1, \cdots, p_n \in \Sigma \]
Assign highest weights $\lambda_1, \cdots, \lambda_n$ to $p_1, \cdots, p_n$.
$\mathcal{H}_j$ : irreducible representations of $\hat{\mathfrak{g}}$ with highest weight $\lambda_j$ at level $K$. 

The space of conformal blocks is defined as
\[ \text{H}^\Sigma(p, \lambda_1) \otimes \cdots \otimes \text{H}^\Sigma(p, \lambda_n) / (g \otimes M_p) \]
where $g \otimes M_p$ acts diagonally via Laurent expansions at $p_1, \cdots, p_n$. 

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$$\mathcal{H}_\Sigma(p, \lambda) = \mathcal{H}_{\lambda_1} \otimes \cdots \otimes \mathcal{H}_{\lambda_n} / (\mathfrak{g} \otimes \mathcal{M}_p)$$

where $\mathfrak{g} \otimes \mathcal{M}_p$ acts diagonally via Laurent expansions at $p_1, \cdots, p_n$. 
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The union

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\bigcup_{p_1, \cdots, p_n} \mathcal{H}_{\Sigma_g}(p, \lambda)
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for any complex structures on \( \Sigma_g \) forms a vector bundle on \( \mathcal{M}_{g,n} \), the moduli space of Riemann surfaces of genus \( g \) with \( n \) marked points.
Conformal block bundle

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This vector bundle is called the **conformal block bundle** and is equipped with a natural **projectively flat connection**. The holonomy representation of the mapping class group is called the quantum representation.
\( \Gamma_{g,n} \): **mapping class group** of the Riemann surface of genus \( g \) with \( n \) marked points (orientation preserving diffeomorphisms of \( \Sigma \) upto isotopy)

\( \Gamma_{g,n} \) is generated by Dehn twists.

Dehn twist along the curve \( C \)

\( \Gamma_{g,n} \) acts on \( \mathcal{H}_\Sigma \): **quantum representation** \( \rho_K \).
A basis of the space of conformal blocks is given by trivalent graphs labelled by highest weights dual to pants decomposition of the surface.

The Dehn twist along $t$ acts as $e^{2\pi i \Delta_m}$
($\Delta_m$ : conformal weight)
A braid and its closure (figure 8 knot)

**genus 0 case**: The flat connection is the **KZ connection**, which is interpreted as **Gauss-Manin connection** via hypergeometric integrals.
The case $g = 0$

$p_1, \cdots, p_{n+1} \in \mathbb{C}P^1$ with $p_{n+1} = \infty$

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We have a flat vector bundle over the configuration space

$$X_n = \{(z_1, \cdots, z_n) \in \mathbb{C}^n ; z_i \neq z_j, \ i \neq j\}.$$

with KZ connection.
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The monodromy representation is the quantum representation of the braid groups.
\{I_\mu\} : \text{orthonormal basis of } \mathfrak{g} \ \text{w.r.t. Killing form.}
\Omega = \sum_\mu I_\mu \otimes I_\mu
r_i : \mathfrak{g} \rightarrow \text{End}(V_i), \ 1 \leq i \leq n \ \text{representations.}
\{I_\mu\} : \text{orthonormal basis of } \mathfrak{g} \text{ w.r.t. Killing form.}

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\Omega_{ij} : \text{the action of } \Omega \text{ on the } i\text{-th and } j\text{-th components of } V_1 \otimes \cdots \otimes V_n.

\omega = \frac{1}{\kappa} \sum_{i,j} \Omega_{ij} d\log(z_i - z_j), \quad \kappa \in \mathbb{C} \setminus \{0\}

\omega \text{ defines a flat connection for a trivial vector bundle over the configuration space } X_n \text{ with fiber } V_1 \otimes \cdots \otimes V_n \text{ since we have}

\omega \wedge \omega = 0
As the holonomy we have representations

$$\theta_\kappa : P_n \to GL(V_1 \otimes \cdots \otimes V_n).$$

In particular, if $V_1 = \cdots = V_n = V$, we have representations of braid groups

$$\theta_\kappa : B_n \to GL(V^{n \otimes}).$$
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\theta_\kappa : B_n \rightarrow GL(V^n). 
$$

We shall express the horizontal sections of the KZ connection: $d\varphi = \omega \varphi$ in terms of homology with coefficients in local system homology on the fiber of the projection map

$$
\pi : X_{m+n} \rightarrow X_n.
$$

$X_{n,m}$ : fiber of $\pi$, \quad $Y_{n,m} = X_{n,m}/S_m$
\( \mathcal{L} \): rank 1 local system over \( Y_{n,m} \)

\[
m = \frac{1}{2}(\lambda_1 + \cdots + \lambda_n - \lambda_{n+1})
\]

\( \mathcal{H}_{n,m} \): local system over \( X_n \) with fiber \( H_m(Y_{n,m}, \mathcal{L}) \)

**Theorem**

There is an injective bundle map from the conformal block bundle

\[
\bigcup \mathcal{H}_{\mathbb{C}P^1}(p, \lambda) \longrightarrow \mathcal{H}_{n,m}
\]

via hypergeometric integrals. The KZ connection is interpreted as Gauss-Manin connection.
Asymptotic faithfulness

Any two elements of the mapping class group are distinguished by the quantum representation for sufficiently large $K$ (J. Andersen).

$B_n[k]$ : normal subgroup of the braid group $B_n$ generated by $\sigma_i^k$, $1 \leq i \leq n - 1$.

\[
\begin{array}{c}
1 & 2 & i & i+1 & n \\
\cdots & \cdots & \sigma_i & \cdots & \\
\end{array}
\]

Theorem (L. Funar and T. Kohno)

For any infinite set $\{k\}$, we have $\bigcap_k B_n[2k] = \{1\}$.

A positive answer to Squier’s conjecture.
The quantum representations are projectively unitary.

\[ \rho_K : \Gamma_g \longrightarrow PU(\mathcal{H}_{\Sigma_g}) \]

The \( k \)-th Johnson subgroup acts trivially on the \( k \)-th lower central series of the fundamental group \( \pi_1(\Sigma_g) \).

The image of the quantum representation is “big” in the following sense.

**Theorem (L. Funar and T. Kohno)**

Suppose \( g \geq 4 \) and \( K \) sufficiently large. Then the image of any Johnson subgroup by \( \rho_K \) contains a non-abelian free group.
Images of braid groups $B_3$ in the mapping class group by the quantum representation $\rho_K$ are related to Schwarz triangle groups.

tessellation of the Poincaré disc by the triangle group
Gilmer and Masbaum show in the case $\kappa = K + 2$ is odd prime, the image of the quantum representation $\rho_K$ is contained in

$$PU(\mathcal{O}_\kappa)$$

where $\mathcal{O}_\kappa$ is the ring of cyclotomic integers:

$$\mathcal{O}_\kappa \subset \mathbb{Q}(e^{2\pi i/\kappa})$$

Suppose $g, K$ sufficiently large.

Theorem (L. Funar and T. Kohno)

$\rho_K(\Gamma_g)$ is of infinite index in $PU(\mathcal{O}_\kappa)$.

Reference: L. Funar and T. Kohno, On images of quantum representations of mapping class groups, arXiv:0907.0568