EXAM II  Solutions

Do all five of the following problems. Show all your work, justify your answers: answers without supporting work will not receive full credit. It is not necessary to simplify your answers.

<table>
<thead>
<tr>
<th>No.</th>
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<tbody>
<tr>
<td>1</td>
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<td>2</td>
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Score
Problem I (20 points)
Determine where the curve
\[ y = x + \sin x, \quad 0 \leq x \leq 2\pi \]
is increasing, decreasing, concave up and concave down. Where are its local extrema and inflection points? Use this information to sketch the curve.

\[ y' = 1 + \cos x \]
\[ y'' = -\sin x \]

- Critical points: \( y' = 0 \Rightarrow 1 + \cos x = 0 \Rightarrow \cos x = -1 \Rightarrow x = \pi \)

Since \( \cos x \in [-1, 1] \), we have that \( 1 + \cos x \geq 0 \) for all \( x \).
So \( y' \geq 0 \) on \([0, 2\pi]\), i.e., \( y \) is increasing over \([0, 2\pi]\) and \( x = \pi \) is not a local extreme.

- \( y'' = 0 \Rightarrow \sin x = 0 \Rightarrow x = 0, \pi, 2\pi \)

<table>
<thead>
<tr>
<th>( x )</th>
<th>0</th>
<th>( \pi )</th>
<th>2( \pi )</th>
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<tbody>
<tr>
<td>( y' )</td>
<td>+</td>
<td>0</td>
<td>+</td>
</tr>
<tr>
<td>( y' )</td>
<td>0</td>
<td>( \pi )</td>
<td>( 2\pi )</td>
</tr>
<tr>
<td>( y'' )</td>
<td>-</td>
<td>0</td>
<td>+</td>
</tr>
</tbody>
</table>

\( y(0) = 0 \)
\( y(\pi) = \pi \)
\( y(2\pi) = 2\pi \)
\( y'' = -\sin x \leq 0 \) for \( x \in [0, \pi] \)
\( \geq 0 \) for \( x \in [\pi, 2\pi] \)

So \( x = \pi \) is an inflection point.
Problem II (20 points) Evaluate the following limits:

a) \[ \lim_{x \to \infty} (\sqrt{x^2 + x + 1} - \sqrt{x^2 - x}) = \lim_{x \to \infty} \frac{(\sqrt{x^2 + x + 1} - \sqrt{x^2 - x})(\sqrt{x^2 + x + 1} + \sqrt{x^2 - x})}{\sqrt{x^2 + x + 1} + \sqrt{x^2 - x}} \]

\[ = \lim_{x \to \infty} \frac{(x^2 + x + 1) - (x^2 - x)}{\sqrt{x^2 + x + 1} + \sqrt{x^2 - x}} \]

\[ = \lim_{x \to \infty} \frac{2x + 1}{x \cdot \sqrt{1 + \frac{1}{x^2}} + \sqrt{1 - \frac{1}{x}}} \]

\[ = \lim_{x \to \infty} \frac{2 + \frac{1}{x}}{\sqrt{1 + \frac{1}{x^2}} + \sqrt{1 - \frac{1}{x}}} \quad \text{so} \quad \frac{2}{1 + 1} = \frac{1}{1} = 1. \]

b) \[ \lim_{x \to 0} \frac{2x}{x + 7\sqrt{x}} = \frac{0}{0} \]

by l'Hôpital: \[ = \lim_{x \to 0} \frac{2}{1 + \frac{7}{2\sqrt{x}}} = \lim_{x \to 0} \frac{4\sqrt{x}}{2\sqrt{x} + 7} = \frac{0}{0 + 7} = 0. \]

c) \[ \lim_{x \to 0} \frac{\sin^2 x}{x \tan 2x} = \frac{0}{0} \]

\[ = \lim_{x \to 0} \frac{\sin x}{x} \cdot \frac{\sin x}{\tan 2x} = \lim_{x \to 0} \frac{\sin x}{x} \cdot \frac{\sin x}{x} \cdot \frac{x}{\tan 2x} \]

\[ = \lim_{x \to 0} \frac{\cos 2x}{\sin 2x} \quad \text{so} \quad \lim_{x \to 0} \frac{x}{\sin 2x} = \frac{1}{2} \]

\[ = \lim_{x \to 0} \frac{2x}{2\cos 2x} = \frac{1}{2} \cdot 2 = 1. \]

d) \[ \lim_{x \to 0} \frac{\tan x}{x + \sin x} = \frac{0}{0} \]

\[ = \lim_{x \to 0} \frac{\tan x}{x} \cdot \frac{x}{x + \sin x} \quad \text{so} \quad \lim_{x \to 0} \frac{\sin x}{x} \cdot \frac{1}{\cos x} \cdot \frac{x}{x + \sin x} \]

\[ = \lim_{x \to 0} \frac{x}{x + \sin x} \quad \text{so} \quad \lim_{x \to 0} \frac{1}{1 + \cos x} = \frac{1}{1 + 1} = \frac{1}{2} \]

\[ = \frac{1}{2} \]
Problem III (20 points)
An isosceles triangle has its vertex at the origin and its base parallel to the x-axis, with the vertices above the axis on the curve $y = 27 - x^2$. Find the largest area the triangle can have.

Let $(x, y)$ be the coordinates of the right vertex of the triangle, which is on the curve $y = 27 - x^2$. Thus $x$ and $y$ are related by $y = 27 - x^2$.

The area of the triangle can be given as: $A = \text{base} \cdot \text{height} / 2$

In our notation: \[ \begin{align*} \text{height} &= y = 27 - x^2 \\ \text{base} &= 2x \end{align*} \]

So $A(x) = x(27 - x^2) = 27x - x^3/2$, gives the area as a function of $x$.

Note that the domain of $x$ is: $0 \leq x \leq 3\sqrt{3}$

In order to maximize $A(x)$, we find its critical points:
$A'(x) = 27 - 3x^2 = 0 \implies x^2 = 9 \implies x = 3$ (the other root $x = -3$ is not in the domain)

Plug-in the endpoints and critical pts in $A$:
$A(0) = 0$
$A(3\sqrt{3}) = 0$
$A(3) = 27 \cdot 3 - 27 = 54$, so the largest area the triangle can have is $\boxed{54}$. 
Problem IV (20 points) Evaluate:

a) \( \int_{-1}^{1} 2x \sin(1 - x^2) \, dx \)

\[ u = 1 - x^2 \Rightarrow du = -2x \, dx \]

\[ x = -1 \Rightarrow u = 0 \quad \text{this signals that the value of the integral is 0.} \]

\[ x = 1 \Rightarrow u = 0 \]

In fact, \( f(x) = 2x \sin(1 - x^2) \) is an odd function, as \( f(-x) = 2(-x) \cdot \sin(1 - (-x)^2) = -2x \cdot \sin(1 - x^2) = -f(x) \).

Since \([-1, 1]\) is symmetric about the origin, we have: \( \int_{-1}^{1} f(x) \, dx = 0 \).

b) \( \int_{1}^{4} \frac{1}{t^{3/2}} \, dt \)

\[ \int_{1}^{4} \frac{1}{t^{3/2}} \, dt = \int_{1}^{4} t^{-3/2} \, dt = \left[ \frac{t^{-1/2}}{-1/2} \right]_{1}^{4} = \left[ \frac{1}{\sqrt{t}} \right]_{1}^{4} = -\frac{2}{\sqrt{4}} + \frac{1}{\sqrt{1}} = 1 \]

Even function

\[ \int_{-\pi/2}^{\pi/2} 15 \sin^4 3x \cos 3x \, dx = 2 \left[ \int_{0}^{\pi/2} 15 \sin^4 3x \cos 3x \, dx \right]_{0}^{\pi/2} = 2 \left[ \int_{0}^{\pi/2} 15 \sin^4 3x \cos 3x \, dx \right]_{0}^{\pi/2} \]

\[ u = \sin 3x \Rightarrow du = 3 \cos 3x \, dx \]

\[ x = 0 \Rightarrow u = 0 \]

\[ x = \frac{\pi}{2} \Rightarrow u = \sin \frac{3\pi}{2} = -1 \]

\[ \text{Get:} \left[ \int_{1}^{4} u^4 \, du = -10 \left[ \int_{0}^{\pi/2} \sin^4 3x \cos 3x \, dx \right]_{0}^{\pi/2} = -2u^5 \right]_{-1}^{1} = -2 \]

\[ d) \int \frac{(t+1)^2 - 1}{t^4} \, dt = \int \frac{t^2 + 2t + 1 - 1}{t^4} \, dt = \int \frac{t^2 + 2t}{t^4} \, dt = \int (t^{-2} + 2t^{-3}) \, dt \]

\[ = -\frac{t^{-1}}{-1} + 2 \cdot \frac{t^{-2}}{-2} + C \]

\[ = -\frac{1}{t} - \frac{1}{t^2} + C. \]
**Problem V** (20 points)
Find the area of the "triangular" region bounded on the left by \( x + y = 2 \), on the right by \( y = x^2 \), and above by \( y = 2 \).

First sketch the region bounded by:
\[
\begin{aligned}
&y = 2 - x \\
&y = x^2 \\
&y = 2
\end{aligned}
\]

Note that \( y = 2 - x \) meets \( y = x^2 \) when:
\[
2 - x = x^2
\]
or
\[
x^2 + x - 2 = 0
\]
or
\[
(x - 1)(x + 2) = 0
\]
\[
\Rightarrow x = 1, x = -2
\]

Also \( y = x^2 \) meets \( y = 2 \) for \( x^2 = 2 \), or \( x = \pm \sqrt{2} \).

The shaded region above is the "triangular" region described in the statement of the problem. Its area is given by:
\[
A = \int_0^1 (2 - (2-x)) \, dx + \int_1^{-\sqrt{2}} (2-x^2) \, dx
\]
\[
= \left[ x \right]_0^1 + \left[ \frac{x^3}{3} \right]_1^{-\sqrt{2}}
\]
\[
= \frac{2}{2} + \left[ \left( 2\sqrt{2} - \frac{2\sqrt{2}}{3} \right) - \left( 2 - \frac{1}{3} \right) \right]
\]
\[
= \frac{1}{2} + \frac{4\sqrt{2}}{3} - \frac{5}{3} = \frac{8\sqrt{2} - 7}{6}
\]