L^2–BETTI NUMBERS OF HYPERSURFACE COMPLEMENTS

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Abstract. In [DJL07] it was shown that if $\mathcal{A}$ is an affine hyperplane arrangement in $\mathbb{C}^n$, then at most one of the L^2–Betti numbers $b_i^{(2)}(\mathbb{C}^n \setminus \mathcal{A}, \mathrm{id})$ is non–zero. In this note we prove an analogous statement for complements of complex affine hypersurfaces in general position at infinity. Furthermore, we recast and extend to this higher-dimensional setting results of [FLM09, LM06] about L^2–Betti numbers of plane curve complements.

1. Introduction

Let M be any topological space and $\alpha : \pi_1(M) \to \Gamma$ an epimorphism to a group Γ (all groups are assumed countable). Then for $i \in \mathbb{N} \cup \{0\}$ we can consider the L^2–Betti number $b_i^{(2)}(M, \alpha) \in [0, \infty]$. We recall the definition and some of the most important properties of L^2–Betti numbers in Section 2.

Let $X \subset \mathbb{C}^n$ $(n \geq 2)$ be a reduced affine hypersurface defined by a polynomial equation $f = f_1 \cdots f_s = 0$, where f_i are the irreducible factors of f. Denote by $X_i := \{f_i = 0\}$, $i = 1, \cdots, s$, the irreducible components of X, and let

$$M_X := \mathbb{C}^n \setminus X$$

be the hypersurface complement. Then M_X has the homotopy type of a finite CW complex of dimension n. It is well–known that $H_1(M_X; \mathbb{Z})$ is a free abelian group generated by the meridian loops γ_i about the non-singular part of each irreducible component X_i of X. Throughout the paper we denote by ϕ the map $\pi_1(M_X) \to \mathbb{Z}$ given by sending each meridian γ_i to 1. This is the same map as the homomorphism $f_* : \pi_1(M_X) \to \pi_1(\mathbb{C}^*) = \mathbb{Z}$ induced by f. We also refer to ϕ as the *total linking number homomorphism*. We call an epimorphism $\alpha : \pi_1(M_X) \to \Gamma$ *admissible* if the total linking number homomorphism ϕ factors through α.

The main result of this note is the following "nonresonance-type" theorem.

Theorem 1.1. *Let $X \subset \mathbb{C}^n$ be a reduced affine hypersurface in general position at infinity, i.e., whose projective completion intersects the hyperplane at infinity transversely. If $\alpha : \pi_1(M_X) \to \Gamma$ is an admissible epimorphism, then the L^2–Betti numbers of the complement M_X are computed by:*

$$b_i^{(2)}(M_X, \alpha) = \begin{cases} 0, & \text{for } i \neq n, \\ (-1)^n \chi(M_X), & \text{for } i = n, \end{cases}$$

where $\chi(M_X)$ denotes the Euler characteristic of M_X.

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In particular,

$$(-1)^n \cdot \chi(M_X) \geq 0.$$

As an immediate consequence we note the following:

Corollary 1.2. *If $X \subset \mathbb{C}^n$ is a reduced affine hypersurface defined by a homogeneous polynomial (i.e., X is the affine cone on a reduced projective hypersurface in $\mathbb{CP}^{n-1}$), and $\alpha : \pi_1(M_X) \to \Gamma$ is an admissible epimorphism, then*

$$b_i^{(2)}(M_X, \alpha) = 0 \ , \ \textit{for all} \ \ i \geq 0.$$

Indeed, it is easy to see that such an affine cone $X \subset \mathbb{C}^n$ is in general position at infinity. Moreover, by the existence of a *global Milnor fibration* [Mi68], the complement M_X is the total space of a fibration over $\mathbb{C}^*$, hence $\chi(M_X) = 0$.

In [DJL07] it was shown that if $\mathcal{A}$ is an affine hyperplane arrangement in $\mathbb{C}^n$, then at most one of the L^2–Betti numbers $b_i^{(2)}(\mathbb{C}^n \setminus \mathcal{A}, \mathrm{id})$ is non–zero. Theorem 1.1 can be seen as an analogous statement for the complement of a hypersurface in $\mathbb{C}^n$ which is in general position at infinity. In particular, we recast and generalize to arbitrary dimensions some results of [LM06, FLM09], where the case $n = 2$ of plane curve complements was considered.

In this note, we are also concerned with the L^2–Betti numbers of the infinite cyclic cover defined by the total linking number homomorphism ϕ. More precisely, given an affine hypersurface $X \subset \mathbb{C}^n$, we denote by $\widetilde{M}_X$ the infinite cyclic cover of M_X corresponding to ϕ. Moreover, for an admissible epimorphism $\alpha : \pi_1(M_X) \to \Gamma$ we let $\widetilde{\Gamma} := \mathrm{Im}\{\pi_1(\widetilde{M}_X) \to \pi_1(M_X) \xrightarrow{\alpha} \Gamma\}$, and we denote the induced map $\pi_1(\widetilde{M}_X) \to \widetilde{\Gamma}$ by $\widetilde{\alpha}$. The L^2–Betti numbers we are interested in are

$$b_i^{(2)}(\widetilde{M}_X, \widetilde{\alpha} : \pi_1(\widetilde{M}_X) \to \widetilde{\Gamma}).$$

In [Ma06], the author showed that for hypersurfaces $X \subset \mathbb{C}^n$ in general position at infinity the ordinary Betti numbers $b_i(\widetilde{M}_X)$ of the infinite cyclic cover $\widetilde{M}_X$ are finite for all $0 \leq i \leq n-1$. In this note, we prove a non-commutative generalization of this fact. More precisely, we show the following:

Theorem 1.3. *Assume that the affine hypersurface $X \subset \mathbb{C}^n$ is in general position at infinity. Then the L^2–Betti numbers $b_i^{(2)}(\widetilde{M}_X, \widetilde{\alpha})$ are finite for all $0 \leq i \leq n-1$.*

As it was already observed in several recent papers, e.g., see [DL06, DM07, Ma06], hypersurfaces in general position at infinity behave much like weighted homogeneous hypersurfaces up to homological degree $n-1$; see Prop.3.1 for a computation of L^2–Betti numbers of weighted homogeneous hypersurface complements. The above Theorem 1.3 comes as a confirmation of this philosophy.

Another motivation for studying L^2–Betti numbers of the infinite cyclic cover $\widetilde{M}_X$ comes from the fact that for appropriate choices of the group Γ these numbers specialize into several classical Alexander-type invariants of the complement. For instance, following work of Cochran and Harvey (cf. [Co04, Ha05]), we can consider the following homomorphism

$$\pi_n : \pi_1(M_X) \to \pi_1(M_X)/\pi_1(M_X)_r^{(m+1)} =: \Gamma_m,$$

where given a group G we denote by $G_r^{(m)}$ the m–th term in the rational derived series of G. The group Γ_m is a poly-torsion-free-abelian (PTFA) group and we define the *higher-order degrees* $\delta_{i,m}(X)$ of X as the dimension of the i-th homology of $\widetilde{M}_X$ with coefficients in the skew field associated to $\widetilde{\Gamma}_m$. Similar invariants were defined and studied in [LM06, LM07] in the case of plane curves. Moreover, as noted in [FLM09], the higher-order degrees $\delta_{i,m}(X)$ can be regarded as L^2–Betti numbers of the infinite cyclic cover $\widetilde{M}_X$, so the results of this note characterize the Cochran-Harvey invariants as well. At this point we want to emphasize that the higher-order degrees of a space M, hence also the L^2–Betti numbers of the infinite cyclic cover $\widetilde{M}$, may as well be infinite, since $\widetilde{M}$ is not in general a finite CW-complex. For example, for a topological space M with $\pi_1(M)$ a free group with at least two generators, the higher-order degrees $\delta_{1,m}$ are infinite (cf. [Ha05, Ex.8.2]). The finiteness results of this note are rigidity properties specific to the complex algebraic setting. We should also mention that if X is irreducible and in general position at infinity, then it is easy to see (cf. [LM06]) that for all $0 \leq i \leq n-1$ the integer $\delta_{i,0}(X)$ equals the degree of the i-th Alexander polynomial of X, or as shown in [Ma06], the i-th Betti numbers of the infinite cyclic cover $\widetilde{M}_X$. In the general reducible case, Libgober pointed out a nice relationship between $\delta_{i,0}(X)$ and the *support* of the i-th universal abelian Alexander module of the complement M_X (for more details, see [LM06] and the references therein).

Our result in Theorem 1.1 is reminiscent of a similar calculation by Jost-Zuo [JZ00] of L^2–Betti numbers of a compact Kähler manifold of non-positive sectional curvature. This was considered in relation to an old question of Hopf whether the Euler characteristic of a compact manifold M of even real dimension $2n$ has sign equal to $(-1)^n$, provided M admits a metric of negative sectional curvature. However, the statement of our Theorem 1.1 is metric independent (and the degree of X is arbitrary). Finally, one should not be misled by these calculations into thinking that the L^2–Betti numbers of finite CW-complexes are always integers, or that most of them usually vanish. In fact, the Atiyah conjecture asserts that these L^2–Betti numbers are always rational; see [Lü02a, Lü02b] for more details on this conjecture and related matters.

This paper is organized as follows. In Section 2 we recall the definition of L^2–Betti numbers and list some of their properties. We also express the Cochran-Harvey higher-order degrees of an affine hypersurface complement as L^2–Betti numbers of the infinite cyclic cover of the complement. The main results, Theorem 1.1 and 1.3 are proved in Section 3.

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2. L^2–Betti numbers

2.1. **The von Neumann algebra and its localizations.** Let Γ be a countable group. Define

$$l^2(\Gamma) := \{f : \Gamma \to \mathbb{C} \mid \sum_{g \in \Gamma} |f(g)|^2 < \infty\}$$

the Hilbert space of square-summable functions on Γ. Then Γ acts on $l^2(\Gamma)$ by right multiplication, i.e., $(g \cdot f)(h) = f(hg)$. This defines an injective map $\mathbb{C}[\Gamma] \to \mathcal{B}(l^2(\Gamma))$, where $\mathcal{B}(l^2(\Gamma))$ is the set of bounded operators on $l^2(\Gamma)$. We henceforth view $\mathbb{C}[\Gamma]$ as a subset of $\mathcal{B}(l^2(\Gamma))$.

The *von Neumann algebra* $\mathcal{N}(\Gamma)$ of Γ is defined as the closure of $\mathbb{C}[\Gamma] \subset \mathcal{B}(l^2(\Gamma))$ with respect to pointwise convergence in $\mathcal{B}(l^2(\Gamma))$. Note that any $\mathcal{N}(\Gamma)$–module $\mathcal{M}$ has a dimension $\dim_{\mathcal{N}(\Gamma)}(\mathcal{M}) \in \mathbb{R}_{\geq 0} \cup \{\infty\}$. We refer to [Lü02b, Def.6.20] for details.

2.2. L^2**–Betti numbers. Definition. Properties.** Let M be a topological space (not necessarily compact) and let $\alpha : \pi_1(M) \to \Gamma$ be an epimorphism to a group. Denote by M_Γ the regular covering of M corresponding to α, and consider the $\mathcal{N}(\Gamma)$–chain complex

$$C_*^{sing}(M_\Gamma) \otimes_{\mathbb{Z}[\Gamma]} \mathcal{N}(\Gamma),$$

where $C_*^{sing}(M_\Gamma)$ is the singular chain complex of M_Γ with right Γ–action given by covering translation, and Γ acts canonically on $\mathcal{N}(\Gamma)$ on the left. The *i-th* L^2*–Betti number of the pair* (M, α) is defined as

$$b_i^{(2)}(M, \alpha) := \dim_{\mathcal{N}(\Gamma)}(H_i(C_*^{sing}(M_\Gamma) \otimes_{\mathbb{Z}[\Gamma]} \mathcal{N}(\Gamma))) \in [0, \infty].$$

We refer to [Lü02b, Def.6.50] for more details. Note that if M is a CW-complex of finite type, then the cellular chain complex $C_*(M_\Gamma)$ can be used in the above definition of L^2–Betti numbers.

In the following lemma we summarize some of the properties of L^2–Betti numbers. We refer to [Lü02b, Thm.6.54, Lem.6.53 and Thm.1.35] for the proofs.

Lemma 2.1. *Let M be a topological space and let $\alpha : \pi_1(M) \to \Gamma$ be an epimorphism to a group.*

1. *$b_i^{(2)}(M, \alpha)$ is a homotopy invariant of the pair (M, α).*
2. *$b_0^{(2)}(M, \alpha) = 0$ if Γ is infinite and $b_0^{(2)}(M, \alpha) = \frac{1}{|\Gamma|}$ if Γ is finite.*
3. *If M is a finite CW–complex, then*
$$\sum_i (-1)^i \, b_i^{(2)}(M, \alpha) = \chi(M),$$
where $\chi(M)$ denotes the Euler characteristic of M.

Remark. For the definition of L^2–Betti numbers of a pair (M, α) we do not need to require that the homomorphism α is surjective. However, we can reduce ourselves to this case since, for an arbitrary homomorphism $\alpha : \pi_1(M) \to \Gamma$, we have that

$$b_i^{(2)}(M, \alpha : \pi_1(M) \to \mathrm{Im}(\alpha)) = b_i^{(2)}(M, \alpha : \pi_1(M) \to \Gamma). \tag{1}$$

A *free* Γ*-CW complex* $\widetilde{M}$ is the same as a regular covering $p : \widetilde{M} \to M$ of a CW complex M with Γ as group of covering transformations. As a generalization of the homotopy invariance of L^2–Betti numbers, we have the following result (see [Lü02b, Thm.1.35(1)]):

Lemma 2.2. *Let $\widetilde{f} : \widetilde{N} \to \widetilde{M}$ be a Γ-map of free Γ-CW complexes of finite type, and denote by $f : N \to M$ the induced map on the corresponding orbit spaces. If the homomorphism $H_i(\widetilde{f}; \mathbb{C}) : H_i(\widetilde{N}; \mathbb{C}) \to H_i(\widetilde{M}; \mathbb{C})$ is bijective for $i \leq d-1$ and surjective for $i = d$, then:*

$$b_i^{(2)}(M; \alpha : \pi_1(M) \to \Gamma) = b_i^{(2)}(N; \alpha \circ f_* : \pi_1(N) \xrightarrow{f_*} \pi_1(M) \xrightarrow{\alpha} \Gamma) \ , \text{ for } i < d, \tag{2}$$

and

$$b_d^{(2)}(M; \alpha) \leq b_d^{(2)}(N; \alpha \circ f_*). \tag{3}$$

Finally, the L^2–Betti numbers provide obstructions for a closed manifold to fiber over the circle S^1. More precisely, by [Lü02b, Thm.1.39], we have the following:

Lemma 2.3. *Let M be a CW complex of finite type, and $f : M \to S^1$ a fibration with connected fiber F. Assume that the epimorphism $f_* : \pi_1(M) \to \pi_1(S^1) = \mathbb{Z}$ admits a factorization $\pi_1(M) \xrightarrow{\alpha} \Gamma \xrightarrow{\beta} \mathbb{Z}$, with α and β epimorphisms . Then*

$$b_i^{(2)}(M, \alpha) = 0 \ , \quad \text{for all } i \geq 0.$$

2.3. **L^2–Betti numbers and Cochran–Harvey invariants.** A group Γ is called *locally indicable* if for every finitely generated non–trivial subgroup $H \subset \Gamma$ there exists an epimorphism $H \to \mathbb{Z}$. In the following we refer to a locally indicable torsion–free amenable group as a LITFA group. We refer to [Lü02b, p.256] for the definition of an *amenable* group, but we note that any solvable group is amenable, while groups containing the free groups on two generators are not amenable. Also, note that a subgroup of a LITFA group is itself a LITFA group.

Denote by S the set of non–zero divisors of the ring $\mathcal{N}(\Gamma)$. By [Re98, Prop.2.8] (see also [Lü02b, Thm.8.22]) the pair $(\mathcal{N}(\Gamma), S)$ satisfies the right Ore condition. The ring $\mathcal{U}(\Gamma) := \mathcal{N}(\Gamma)S^{-1}$ is called the *algebra of operators affiliated to* $\mathcal{N}(\Gamma)$. For any $\mathcal{U}(\Gamma)$–module $\mathcal{M}$ we also have a dimension $\dim_{\mathcal{U}(\Gamma)}(\mathcal{M})$. By [Lü02b, Thm.8.31] we have

$$b_i^{(2)}(M, \alpha) = \dim_{\mathcal{U}(\Gamma)}(H_i(C_*^{sing}(M_\Gamma) \otimes_{\mathbb{Z}[\Gamma]} \mathcal{U}(\Gamma))).$$

We collect below some properties of LITFA groups (see [FLM09, Thm.2.2] and the references therein):

Theorem 2.4. *Let Γ be a LITFA group.*

(1) *All non–zero elements in $\mathbb{Z}[\Gamma]$ are non–zero divisors in $\mathcal{N}(\Gamma)$.*
(2) *$\mathbb{Z}[\Gamma]$ is an Ore domain and embeds in its classical right ring of quotients $\mathbb{K}(\Gamma)$, a skew-field.*
(3) *$\mathbb{K}(\Gamma)$ is flat over $\mathbb{Z}[\Gamma]$.*
(4) *There exists a monomorphism $\mathbb{K}(\Gamma) \to \mathcal{U}(\Gamma)$ which makes the following diagram commute*

$$\begin{array}{ccc} \mathbb{Z}[\Gamma] & \longrightarrow & \mathbb{K}(\Gamma) \\ & \searrow & \downarrow \\ & & \mathcal{U}(\Gamma). \end{array}$$

Remark. Since for a LITFA group Γ, $\mathbb{K}(\Gamma)$ is a skew-field, it follows that any $\mathbb{K}(\Gamma)$-module is free. In particular, $\mathcal{U}(\Gamma)$ is flat as a $\mathbb{K}(\Gamma)$-module.

The following result of [FLM09] relates L^2–Betti numbers to ranks of modules over skew fields.

Proposition 2.5. *(*[FLM09, Prop.2.3]*) Let $\alpha : \pi_1(M) \to \Gamma$ be an epimorphism to a LITFA group Γ. Then*

$$b_i^{(2)}(M, \alpha) = \dim_{\mathbb{K}(\Gamma)}(H_i(M; \mathbb{K}(\Gamma))).$$

A group Γ is called *poly–torsion–free–abelian* (PTFA) if there exists a normal series

$$1 = \Gamma_0 \subset \Gamma_1 \subset \cdots \subset \Gamma_{n-1} \subset \Gamma_n = \Gamma$$

such that Γ_i/Γ_{i-1} is torsion free abelian. It is easy to see that PTFA groups are LITFA. Note that the quotients $\pi/\pi_r^{(k)}$ of a group by terms in the rational derived series are PTFA (cf. [Co04, Ha05]).

We now recall the definition of the Cochran–Harvey invariants (which in the context of complex algebraic geometry were first studied in [LM06], for plane curve complements). Let X be a hypersurface in $\mathbb{C}^n$, with complement $M_X := \mathbb{C}^n \setminus X$. Furthermore let $\alpha : \pi_1(M_X) \to \Gamma$ be an *admissible* epimorphism to a LITFA group. Recall that "admissible" means that there exists a map $\widetilde{\phi} : \Gamma \to \mathbb{Z}$ such that the following diagram commutes

$$\begin{array}{ccc} \pi_1(M_X) & \xrightarrow{\ \alpha\ } & \Gamma \\ & {\scriptstyle \phi} \searrow \quad \swarrow {\scriptstyle \widetilde{\phi}} & \\ & \mathbb{Z} & \end{array}$$

where $\phi : \pi_1(M_X) \to \mathbb{Z}$ is the total linking homomorphism. Denote by $\widetilde{M}_X$ the infinite cyclic cover of M_X defined by the kernel of the total linking number homomorphism ϕ. Let $\widetilde{\Gamma}$ be the kernel of $\widetilde{\phi} : \Gamma \to \mathbb{Z}$ and denote the induced homomorphism $\pi_1(\widetilde{M}_X) \to \widetilde{\Gamma}$ by $\widetilde{\alpha}$.

Now consider the homomorphism $\pi_1(M_X) \to \pi_1(M_X)/\pi_1(M_X)_r^{(m+1)} =: \Gamma_m$. It is easy to see that this homomorphism is admissible. As in [LM06] we now define

$$\delta_{i,m}(X) = \dim_{\mathbb{K}(\widetilde{\Gamma}_m)}(H_i(\widetilde{M}_X; \mathbb{K}(\widetilde{\Gamma}_m))).$$

The following result, which is an immediate corollary to Prop.2.5, shows that the L^2–Betti numbers of $\widetilde{M}_X$ can be viewed as a generalization of the Cochran–Harvey invariants of affine hypersurface complements.

Theorem 2.6. *Let $X \subset \mathbb{C}^n$ be an affine hypersurface with complement M_X, and let $\alpha : \pi_1(M_X) \to \Gamma$ be an admissible epimorphism to a LITFA group. Then, in the above notations, we have*

$$\dim_{\mathbb{K}(\widetilde{\Gamma})}(H_i(\widetilde{M}_X; \mathbb{K}(\widetilde{\Gamma}))) = b_i^{(2)}(\widetilde{M}_X, \widetilde{\alpha} : \pi_1(\widetilde{M}_X) \to \widetilde{\Gamma}).$$

3. Vanishing of L^2-Betti numbers of hypersurface complements

Let X be a reduced hypersurface in $\mathbb{C}^n$ ($n \geq 2$), defined by the equation $f = f_1 \cdots f_s = 0$, where f_i are the irreducible factors of f, and let $X_i = \{f_i = 0\}$ denote the irreducible components of X. Embed $\mathbb{C}^n$ in $\mathbb{CP}^n$ by adding the hyperplane at infinity, H, and let $\bar{X}$ be the projective hypersurface in $\mathbb{CP}^n$ defined by the homogenization f^h of f. Let M_X denote the affine hypersurface complement

$$M_X := \mathbb{C}^n \setminus X.$$

Alternatively, M_X can be regarded as the complement in $\mathbb{CP}^n$ of the divisor $\bar{X} \cup H$. Then $H_1(M_X)$ is free abelian, generated by the meridian loops γ_i about the non-singular part of each irreducible component X_i, for $i = 1, \cdots, s$ (cf. [Di92], (4.1.3), (4.1.4)).

Since M_X is a n-dimensional affine variety, it has the homotopy type of a finite CW-complex of dimension n (e.g., see (cf. [Di92], (1.6.7), (1.6.8)). Hence

$$b_i^{(2)}(M_X, \alpha) = 0\ , \quad \text{for all} \quad i > n. \tag{4}$$

Let us now recall our notations. We start with an admissible epimorphism $\alpha : \pi_1(M_X) \to \Gamma$ to a group Γ, and consider the induced epimorphism $\widetilde{\alpha} : \pi_1(\widetilde{M}_X) \to \widetilde{\Gamma}$, where $\widetilde{\Gamma} := \mathrm{Ker}(\widetilde{\phi} : \Gamma \to \mathbb{Z})$, and $\widetilde{M}_X$ is the infinite cyclic cover of M_X defined by the total linking number homomorphism ϕ. As already noted in the introduction, ϕ coincides with the homomorphism $\pi_1(M_X) \to \pi_1(\mathbb{C}^*) = \mathbb{Z}$ induced by the polynomial map f (cf. [Di92, p.76-77]). The admissibility assumption implies that the Γ-cover of M_X defined by α factors through the infinite cyclic cover $\widetilde{M}_X$.

We begin our investigation of L^2–Betti numbers of hypersurface complements with the following special case:

Proposition 3.1. *Let X be an affine hypersurface defined by a weighted homogeneous polynomial $f : \mathbb{C}^n \to \mathbb{C}$. Then*

(1) *All L^2–Betti numbers $b_i^{(2)}(M_X, \alpha)$ of the complement M_X vanish.*
(2) *All L^2–Betti numbers $b_i^{(2)}(\widetilde{M}_X, \widetilde{\alpha})$ of the infinite cyclic cover $\widetilde{M}_X$ are finite, and $b_i^{(2)}(\widetilde{M}_X, \widetilde{\alpha}) = 0$ for $i \geq n$.*

Proof. Since the defining polynomial f of X is weighted homogeneous, there exist a global *Milnor fibration* (e.g., see [Mi68] or [Di92], (3.1.12)):

$$F = \{f = 1\} \hookrightarrow M_X = \mathbb{C}^n \setminus X \xrightarrow{f} \mathbb{C}^*.$$

Moreover, the fiber F has the homotopy type of a finite CW-complex of dimension $n-1$, and F is $(n-s-2)$-connected, where s is the dimension of the singular locus of the hypersurface singularity germ $(X, 0)$. In particular, since X is reduced, F is connected. The vanishing of L^2–Betti numbers $b_i^{(2)}(M_X, \alpha)$ of the complement follows now from Lemma 2.3.

Note the Milnor fiber F is homotopy equivalent to the infinite cyclic cover $\widetilde{M}_X$ of M_X corresponding to the kernel of the total linking number homomorphism. It follows that $\widetilde{M}_X$ has the homotopy type of a *finite* CW complex of dimension $n-1$, so the claim about the finiteness of the L^2–Betti numbers of $\widetilde{M}_X$ follows readily. □

Example. If X is a *central* hyperplane arrangement in $\mathbb{C}^n$ (i.e., all hyperplanes pass through the origin), then Prop.3.1 yields that all L^2–Betti numbers $b_i^{(2)}(M_X, \alpha)$ of the complement M_X vanish; this fact also follows from [DJL07].

Affine hypersurfaces defined by homogeneous polynomials are basic examples of *hypersurfaces in general position at infinity*, i.e., hypersurfaces $X \subset \mathbb{C}^n$ for which the hyperplane at infinity H is transversal in a stratified sense to the projective completion $\bar{X} \subset \mathbb{CP}^n$. In [Ma06], the author showed that for affine hypersurfaces X in general position at infinity the ordinary Betti numbers $b_i(\widetilde{M}_X)$ of the infinite cyclic cover $\widetilde{M}_X$ are finite for all $0 \leq i \leq n-1$. In this paper we give a non-commutative generalization of this fact. We begin with the following comparison result.

Theorem 3.2. *Let X be a hypersurface in $\mathbb{C}^n$, and let S^∞ be a $(2n-1)$-sphere in $\mathbb{C}^n$ of a sufficiently large radius (that is, the boundary of a small tubular neighborhood in $\mathbb{CP}^n$ of the hyperplane H at infinity). Denote by $X^\infty = S^\infty \cap X$ the link of X at infinity, and by $M_X^\infty = S^\infty - X^\infty$ its complement in S^∞. Let α^∞ be the composition map*

$$\pi_1(M_X^\infty) \to \pi_1(M_X) \to \Gamma.$$

Denote by $\widetilde{M}_X^\infty$ *the infinite cyclic cover of* M_X^∞ *defined by the composition*

$$\pi_1(M_X^\infty) \to \pi_1(M_X) \xrightarrow{\phi} \mathbb{Z},$$

and let $\widetilde{\alpha}^\infty : \pi_1(\widetilde{M}_X^\infty) \to \widetilde{\Gamma}$ *be the induced homomorphism to* $\widetilde{\Gamma} := Ker\{\widetilde{\phi} : \Gamma \to \mathbb{Z}\}$*. Finally, let* $b_i^{(2)}(M_X^\infty; \alpha^\infty)$ *and* $b_i^{(2)}(\widetilde{M}_X^\infty; \widetilde{\alpha}^\infty)$ *be the* L^2*–Betti numbers of* M_X^∞ *and* $\widetilde{M}_X^\infty$*, respectively.*

Then for all $i \leq n-1$ *we have the inequalities*

$$b_i^{(2)}(\widetilde{M}_X, \widetilde{\alpha}) \leq b_i^{(2)}(\widetilde{M}_X^\infty; \widetilde{\alpha}^\infty) \tag{5}$$

and

$$b_i^{(2)}(M_X, \alpha) \leq b_i^{(2)}(M_X^\infty; \alpha^\infty), \tag{6}$$

with equalities in (5) and (6) if $i < n-1$*.*

Proof. First, it is clear that α^∞ is an admissible map. Next, note that M_X^∞ is homotopy equivalent to $T(H) \setminus \bar{X} \cup H$, where $T(H)$ is the tubular neighborhood of H in $\mathbb{CP}^n$ for which S^∞ is the boundary. Then a classical argument based on the Lefschetz hyperplane theorem yields that the homomorphism $\pi_i(M_X^\infty) \to \pi_i(M_X)$ is an isomorphism for $i < n-1$ and it is surjective for $i = n-1$; see [DL06][Section 4.1] for more details. In particular, α^∞ is an epimorphism, as is the composite homomorphism $\pi_1(M_X^\infty) \to \mathbb{Z}$.

From the above considerations, it follows that

$$\pi_i(M_X, M_X^\infty) = 0 \ , \text{ for all } \ i \leq n-1, \tag{7}$$

hence M_X has the homotopy type of a complex obtained from M_X^∞ by adding cells of dimension $\geq n$. So the same is true for any covering, and in particular for the corresponding Γ-coverings. So the group homomorphisms

$$H_i((M_X^\infty)_\Gamma; \mathbb{Z}) \to H_i((M_X)_\Gamma; \mathbb{Z}) \tag{8}$$

are isomorphisms if $i < n-1$ and surjective for $i = n-1$. Since these homorphisms are induced by an embedding map, they are in fact homomorphisms of $\mathbb{Z}\Gamma$-modules. The (in)equalities in (6) follow now from Lemma 2.2.

Next note that the Γ-cover $(M_X)_\Gamma$ of M_X is a $\widetilde{\Gamma}$-cover of the infinite cyclic cover $\widetilde{M}_X$. Similar considerations apply to the covers of M_X^∞. So (8) can be restated as saying that the group homomorphisms

$$H_i((\widetilde{M}_X^\infty)_{\widetilde{\Gamma}}; \mathbb{Z}) \to H_i((\widetilde{M}_X)_{\widetilde{\Gamma}}; \mathbb{Z}) \tag{9}$$

are isomorphisms if $i < n-1$ and surjective for $i = n-1$. And, as before, these are in fact homomorphisms of $\mathbb{Z}\widetilde{\Gamma}$-modules. Another application of Lemma 2.2 yields the (in)equalities of (5).

□

In the next result, we restrict our attention to the case of hypersurfaces in general position at infinity. As it was already observed in a sequence of papers, e.g., see [DL06, DM07, Ma06], such hypersurfaces behave much like weighted homogeneous hypersurfaces up to homological degree $n-1$.

Theorem 3.3. *Assume that the affine hypersurface* $X \subset \mathbb{C}^n$ *is in general position at infinity, i.e., the hyperplane at infinity* H *is transversal in the stratified sense to the projective completion* $\bar{X}$*. Then*

(1) *The L^2–Betti numbers $b_i^{(2)}(\widetilde{M}_X, \widetilde{\alpha})$ of the infinite cyclic cover $\widetilde{M}_X$ are finite for all $0 \le i \le n-1$. In particular, the Cochran-Harvey higher-order degrees $\delta_{i,m}(X)$ are finite for $0 \le i \le n-1$ and all integers m.*

(2) *The L^2–Betti numbers of the complement M_X are computed by*

$$b_i^{(2)}(M_X, \alpha) = \begin{cases} 0, & \text{for } i \neq n, \\ (-1)^n \chi(M_X), & \text{for } i = n. \end{cases}$$

In particular,

$$(-1)^n \cdot \chi(M_X) \ge 0.$$

Proof. For the first part of the theorem, by Thm.3.2 it suffices to show that the L^2–Betti numbers $b_i^{(2)}(\widetilde{M}_X^\infty; \widetilde{\alpha}^\infty)$ are finite for all $0 \le i \le n-1$. (This was proved in [FLM09] for $n = 2$.) Note that since $\bar{X}$ is transversal to H, the space M_X^∞ is a circle fibration over $H \setminus \bar{X} \cap H$ which is homotopy equivalent to the complement in $\mathbb{C}^n$ to the affine cone over the projective hypersurface $\bar{X} \cap H \subset H = \mathbb{CP}^{n-1}$ (for a similar argument see [DL06], Section 4.1). Let $\{h = 0\}$ be the polynomial defining $\bar{X} \cap H$ in H. Then the infinite cyclic cover $\widetilde{M}_X^\infty$ of M_X^∞ is homotopy equivalent to the Milnor fiber $\{h = 1\}$ of the (homogeneous) hypersurface singularity at the origin defined by h. In particular, $\widetilde{M}_X^\infty$ has the homotopy type of a finite CW-complex. So the claim about the finiteness of $b_i^{(2)}(\widetilde{M}_X^\infty; \widetilde{\alpha}^\infty)$ follows now from definition. Similarly, the finiteness of the Cochran-Harvey higher-order degrees $\delta_{i,m}(X)$ in the relevant range follows from the considerations of Section 2.3, where these degrees are realized as L^2–Betti numbers of the infinite cyclic cover $\widetilde{M}_X$.

For the second part of the theorem, note that by the above considerations M_X^∞ is homotopic to the total space of a fibration over S^1, namely the Milnor fibration at the origin corresponding to the homogeneous polynomial h. So, by Lemma 2.3 we obtain that all L^2–Betti numbers $b_i^{(2)}(M_X^\infty; \alpha^\infty)$ vanish. The claim about the L^2–Betti numbers of M_X now follows from the inequalities (6) of Thm.3.2, together with Lem.2.1(3).

□

Remark. If Γ is a LITFA group as in Section 2.3, then the L^2–Betti numbers are determined by ranks of homology modules over skew-fields. In this case, the flatness of certain rings involved shows that the finiteness of the L^2–Betti number $b_i^{(2)}(\widetilde{M}_X, \widetilde{\alpha})$ of the infinite cyclic cover is equivalent to the vanishing of the L^2–Betti number $b_i^{(2)}(M_X, \alpha)$ of the complement.

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