

Forall Exists

The formal definition of the limit has a complex logical structure. For each of the following definitions, try to focus on the *structure* of the statement and the corresponding arguments rather than the specific content point.

1. A number system¹ forms a group under multiplication if and only if [**for all** a] [**there exists** b] such that $[a \times b = 1]$.
 - (a) Do the integers form a group under multiplication? Do the rationals?
 - (b) What does it take to refute a statement of this form - that is, what does it take to show a number system is not a group under multiplication?
 - (c) What does it take to *prove* that a number system meets this definition? Remember that an example is not a proof.
2. A number system is called dense if and only if [**for all pairs of numbers** $a < b$] [**there exists another number** c] such that $[a < c < b]$.
 - (a) Are the integers dense? That is, if integers are the only “numbers” available, is the preceding definition true? Are the rationals dense?
 - (b) What does it take to *refute* a statement of this form - that is, what does it take to show a number system is not dense?
 - (c) What does it take to *prove* that a number system meets this definition? Remember that an example is not a proof.
 - (d) If you haven't already, draw parallels between parts (b) and (c) in questions 1 and 2.
3. The function f has limit L at $x = a$ (in symbols, $\lim_{x \rightarrow a} f(x) = L$) if and only if [**for all** $\epsilon > 0$] [**there exists** $\delta > 0$] such that [*whenever* $|x - a| < \delta$ *then* $|f(x) - L| < \epsilon$].
 - (a) Rewrite the formal definition of the limit using English phrases instead of absolute values.
 - (b) Looking at the structure of this definition and your responses to questions 1 and 2, what does it take to refute this statement - that is, what does it take to show that L is not the limit of $f(x)$ as x approaches a ?
 - (c) What would it take to prove that f has limit L as x approaches a - that is, what would it take to show $\lim_{x \rightarrow a} f(x) = L$?

¹that happens to satisfy some other properties too