Problems 9

Please, do problems 1, 2, 4, 6, 8, 10, 11.

1. Consider the sequence defined recursively by

\[ x_1 = \sqrt{2} \]

\[ x_{n+1} = \sqrt{2 + x_n} \]

(a) Show by induction that \( x_n < 2 \), for all \( n \).

(b) Show by induction that \( x_n < x_{n+1} \), for all \( n \).

2. Suppose \( A, B \) are nonempty sets of real numbers, such that \( a \leq b \), for all \( a \in A, b \in B \).

(a) Prove \( \sup A \leq b \) for all \( b \in B \).

(b) Prove \( \sup A \leq \inf B \).

3. In the proof of Theorem 7.1, show that there is a largest \( x \) in \((a, b)\) with \( f(x) = 0 \).

4. Given \( f \) continuous, nonnegative on \([a, b]\), \( f(a) = f(b) = 0 \) and there exists an \( x_0 \), such that \( f(x_0) > 0 \).

Then there exist \( c, d \), such that

\[ a \leq c < x_0 < d \leq b \]

and

\[ f(c) = f(d) = 0 \]

and for all \( x \) in \((c, d)\): \( f(x) > 0 \).

5. Nested Interval Theorem:

Consider a sequence of closed intervals \( I_1 = [a_1, b_1], I_2 = [a_2, b_2], \ldots \), where

\[ a_1 \leq \ldots \leq a_n \leq \ldots \leq b_m \leq \ldots \leq b_1. \]

Prove that there is a point \( x \) which is in every \( I_n \).

Also show that this conclusion is false for open intervals.

6. Suppose \( f \) is continuous on \([a, b]\) and \( f(a) < 0, f(b) > 0 \), then either

(i) \( f\left(\frac{a+b}{2}\right) = 0 \)

(ii) \( f(a) < 0 < f\left(\frac{a+b}{2}\right) \) (\(*\))

(iii) \( f\left(\frac{a+b}{2}\right) < 0 < f(b) \)

Suppose \((*)\). Apply the same argument to \([a, \frac{a+b}{2}]\).

Claim: This process will lead to \( x \in [a, b] \) such that \( f(x) = 0 \).

7. Show: Every point in \([0,1]\) is an accumulation point.

8. Show: No point at all is an accumulation point of the set of natural numbers \( \mathbb{N} \).
9. Show: every point on the real line, both rational and irrational, is an accumulation point of the set \( \mathbb{Q} \).

10. Show: every point of a closed interval \([a, b]\) is an accumulation point of \((a, b)\). No point outside can be.

11. Find the sup, inf, maxima, minima (if they exist), accumulation points and limits superior and inferior, of:
   
   (a) \( A = \left\{ \frac{1}{n} \mid n \in \mathbb{N} \right\} \).
   
   (b) \( A = \{ x \mid 0 \leq x \leq \sqrt{2}, x \in \mathbb{Q} \} \).
   
   (c) \( A = \left\{ \frac{1}{n} + (-1)^n n \mid n \in \mathbb{N} \right\} \).

12. Are they uniformly continuous?
   
   (a) \( f(x) = \frac{1}{x} \) on \((0, 1)\)
   
   (b) \( f(x) = \frac{1}{x} \) on \((\frac{1}{n}, 1)\)
   
   (c) \( f(x) = \sin x \) on \(\mathbb{R}\).
   
   (d) \( f(x) = x^3 \) on \([-100, 100]\)
   
   (e) \( f(x) = x \)
   
   (f) \( f(x) = e^x \)
   
   (g) \( f(x) = e^{-x} \) on \(\mathbb{R}^+\)

13. Prove that if \( f \) is continuous and \( f(x) = 0 \) for all \( x \) in the dense set \( A \), then \( f(x) = 0 \) for all \( x \).

14. Suppose that \( f \) is a function such that \( f(a) \leq f(b) \) whenever \( a < b \).
   
   (a) Prove that \( \lim_{x \to a^-} f(x) \) and \( \lim_{x \to a^+} f(x) \) both exist.
   
   (b) Prove that \( f \) never has a removable discontinuity.