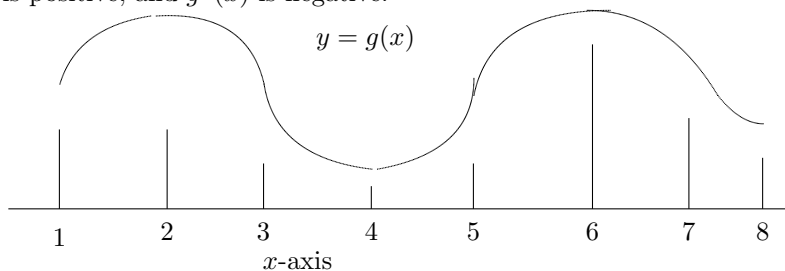


curve sketching

- Let $f(x) = 3x^5 + 5x^4$. Find the intervals where $f(x)$ is concave up and where it is concave down. Find the inflection points of the function.
- Let $f(x) = \frac{x}{1+x^2}$. Find the intervals where $f(x)$ is increasing and where it is decreasing. Find the critical points of $f(x)$ and indicate which are local maxima and which are local minima.
- Let $f(x) = x^3 + \frac{3}{x}$. On which intervals is $f(x)$ increasing and on which intervals is $f(x)$ decreasing? On which intervals is $f(x)$ concave up and on which intervals is $f(x)$ concave down? Does the function $f(x)$ have any horizontal or vertical asymptotes? If it does find them. Sketch the graph of $y = f(x)$.
- A particle moving in the plane has x and y coordinates given by the equations: $x = t^2 + t$, $y = 1 - t$. At time $t = 1$, find the slope of the tangent line to the path of the particle and determine whether the path is concave up or down at that time.
- The function $g(x)$ on the interval $(1,8)$ has both a first and second derivative at every point. Find the intervals where $g'(x)$ is positive, $g'(x)$ is negative, $g''(x)$ is positive, and $g''(x)$ is negative.

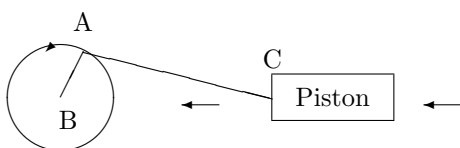


max-min

- A man has 112 miles of fencing for inclosing two separate lots, one of which is to be a square and the other a rectangle which is three times as long as it is wide. Find the dimensions of each lot so that the total area which is inclosed shall be a minimum.
- The intensity of illumination at any point is proportional to the product of the strength of the light source and the inverse square distance from the source. If two sources of strengths a and b are at a distance c apart, at what point on the line joining them will the intensity be a minimum? Assume the intensity at any point is the sum of intensities from the two sources.
- Find the area of the largest rectangle with lower base on the x -axis and upper vertices on the curve $y = 9 - x^2$.

related rates

9. A piston is being driven back and forth by a crankshaft which is turning (see diagram). The points A,B,C are swivel joints. The point B which is the axis of the crankshaft is held fixed while the point A rotates around counterclockwise exactly once per unit of time. The point C on the piston moves horizontally back and forth. The two connecting arms AB and AC are rigid and of length 1 unit and 3 units respectively. Find the speed the piston (point C) is moving when the point A is directly above the point B. What is the speed when the angle ABC is 45 degrees?



10. A particle is constrained to move along a parabola whose equations is $y = x^2$. At what point on the curve are the x and y coordinates changing at the same rate?

11. A 6 foot man walks away from a 10 foot high lamp at the rate of 3 ft per sec. How fast is the tip of his shadow moving?

limits

12. $\lim_{x \rightarrow 0^+} \frac{\sin(x^2)}{x^3}$

13. $\lim_{x \rightarrow \infty} x \sin(\frac{1}{x})$

14. $\lim_{x \rightarrow 0} \frac{\sqrt{x+1}-1}{\sqrt{x+4}-2}$

15. $\lim_{x \rightarrow 0} \frac{1}{x} (\frac{1}{\sqrt{1+x}} - 1)$.

16. $\lim_{y \rightarrow -\infty} \frac{y-1}{\sqrt{y^2-1}}$

a proof

17. Prove the first part of the fundamental theorem of calculus:

Suppose that $f(t)$ is a continuous function on the interval $[a,b]$ and $F(x) = \int_a^x f(t) dt$, then $F'(x) = f(x)$.

18. Evaluate $\frac{d}{dx} \int_x^{x^2} \sin(t^3) dt$

19. Evaluate $\int_2^3 \frac{d}{dt} (\sin(t^3)) dt$

20. $f(x) = \int_1^x \sqrt{1+t^2} dt$, $g(x) = x^2$, $h(x) = f(g(x))$, find $h'(x)$.

integrals

21. $\int \frac{1+y\sqrt{y}}{y^2} dy$

22. $F'(x) = \sqrt{1+x}$ and $F(0) = 1$.

23. $\int \cos^3(\theta) d\theta$

24. $\int \cos(4-2x) dx$

25. $g'(x) = \frac{1}{x^2\sqrt{1+\frac{1}{x}}}$

26. $\int \frac{y^3}{(y^2+1)^3} dy$

27. $\int (1+x^2)^{14} 2x dx$

28. $\int \cos(x)\sqrt{1+\sin(x)} dx$

29. $\int \frac{x dx}{\sqrt{1+x^2+\sqrt{(1+x^2)^3}}}$

30. Find the area under the curve $f(x) = \frac{x}{(x^2+1)^2}$, above the x -axis, and between the lines $x = 0$ and $x = 1$.

31. If $\int_0^1 f(x) dx = 1$ and $\int_0^2 f(x) dx = 3$, then what is $\int_1^2 (5-2f(x)) dx$?

32. $\int_{-1}^1 x f(x^2+1) dx$

33. Use the trapezoidal rule to approximate the integral $\int_0^1 \sin(x^2) dx$ using the step size $\Delta x = .25$. Find an estimate for the error.

34. Use Simpson's rule to approximate the integral $\int_0^1 \sin(x^2) dx$ using the step size $\Delta x = .25$.

True or False

35. Suppose $g'(x) = 0$ for all x , then g is a constant function.

36. Between any two zeros of a differentiable function lies a zero of its derivative.

37. If the derivative of a function is always positive then the function cannot have an inflection point.

38. If $f(x) = x^3 + x + 1$ then there must be a real number c with $0 < c < 1$ such that $f'(c) = 2$.

39. If $f(x) = x^3 + x + 1$ then there must be a real number c with $-1 < c < 1$ such that $f(c) = 0$.

40. Suppose $g'(x) > 0$ on an interval (a,b) then $g(x)$ is a decreasing function.
41. If an interval contains three distinct zeros of a twice differentiable function then it also contains a zero of the function's second derivative.
42. A continuous function f on an interval $[a,b]$ must achieve its extreme values (minimum and maximum).
43. If $f(c)$ is the maximum value of a differentiable function f , then $f'(c) = 0$.
44. Suppose $f(x)$ is a continuous function on $[a,b]$ such that for every x in $[a,b]$, $f(x)$ is in $[a,b]$. Then for some c in $[a,b]$, $f(c)=c$.

From now on assume f , g , and f_k are continuous functions on the interval $[a,b]$.

45. $\int_a^b f(x)g(x)dx = \int_a^b f(x)dx \int_a^b g(x)dx$
46. $\int_a^b [c_1f(x) + c_2g(x)]dx = c_1 \int_a^b f(x)dx + c_2 \int_a^b g(x)dx$ where c_1, c_2 are constants.
47. $\int_a^b [\sum_{k=1}^n c_k f_k(x)]dx = \sum_{k=1}^n [c_k \int_a^b f_k(x)dx]$ where c_1, \dots, c_n are constants.
48. $\int_a^b [f(x)]^n dx = [\int_a^b f(x)dx]^n$
49. $\int_a^b \sqrt{f(x)}dx = \sqrt{\int_a^b f(x)dx}$
50. if $a < c < b$, then $\int_a^c f(x)dx = \int_a^b f(x)dx - \int_c^b f(x)dx$
51. if $a < c < b$, then $\int_a^c f(x)dx \leq \int_a^b f(x)dx$
52. Suppose for every x in $[a,b]$, $f(x) \leq g(x)$, then $\int_a^b f(x)dx \leq \int_a^b g(x)dx$
53. Suppose for every x in $[a,b]$, $f(x) \leq g(x)$, then for every x in $[a,b]$, $f'(x) \leq g'(x)$.

Exam from April 1989

1. Consider the function $y = x^3 - 3x^2 + 4$.
What are the critical points? Determine whether they are local mins, local maxs, or saddle points.
What are the intervals where the function is increasing?
What are the intervals where the function is decreasing?
What are the inflection points?
What are the intervals where the function is concave up?
What are the intervals where the function is concave down?
Are there any vertical or horizontal asymptotes?
Sketch the graph of the function.
2. You are about to make a one-quart oil can shaped like a right circular cylinder. What dimensions will use the least material? (Note that one quart is approximately 57.75 cubic inches.)
3. A baseball diamond is a square 90 ft on a side. A player runs from third base to home at a rate of 20 ft/sec. At what rate is the player's distance from second base increasing when the player is 10 ft from home base?
4. Find the limit
$$\lim_{x \rightarrow 0} \frac{x(\cos(x) - 1)}{\sin(2x) - 2x}$$
5. Evaluate the following indefinite integrals:
 - a. $\int \frac{\sin(\sqrt{x})}{\sqrt{x}} dx$
 - b. $\int (r^2 - 1)^7 r dr$
 - c. $\int \tan^2(\theta) d\theta$
 - d. $\int \cos^2(\theta) d\theta$
6. Prove the first part of the fundamental theorem of calculus:
Suppose that $f(t)$ is a continuous function on the interval $[a, b]$ and $F(x) = \int_a^x f(t) dt$, then $F'(x) = f(x)$.
7. State the second part of the fundamental theorem of calculus, but do not prove it.
8. Evaluate the definite integral $\int_1^3 \frac{dx}{(3x-1)^2}$.
9. Approximate the integral $\int_1^3 \frac{1}{x} dx$ with the $n = 4$ steps using the trapezoidal rule. Estimate the size of n needed to approximate this integral using the trapezoidal rule with an error of absolute value less than 10^{-2} .

Answers

1. $(-\infty, -1)$ concave down, $(-1, \infty)$ concave up. -1 is the only inflection point.
2. increasing on $(-1, 1)$, decreasing on $(-\infty, -1)$ and on $(1, \infty)$, -1 is a local min and 1 is a local max.
3. decreasing on $(-1, 0)$ and $(0, 1)$, increasing on $(-\infty, -1)$ and $(1, \infty)$, concave up on $(0, \infty)$ concave down on $(-\infty, 0)$, vertical asymptote at $x = 0$, no horizontal asymptotes.
4. slope is $-\frac{1}{3}$, second derivative is $\frac{2}{27}$ hence curve is concave up.
5. $g'(x)$ is positive: $(1, 2), (4, 6)$, $g'(x)$ is negative: $(2, 4), (6, 8)$, $g''(x)$ is positive: $(3, 5), (7, 8)$, and $g''(x)$ is negative: $(1, 3), (5, 7)$.
6. 12×12 and 8×24
7. $c(\frac{a^{1/3}}{a^{1/3}+b^{1/3}})$ units from the source of strength a .
8. $12\sqrt{3}$
9. -2π and $-\pi(\sqrt{2} + \frac{1}{\sqrt{8.5}})$.
10. $(\frac{1}{2}, \frac{1}{4})$
11. $\frac{15}{2}$ ft per sec.
12. ∞
13. 1
14. 2
15. $-\frac{1}{2}$
16. 1
17. Does the word "fundamental" make you think this theorem might be important?
18. $(\sin(x^6))(2x) - \sin(x^3)$
19. $\sin(27) - \sin(8)$
20. $2x\sqrt{1+x^4}$
21. $-\frac{1}{y} + 2\sqrt{y} + C$
22. $F(x) = \frac{2}{3}(1+x)^{3/2} + \frac{1}{3}$
23. $\sin(\theta) - \frac{\sin^3(\theta)}{3} + C$
24. $-\frac{1}{2}\sin(4-2x) + C$
25. $g(x) = -2\sqrt{1+\frac{1}{x}} + C$
26. $\frac{1}{4}(\frac{1}{(y^2+1)^2}) - \frac{1}{2}(\frac{1}{y^2+1}) + C$
27. $\frac{(1+x^2)^{15}}{15} + C$
28. $\frac{2}{3}(1+\sin(x))^{\frac{3}{2}} + C$
29. $2\sqrt{1+\sqrt{1+x^2}} + C$ using substitution $u^2 = 1+x^2$.
30. $\int_0^1 \frac{x}{(x^2+1)^2} dx = \frac{1}{4}$
31. 1
32. 0

33. Trapezoid rule gives .315975. $f''(x) = 2 \cos(x^2) - 4x^2 \sin(x^2)$. Since cosine and sine in absolute value never exceed 1, and $|x|$ and $|x^2|$ are bounded by 1 on the interval $[0, 1]$, we can estimate that on the interval $[0, 1]$:

$$\begin{aligned} |f''(x)| &= |2 \cos(x^2) - 4x^2 \sin(x^2)| \leq |2 \cos(x^2)| + |4x^2 \sin(x^2)| \\ &\leq |2| |\cos(x^2)| + 4|x^2| |\sin(x^2)| \leq 2 + 4 = 6 \end{aligned}$$

Consequently the error in the trapezoidal rule is bounded by $\frac{1}{32} = .03125$ and the true value of this integral is between .284725 and .347225.

34. .309944

35. True, since by the mean value theorem if there exists a and b where $g(a) \neq g(b)$ then for some c in between a and b , $g'(c) = \frac{g(b)-g(a)}{b-a}$, and hence $g'(c) \neq 0$.

36. True, this is Rolle's Theorem.

37. False, consider the function $f(x) = x^3 + x$.

38. True, by the mean value theorem and the fact that $\frac{f(1)-f(0)}{1-0} = 2$

39. True, by the intermediate value theorem and the fact that $f(-1)$ is negative and $f(1)$ is positive.

40. False, consider the function $g(x) = x$ on the interval $(1, 2)$.

41. True, between the first two zeros of $f(x)$ and between the second and third zero lies a zero of $f'(x)$ by Rolle's Theorem. Applying Rolle's Theorem to $f'(x)$ and its two zeros we see that its derivative namely $f''(x)$ must have a zero.

42. True

43. True

44. True, apply the intermediate value theorem to the function $g(x) = f(x) - x$.

45. False, let $f(x) = g(x) = 1$ and $(a, b) = (0, 2)$ then $2 \neq 4$.

46. True

47. True

48. False, let $f(x) = 1$, $(a, b) = (0, 2)$, and $n = 2$ then $2 \neq 4$.

49. False, let $f(x) = 1$ and $(a, b) = (0, 2)$ then $2 \neq \sqrt{2}$.

50. True

51. False, consider $f(x) = -1$, $a = -1$, $c = 0$, $b = 1$

52. True

53. False, let $f(x) = x$, $g(x) = 100 - x$, and $[a, b] = [0, 1]$.

Answers to Exam from April 89

1. Local max at $x = 0$ and local min at $x = 2$. Increasing on $(-\infty, 0)$ and $(2, \infty)$. Decreasing on $(0, 2)$. Inflection point at $x = 1$. Concave up on $(1, \infty)$. Concave down on $(-\infty, 1)$. No vertical or horizontal asymptotes.
2. radius = 2.1 in and height = 4.2 in (rounded off)
3. 13.2872 ft per sec
4. $3/8$
5. a. $-2 \cos(\sqrt{x}) + C$ b. $(r^2 - 1)^8/16 + C$ c. $\tan(\theta) - \theta + C$ d. $\sin(2\theta)/4 + \theta/2 + C$
8. $1/8$
9. trap = $67/60 = 1.1167$. $n \geq 10\sqrt{2/3}$.