Show all work. Circle your answers.

Hand in to your instructor. Circle your instructor’s name:

Tim Daniel    Michael Iltis    Arnold Miller

Name ________________________________

Score
1. ____ (15)
2. ____ (15)
3. ____ (14)
4. ____ (14)
5. ____ (14)
6. ____ (14)
7. ____ (14)

Total ____ (100)
1. A sample of radioactive iodine has a half life of 20 hours. If initially four nanograms (ng) are ingested by a patient, how long is it until .25 nanograms remain?
2. Find the solution of

\[ \frac{dy}{dt} = t(y + 1)^2 \quad (0, 1) \]

and determine the interval on which it is valid.
3. Find the orthogonal trajectories of the family of curves $y = C \tan(x)$. 
4. A tank contains 5 gallons of water in which 1 pound of solvent is dissolved. Water containing .5 pounds of solvent per gallon enters at a rate of 3 gallons per minute. The mixture is kept well-stirred and leaves the tank at the rate of 2 gallons per minute. Find the amount of solvent in the tank at time $t$. 
5. Consider the problem of numerically solving the differential equation $y' = t + yt$ with initial point $(t_0, y_0) = (0, 1)$ and step size $h = .1$. What is $t_3$? What is $y_3$ if you use Euler’s method? What is $y_3$ if you use the modified Euler’s method (use Euler’s method on the first step)?
6. Does the Existence-Uniqueness Theorem guarantee that \( y' = (1 - y^2)^{1/3} \) has a unique solution through \((0, 1)\)? Why or why not?
7. Let \( p(t) \) and \( q(t) \) be functions which are continuous for all real numbers \( t \).
Let \((t_0, y_0)\) be any fixed point in the plane. Define \( \mu(t) \) and \( \phi(t) \) as follows:

\[
\mu(t) = e^{\int_{t_0}^{t} p(s) ds} \\
\phi(t) = \frac{1}{\mu(t)} \left[ \int_{t_0}^{t} \mu(s) q(s) ds + y_0 \right]
\]

Show that the function \( \phi(t) \) is a solution to the initial value problem

\[
y' + py = q, \quad (t_0, y_0)
\]

Determine its interval of validity.
median score: 80

1. A sample of radioactive iodine has a half life of 20 hours. If initially four nanograms (ng) are ingested by a patient, how long is it until .25 nanograms remain?

\[ I'(t) = kI \text{ hence } I(t) = I_0e^{kt}. \]

\[ I_0 = 4 \text{ and } I(20) = 2. \]

\[ 2 = I(20) = 4e^{k20} \]

\[ e^{k20} = 1/2 \text{ hence } e^k = (1/2)^{1/20} \text{ hence } I(t) = 4(1/2)^{t/20} \]

so .25 = 1/4 = 4(1/2)^{t/20} and 1/16 = (1/2)^{t/20} so t/20 = 4 and t = 80.

2. Find the solution of

\[ \frac{dy}{dt} = t(y + 1)^2 \quad (0, 1) \]

and determine the interval on which it is valid.

\[ \frac{dy}{(y + 1)^2} = t \ dt \]

\[ -\frac{1}{y + 1} = \frac{t^2}{2} + C \]

\[ -\frac{1}{2} = 0 + C \]

\[ -\frac{1}{y + 1} = \frac{t^2}{2} + -\frac{1}{2} \]

\[ -\frac{2}{t^2 - 1} - 1 = y \]

The valid interval is \(-1 < t < 1\).

3. Find the orthogonal trajectories of the family of curves \(y = C \tan(x)\).

\[ \frac{dy}{dx} = C \sec^2(x) = \frac{y \sec^2(x)}{\tan(x)} = y \frac{1/\cos^2(x)}{\sin(x)/\cos(x)} = \frac{y}{\sin(x) \cos(x)} \]

since \(C = \frac{y}{\tan(x)}\). Hence orthogonal trajectories satisfy:

\[ \frac{dy}{dx} = \frac{-\sin(x) \cos(x)}{y} \]
\[ y \, dy = -\sin(x) \cos(x) \, dx \]

substituting \( u = \cos(x) \) and \( du = -\sin(x) \, dx \) we get:

\[ y^2/2 = \cos^2(x)/2 + C \]

\[ y = \pm \sqrt{\cos^2(x) + C} \]

4. A tank contains 5 gallons of water in which 1 pound of solvent is dissolved. Water containing .5 pounds of solvent per gallon enters at a rate of 3 gallons per minute. The mixture is kept well-stirred and leaves the tank at the rate of 2 gallons per minute. Find the amount of solvent in the tank at time \( t \).

\[ S(t) = \text{lb of solvent in the tank at time } t. \quad S' = \text{rate in} - \text{rate out}. \]
\[ \text{rate in} = (0.5 \text{lb/gal})(3 \text{gal/min}) = 1.5 \text{lb/min}. \]
\[ \text{rate out} = (S(t) \text{lb}/V(t) \text{gal})(2 \text{gal/min}) \text{ where } V(t) \text{ is the number of gallons of mixture in the tank at time } t. \]

Noting that \( V(0) = 5 \) and \( V \) increases one gallon per minute, we see that \( V(t) = 5 + t. \) The initial value problem to solve is:

\[ S' = 1.5 - \frac{S}{t+5}^2 \quad S(0) = 1 \]

\[ S' + \frac{S}{t+5}^2 = 1.5 \]

\[ \mu(t) = e^{2\ln(t+5)} = (t + 5)^2 \]

\[ S(t) = \frac{1}{(t+5)^2} \left( \int 1.5(t+5)^2 \, dt + C \right) \]

\[ S(t) = \frac{1}{2} (t + 5) + \frac{C}{(t + 5)^2} \]

\( S(0) = 1 \) gives \( C = \frac{-75}{2} \)

5. Consider the problem of numerically solving the differential equation \( y' = t + yt \) with initial point \( (t_0, y_0) = (0, 1) \) and step size \( h = .1. \) What is \( t_3? \) What is \( y_3 \) if you use Euler’s method? What is \( y_3 \) if you use the modified Euler’s method (use Euler’s method on the first step)?

\[ t_0 = 0, t_1 = .1, t_2 = .2, t_3 = .3 \]

Euler method:

\[ y_{n+1} = y_n + h f(t_n, y_n) = y_n + .1(t_n + y_n t_n) \]
\[ y_{n+1} = y_n + .1(t_n + y_n t_n) \]
\[ y_0 = 1 \]
\[ y_1 = y_0 + .1(t_0 + y_0 t_0) = 1 + .1(0 + 1 \times 0) = 1 \]
\[ y_2 = y_1 + .1(t_1 + y_1 t_1) = 1 + .1(1 + 1 \times 1) = 1.02 \]
\[ y_3 = y_2 + .1(t_2 + y_2 t_2) = 1.02 + .1(2 + 2 \times 1.02) = 1.0604 \]

Modified Euler method:
\[ y_{n+1} = y_n - 1 + 2hf(t_n, y_n) = y_n - 1 + .2(t_n + y_n t_n) \]
\[ y_0 = 1 \]
\[ y_1 = 1 \]
\[ y_2 = y_0 + .2(t_1 + y_1 t_1) = 1 + .2(1 + 1 \times 1) = 1.04 \]
\[ y_3 = y_1 + .2(t_2 + y_2 t_2) = 1 + .2(2 + 1.04 \times 2) = 1.0816 \]

6. Does the Existence-Uniqueness Theorem guarantee that \( y' = (1 - y^2)^{1/3} \) has a unique solution through (0, 1)? Why or why not?

Let \( f(t, y) = (1 - y^2)^{1/3} \), then \( f(t, y) \) is continuous in the whole plane. However
\[ f_y(t, y) = \frac{-2y}{(1-y^2)^{2/3}} \]
which is not continuous at any point \((t, y)\) with \( y = \pm 1 \). Therefore the Existence-Uniqueness Theorem does not apply to the point \((0, 1)\) and does not guarantee that \( y' = (1 - y^2)^{1/3} \) has a unique solution thru this point.

7. Let \( p(t) \) and \( q(t) \) be functions which are continuous for all real numbers \( t \). Let \((t_0, y_0)\) be any fixed point in the plane. Define \( \mu(t) \) and \( \phi(t) \) as follows:
\[ \mu(t) = e^{ \int_{t_0}^{t} p(s) ds } \]
\[ \phi(t) = \frac{1}{\mu(t)} \left[ \int_{t_0}^{t} \mu(s)q(s) ds + y_0 \right] \]

Show that the function \( \phi(t) \) is a solution to the initial value problem
\[ y' + py = q, \ (t_0, y_0) \]

Determine its interval of validity.
\[ \mu(t_0) = e^{\int_{t_0}^{t} p(s) ds} = e^0 = 1 \]
\[ \phi(t_0) = \frac{1}{\mu(t_0)} \left[ \int_{t_0}^{t_0} \mu(s)q(s) ds + y_0 \right] = \frac{1}{1}[0 + y_0] = y_0 \]

Hence \( \phi(t_0) = y_0 \). Using the chain rule:

\[ \mu' = e^{\int_{t_0}^{t} p(s) ds} \frac{d}{dt} \int_{t_0}^{t} p(s) ds \]

By the fundamental theorem of calculus:

\[ \frac{d}{dt} \int_{t_0}^{t} p(s) ds = p(t) \]

Hence

\[ \mu' = \mu p \]

Using the product rule, and the definition of \( \phi \):

\[ \mu' \phi + \mu \phi' = (\mu \phi)' = \frac{d}{dt} \left[ \int_{t_0}^{t} \mu(s)q(s) ds + y_0 \right] \]

Again by the fundamental theorem of calculus and since the derivative a constant is zero:

\[ \frac{d}{dt} \left[ \int_{t_0}^{t} \mu(s)q(s) ds + y_0 \right] = \mu(t)q(t) \]

Hence

\[ \mu' \phi + \mu \phi' = \mu q \]

substituting \( \mu' = \mu p \)

\[ \mu p \phi + \mu \phi' = \mu q \]

Noting that \( \mu \) is an exponential function we see that for all \( t \) \( \mu(t) \neq 0 \) so we can divide it out and get:

\[ p \phi + \phi' = q \]

as was to be shown. Since \( \mu \) is never zero and since \( p \) and \( q \) are everywhere continuous we see that the valid interval is \( -\infty < t < \infty \).