Make your own practice exam. Choose three easy, three medium, and three hard. Don’t forget the assigned problems from the text.

Easy

1. Find the maximum and minimum of \( f(x, y) = 2x + y \) subject to the the constraint \( x^2 + 2y^2 = 2 \).

2. Given \( z = x^2 - xy + y^2 + x \) the point \((-2, -1)\) is:
   a. not a critical point b. a local maximum c. a local minimum d. a saddle point e. none of above

3. Let \( c(t) = x(t)i + y(t)j \) and let \( v(t) = (dx/dt)i + (dy/dt)j \). Suppose that for every \( t \) we have \( g(x(t), y(t)) = 0 \) for some curve given by \( g \). What will be the angle between \( \nabla g \) and the vectors \( v(t) \)? Verify this for \( g(x, y) = x^2 + y^2 - 1 \) and \( c(t) = \cos(t)i + \sin(t)j \).

4. Explain what is meant by a function \( f(x, y) \). What does the graph of \( f(x, y) \) look like? How is \( \partial f/\partial x \) and \( \partial f/\partial y \) defined at a point. What is \( \nabla f \)? What is the physical significance of \( \nabla f \)? If \( u \) is a unit vector what is \( D_u f \). How is it computed in practice? Answer the same questions for a function of three variables.

5. Find the equation of the tangent plane to the surface \( x^2 + 2xy - z^2 = -3 \) at the point \((1, 0, 2)\).

6. If \( x^2 + 2y^2 + 2z^2 = 150 \) what is the largest possible value of \( xyz \)?

7. The function \( w = x^2 + 2y^2 + 3z^2 \) has a minimum value on the plane \( x + 3y + 3z = 8 \). A minimum necessarily exists because always \( w \geq 0 \) and \( w \) gets large if any of \( x, y, z \) gets large. Find the point or points where the minimum occurs.

8. Does there exist a function \( w = f(x, y) \) such that \( \partial w/\partial x = x(y^2e^{x^2} + e^{y^2} + 2) \) and \( \partial w/\partial y = y(x^2e^{y^2} + e^{x^2} - 2) \)? If yes, find such a function and if no, explain why not.

9. Let \( z = y^2 - xe^{xy} \). (a) compute \( \nabla z \). (b) Find the directional derivative of \( z \) at the point \((0, 2)\) in the direction toward \((5, -10)\). I move in the plane and so I see changing values of \( z \). At time \( t \), I am at \( x = e^t - 1, y = t^2 + 2t + 2 \). At what rate do I see \( z \) changing when \( t = 0? \)

10. The point \((3, 2, 1)\) lies on the surface \( xyz = x^3 - y^3 - z^3 - 12 \). Find the equation of the plane tangent to this surface at \((3, 2, 1)\).

11. Find and classify the critical points of \( z = x^3 + y^3 - 3xy \) and \( z = (x - y + 1)^2 \).

12. An equation of the surface of a mountain is \( z = 1200 - 3x^2 - 2y^2 \). A mountain climber is at \((-10, 5, 850)\). What is the direction of steepest ascent?

Medium

13. Let \( z = f(x, y), x = \cos(t), \) and \( y = \sin(2t) \). Given that \( \nabla f|_{(0,0)} = (4, -3) \) the value of \( \partial f/\partial x \) when \( t = \pi/2 \) must be:
   a. -10 b. 2 c. 6 d. -4 e. (1,0) f. (0,0) g. none of above

14. Let \( f(x, y) = (\ln(y))^{\ln(x)} \). Find \( \partial f/\partial y =? \partial f/\partial x =? \partial^2 f/\partial y \partial x =? \)
15. Explain why the Lagrange multiplier method works, in particular why does looking for a solution to \( \nabla f = \lambda \nabla g \) lead to optimizing \( f(x, y) \) subject to the constraint \( g(x, y) = 0 \)?

16. Suppose \( w = 3xy + 2yz \) and \( P \) is the point \( x = 1, y = -1, z = 1 \). Find \( \nabla w \) at \( P \). Along which unit vector leaving \( P \) does \( w \) decrease most rapidly? Find the directional derivative of \( w \) at \( P \) in the direction from \( P \) to the origin. The equations \( x = \cos(t), y = -1 + t, z = e^t \) for \(-\infty < t < \infty \) define a curve passing through the point \( P \). Find the directional derivative of \( w \) along the curve at \( P \).

17. Let \( w = xyz + x^3 \). Compute the directional derivative of \( w \) at the point \((2, -6, 3)\) in the direction toward the origin. Give the coordinates of a point with the property that the directional derivative of \( w \) at \((2, -6, 3)\) in the direction toward that point is as large as possible. Find the equation of the plane tangent to the level surface of \( w \) at the point \((2, -6, 3)\). View \( w, x, y \) as independent variables and \( z \) as dependent. Find the \( \partial z / \partial x \).

18. Let \( f(u, v) \) have partial derivatives \( f_u \) and \( f_v \). Define a new function \( g \) by \( g(x, y, z) = f(x + yz, y + x^2) \). Given that \( f_u(7, 3) = -2 \) and \( f_v(7, 3) = -5 \) compute \( g_y(1, 2, 3) \).

19. At the point \((1, 2)\) the directional derivative of the function \( z = f(x, y) \) in the direction toward the point \((5, -1)\) is \( 2 \). Also, \( f_z = 1 \) at the point \((1, 2)\). Find \( \nabla w \) at \((1, 2)\).

20. I have a function \( w = f(x, y) \). Of the following three functions, one is \( \partial w / \partial x \) and one is \( \partial w / \partial y \). Identify them. The functions are: \( e^x + e^y, e^x + y, \) and \( xe^y + y \).

21. The temperature at point \((x, y, z)\) of the ball \( x^2 + y^2 + z^2 \leq 3 \) is given by \( T = xy + z \). Find the hottest and coldest points of the ball.

22. Let \( w = xy - yz^2 - z \). At the point \( x = -1, y = -1, \) and \( z = -1 \), we see that \( w = 3 \). For which of the three independent variables will a small error in measurement have the greatest effect on \( w \)? For which will the effect be least? Explain.

23. Find the directional derivative of \( f(x, y, z) = x^2 + y^2 - z^2 \) at \((3, 4, 5)\) in the direction along the curve of intersection of the two surfaces \( 2x^2 + 2y^2 - z^2 = 25 \) and \( x^2 + y^2 = z^2 \).

24. A function \( f(x, y) \) has at the point \((1, 2)\) directional derivatives \(+2\) in the direction toward \((2, 2)\) and \(-2\) in the direction toward \((1, 1)\). Find \( \nabla f \rvert_{(1,2)} \) and compute the directional derivative in the direction toward \((4, 6)\).

**Hard**

25. Let \( f(X) = ||X||^4 \) where \( X \) is a vector in three dimensional space. Find the directional derivative of \( f \) at the point \((3, 4, 5)\) in either of the two directions along the curve of intersection of the two surfaces \( x^2 + y^2 = z^2 \) and \( -x^2 + y + z^2 = 2 \).

26. Does the \[ \lim_{(x, y) \to (0, 0)} \frac{x^2 y}{x^4 + y^2} = 0? \]

Explain why or why not.

27. Can a function \( z = f(x, y) \) have partial derivatives at every point but not be continuous?

28. Let \( f = f(x, y), \) \( R \) any bounded region in the \( xy \)-plane on which \( f \) is continuous and differentiable, and let \( M = \max D_u f \) over the region \( R \) and any direction \( u \). Can \( M < 0? \) Can \( M = 0? \) Explain.

29. (Challenge your TA with this one.) Find a second derivative test for \( w = f(x, y, z) \) which uses \( f_{xx}, f_{yy}, f_{zz}, f_{xy}, f_{xz}, f_{yz} \).

30. I had a function \( f(x, y) \) and I computed its partial derivatives \( f_x \) and \( f_y \). All I remember now is that \( f_x \) is either \((y + 1)e^x \) or \( e^x \) or \( ye^x + y \) and \( f_y \) is either \( 2ye^x + e^y \) or \( ye^x \) or \((x + 1)e^y \). I also recall that \( f(0, 0) = 1 \). What is my function?

31. Find the maximum and minimum values of the function \( w = x + 2y + 2z \) on that part of the surface \( z = x^2 + y^2 \) where \( z \leq 1 \).

32. Given: \( xy^2 + xz + yz^2 = 25 \). This defines \( z \) as a function \( z = f(x, y) \) with \( f(1, 2) = 3 \). Compute the partial derivative \( f_x(1, 2) \).
33. The point \((1, 2, 3)\) lies on the surfaces \(xyz = 6\) and \(x^2 + y^2 + z^2 = 14\). Find equations for the line through \((1, 2, 3)\) which is tangent to both surfaces.

34. The temperature at the point \((x, y, z)\) in space is given by the function \(w = f(x, y, z)\). At the point \((3, 2, 1)\) the value of \(\nabla w\) is \(i - 2j - k\). (a) Compute the directional derivative of \(w\) at \((3, 2, 1)\) in the direction toward the point \((1, 3, 3)\). (b) Find the maximum and minimum values the directional derivative of \(w\) can have at \((3, 2, 1)\). (c) Given \(f(3, 2, 1) = 5\) find a reasonable approximation for \(f(3.01, 1.98, 1.03)\). A particle moves through space according to \(x = t^3 + 2, y = 3 - t^2, z = 2t - 1\), where \(t\) is time. Note that at \(t = 1\), the particle is at \((3, 2, 1)\). On the particle there is a thermometer measuring the temperature of space at the location of the particle. (d) Compute the rate of change of the thermometer reading with respect to time at \(t = 1\). The equation \(f(x, y, z) = 5\) is viewed as defining \(z\) implicitly as a function of \(x\) and \(y\). Write \(z = g(x, y)\) so that \(g(3, 2) = 1\) since \(f(3, 2, 1) = 5\). (e) Compute \(g_y(3, 2)\).

35. The function \(w = x^2 + y - yx\) is defined on the area bounded by the curve \(y = 9 - x^2\) and the \(x\)-axis. Find the maximum and minimum values of \(w\) on this area and the points where they occur.

36. Let \(f(u, v)\) have partial derivatives \(f_u\) and \(f_v\). Suppose \(f_u(1, 1) = 1\) and \(f_v(1, 1) = 2\). We define a new function \(g(x, y, z)\) by \(g(x, y, z) = f(x/y, y/z)\). Compute \(g_y(1, 1, 1)\).

37. Given functions \(f(r, s)\) and \(g(x, y)\), we write \(w = f(y^2, g(x, y))\). Given the following data, compute \(\frac{\partial w}{\partial x}\) and \(\frac{\partial w}{\partial y}\) at \(x = 1\) and \(y = 2\).

\[
\begin{align*}
g(1, 2) &= 3, g_x(1, 2) = 4, g_y(1, 2) = 5, f(4, 3) = 6, f_r(4, 3) = 7, f_s(4, 3) = 2, f(1, 2) = 1, f_r(1, 2) = 3, f_s(1, 2) = 9
\end{align*}
\]

38. Let \(g(x, t) = \frac{x}{\sqrt{kt}}\) and \(f(x, t) = \int_0^{g(x,t)} e^{-u^2} du\). Show

\[
k\frac{\partial^2 f}{\partial x^2} = \frac{\partial f}{\partial t}
\]

Hint: Use the chain rule.

39. Find the path of a heat seeking particle starting at the point \((1, 2)\) if the temperature at \((x, y)\) is given by \(T(x, y) = x - y^2 + 2\).
Answers

2. a
8. \( w = \frac{1}{2}y^2e^{x^2} + \frac{x^2}{2}e^y + x^2 - y^2 \)
10. \( 25x + 9y - 3z = 90 \)
14. Use that \( f(x, y) = e^{\ln(x)\ln(\ln(y))} \)
23. 0
26. No, consider the curve \( y = x^2 \) and the \( x \)-axis.
32. \(-\frac{7}{13}\)
35. \( (\pm \sqrt{3}, 6), -1.39, 19.39 \)
39. \( y = 2e^{2-2x} \)