Make your own practice exam, pick one from each section.

6.9

1. In some chemical reactions the rate at which the amount of a substance changes with time is proportional to the amount present. For the change of δ-glucono lactone into gluconic acid

\[ \frac{dy}{dt} = -0.6y \]

(t is measured in hours and y in grams.) If there are 100 grams of δ-glucono lactone present when \( t = 0 \) how many grams will be left after the first hour?

2. a) If λ is the radioactive decay constant for \( C^{14} \), write the differential equation for the radioactive decay of \( C^{14} \). \( C^{14} \) has a half life of 5700 years
   
   b) solve the differential equation obtained in part (a) where \( y_0 \) is the initial amount of \( C^{14} \).
   
   c) In charcoal from a beam found at an archeological site, the \( C^{14} \) decays at a rate of 4.09 disintegrations per minute per gram of carbon, and in living wood the rate is 6.68 dpmg. Estimate the age of the beam.

1.2

3. Solve the initial value problem

\[ y' + (1 + y^2)\cos(t) = 0, \quad y(0) = 1. \]

4. Solve

\[ x^2(y^3 + 1) \, dx + y^2\sqrt{x^3 + 3} \, dy = 0; \quad y(1) = 0 \]

5. Find all solutions

\[ \frac{dy}{dt} = t(y + 1) \]

1.3

6. Find the path of a heat seeking particle starting at the point (1, 2) if the temperature at (x, y) is given by

\[ T(x, y) = x - y^2 + 2 \]

Hint: orthogonal trajectories to the contour curves of \( T \).

7. Find the orthogonal trajectories of the family of curves

\[ y = C\sin(x). \]

8. Compute orthogonal trajectories for the family

\[ y + (1/3)y^3 + (1/2)x^2\ln x - (1/4)x^2 = C \]

1.4

9. Solve the

\[ ty' + (1 + t)y = te^t \quad y(1/2) = \sqrt{e}. \]

10. Find solution of

\[ y' + 2y = 2 \] thru initial point \((0,1/2)\) and determine the interval on which it is valid.
11. Solve
\[ x \frac{dy}{dx} + 3y = \left( \frac{\pi/2}{x^2} \cos\left( \frac{\pi/2}{x} \right) \right); \quad y(1) = 2 \]

Give the interval of the solution.

12. Let
\[ \mu(t) = e^{\int_0^t e^s \, ds} \]
and let
\[ \phi(t) = \frac{1}{\mu(t)} \left[ \int_0^t \mu(s) s \, ds + 1 \right] \]

Show that the function \( \phi(t) \) is a solution to the initial value problem \( y' + e^t y = t, \quad (0,1) \). Determine its interval of validity.

1.5

13. A parachutist weighing 160 lbs falls from rest towards earth. After counting to 5 (ie. 5 seconds) the parachutist opens the parachute. Before opening the parachute the force due to air resistance is \( (1/2)v \), and after it is \( 25v \).

a) write the differential equation for the parachutists motion before and after the parachute opens,

b) compute \( v(t) \) (the parachutists velocity \( t \) secs after opening the parachute).

14. A tank has 10 gallons of salt water containing 2 pounds of dissolved salt. Salt water with 1.5 pounds of salt per gallon enters the tank at the rate of 3 gallons per minute. The mixture is kept well-stirred and leaves the tank at the rate of 4 gallons per minute. Find the amount of salt in the tank at any time \( t \) with \( 0 \leq t \leq 10 \).

1.6

15. Sketch the direction field of \( y' = y - t/2 \) in the first quadrant.

16. Sketch the direction field of \( y' = 1 - y^2 \) on the grid determined by \( t = 0, .25, .5, \ldots, 1.75, 2 \) and \( y = -2, -1.5, -1, \ldots, 2 \). Identify the constant solutions and sketch some other solutions.

1.7

17. Using Euler’s method, with \( h = .1 \) solve \( y' + y^2 = \sin(t) \) for \( 0 \leq t \leq 1 \) with \( t = 0 \) and \( y = 0 \) the initial value.

18. Numerically solve the system \( y' = y \) with initial point (0,1) on the interval [0,1] with step size \( h = .1 \). Express your output in a table with 11 rows and 6 columns; where column 1 is \( t \) (for \( t = 0, .1, .2, \ldots, 9, 1 \)), column 2 is \( e^t \), column 3 is Euler’s method, column 4 is improved Euler’s method, column 5 is the error in column 3, column 6 is the error in column 4.

19. Solve problem 31 Bruer-Nohel p.80 numerically. Use Euler’s method to calculate the velocity \( v \) at time \( t \) with a step size of \( h = 1/4 \). If \( x \) is the height of the ball at time \( t \) use Euler’s method to approximate \( x \), i.e. \( x_0 = 500 \) and
\[ x_{n+1} = x_n + hv_n \]
Stop calculating when \( x_n < 0 \), i.e. the ball hits the ground. Repeat using improved Euler’s method, for \( x \) use
\[ x_{n+1} = x_n + \frac{h}{2}(v_n + v_{n+1}) \]

Compare your answers with the analytic solution.

20. For the initial value problem \( y' = e^{x \sin y} \quad y(0) = 1 \), use Euler’s (or modified Euler’s) method to approximate \( y(1) \) (use step size= 0.2).
21. Using your calculator find the smallest integer $n$ such that your calculator thinks that $1 + 10^{-n} = 1$, i.e. punch in $\frac{1}{10^n}$, add 1, subtract 1, and see if you get zero.

1.8

22. a) Show that the initial value problem $y' = e^{-t} + \ln(1 + y^2)$, $y(0) = 0$, has a unique solution on some interval containing 0.

b) Can you conclude by the Existence-Uniqueness Theorem that the initial value problem

$$y' = \begin{cases} 
  e^{-t} + \ln(1 + y^2) & y \geq 0 \\
  e^{-t} + e^y - 1 & y < 0 
\end{cases}, y(0) = 0$$

has a unique solution on some interval containing 0? Justify your answer.

23. Consider the differential equation $y' = 1 + y^2$. Conclude by the Existence-Uniqueness Theorem that the region $D$ is the entire $ty$-plane. Find all solutions of this differential equation. Show that no solution is valid for all $t$. Explain why this does not contradict the Existence-Uniqueness Theorem.

24. Show that the differential equation $y' = -2t\sqrt{1 - y^2}$ has two distinct solutions going thru the initial point $(0, 1)$. Find two that are valid in the interval $(-\infty, \infty)$. (Note: answer in the back for #15 p.13 is incorrect.) Show that this equation has a unique solution thru the point $(0, 0)$ and find an appropriate region $D$ to apply Brunaer-Nohel Theorem 1 p73.
Answers

1. $y = 100e^{-6t}$ and $y(1) = 100e^{-6} = 54.88$
2. $y = Ce^{-t/2} - 1$
3. $y = 2e^{2-2x}$
4. $y = \pm \sqrt{2\ln(|\cos(x)|)} + C$
5. $S = (1.5)(10 - t) + C(10 - t)^4$ where $C = -0.0013$
6. $y = \tan(t + C)$
7. $y_1 = 1, y_2 = \sin(t^2 + \frac{\pi}{2})$