Show all work. Circle your answer.
No notes, no books, no calculator, no cell phones, no pagers, no electronic devices at all.
Solutions will be posted shortly after the exam: www.math.wisc.edu/~miller/m240

Name______________________________

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1. (6 pts) Show that $p \to q$ and $(\neg q) \to (\neg p)$ are logically equivalent.

2. (6 pts) Construct a truth table for the following propositional sentence:

$$(p \lor q) \to (p \land r)$$
3. (6 pts) Find a statement which is logically equivalent to
\[ \neg[ (\exists x P(x)) \rightarrow (\forall y Q(y)) ] \]
but in which the negation sign appears (if at all) only in front of the predicate symbols.

4. (6 pts) Let \( A = \{1, 2\} \) and \( B = \{1, 3, 5\} \). Find \( A \times B \).
5. (6 pts) Let $A = \{1, 3, 4\}$, $B = \{2, 4\}$, and $C = \{3, 4, 5, 6\}$. Find
   
   (a) $|A|$

   (b) $B \setminus C$

   (c) $(A \cup B) \cap C$

6. (6 pts) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

   \[ f(x) = \lfloor x \rfloor \]

   Let $S = \{2, 5\}$. Find $f^{-1}(S)$. 
7. (6 pts) Compute the following double sum:

\[ \sum_{i=1}^{2} \sum_{j=1}^{3} (i + j) \]

8. (5 pts) How large a problem can be solved in 10 seconds or less using an algorithm that when input \( n \) requires \( f(n) = n^2 \) bit operations where each bit operation is carried out in \( 10^{-9} \) seconds?
9. (5 pts) The value of the Euler $\phi$-function at the positive integer $n$ is defined to be the number of positive integers less than or equal to $n$ that are relatively prime to $n$. Find $\phi(12)$.

10. (8 pts) What is smallest positive integer $k$ such that the function

$$f(n) = (n \log(n) + 1)^2$$

is $O(n^k)$?
11. (8 pts) Show that $2^n > 2n + 1$ for all integers $n \geq 3$.

12. (8 pts) Let $f_n$ be the $n^{th}$ element of the Fibonacci sequence. Prove that

$$f_1 + f_3 + f_5 + \cdots + f_{2n-1} = f_{2n}$$

for every positive integer $n$. 
13. (8 pts) Find a $2 \times 2$ matrix $A$ so that

$$A \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix}$$

Hint: Solve a system of linear equations.
14. (8 pts) Let $n = (\ldots a_k a_{k-1} \ldots a_2 a_1 a_0)_2$ be any positive integer written in binary.
   Let $E$ be the set of even $k$ such that $a_k = 1$ and
   let $O$ be the set of odd $k$ such that $a_k = 1$.
   Show that $n$ is divisible by 3 if and only if $|E| + 2|O|$ is divisible by 3.
15. (8 pts)

(a) Find \( d = \text{gcd}(54, 17) \) using the Euclidean Algorithm.

(b) Find integers \( a \) and \( b \) such that \( d = a \cdot 54 + b \cdot 17 \)
Answers

1. $p \rightarrow q$ is false iff $p$ is true and $q$ is false.
   
   $(\neg q) \rightarrow (\neg p)$ is false iff $\neg q$ is true and $\neg p$ is false.
   
   Hence they have the same truth table.

2. 

   \[
   \begin{array}{c|c|c|c|c|c|c}
   p & q & r & (p \lor q) & (p \land r) \\
   \hline
   T & T & T & T & T & T \\
   T & T & F & T & F & F \\
   T & F & T & T & T & T \\
   T & F & F & T & F & F \\
   F & T & T & T & F & F \\
   F & T & F & T & F & F \\
   F & F & T & F & T & F \\
   F & F & F & F & T & F \\
   \end{array}
   \]

3. $(\exists x P(x)) \land (\exists y \neg Q(x))$

4. $\{(1, 1), (1, 3), (1, 5), (2, 1), (2, 3), (2, 5)\}$

5. $|A| = 3$, $B \setminus C = \{2\}$, $(A \cup B) \cap C = \{3, 4\}$

6. $\{x : 2 \leq x < 3 \text{ or } 5 \leq x < 6\} = [2, 3) \cup [5, 6)$

7. 21

8. $n = 10^5$

9. 4

10. $k = 3$ because $\log(n)$ is $O(n^\epsilon)$ for any real $\epsilon > 0$.

11. This is proved by induction.

   Basis: $n = 3$ this true because $2^3 = 8 > 7 = 2 \cdot 3 + 1 = 7$

   Inductive step. Assume $2^n > 2n + 1$ and $n \geq 3$. Then

   \[
   2^{n+1} = 2 \cdot 2^n = 2^n + 2^n > 2n + 1 + 2^n > 2n + 1 + 8 > 2(n + 1) + 1
   \]

   By inductive hypothesis and since $2^n \geq 2^3 = 8$. Hence

   \[
   2^{n+1} > 2(n + 1) + 1
   \]

   as we needed to show.
12. This is proved by induction.
Basis: $f_1 = 1 = f_2$
Inductive Step: Assume true for $n$. Then
\[ f_1 + f_3 + f_5 + \cdots + f_{2n-1} + f_{2n+1} = f_{2n} + f_{2n+1} \]
by inductive hypothesis. But by the definition of Fibonacci sequence
\[ f_{2n} + f_{2n+1} = f_{2n+2} = f_{2(n+1)} \]
and so noting that $2(n+1) - 1 = 2n + 1$
\[ f_1 + f_3 + f_5 + \cdots + f_{2(n+1)-1} = f_{2(n+1)} \]
as was to be shown.

13.
\[ A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \]

14. If $k$ is even, say $k = 2l$ then
\[ 2^k = 2^{2l} = 4^l \]
But $4 \equiv_3 1$ so it follows that
\[ 2^k \equiv_3 4^l \equiv_3 1^l \equiv_3 1 \]
If $k$ is odd, then $k - 1$ is even and hence
\[ 2^k = 2 \cdot 2^{k-1} \equiv_3 2 \]
Now since $n = \sum_{k \in E} 2^k + \sum_{k \in O} 2^k$ it follows from above that
\[ n \equiv_3 \sum_{k \in E} 1 + \sum_{k \in O} 2 \]
but
\[ \sum_{k \in E} 1 + \sum_{k \in O} 2 = |E| + 2|O| \]
so $3$ divides $n$ iff $n \equiv_3 0$ iff $|E| + 2|O| \equiv_3 0$ iff $3$ divides $|E| + 2|O|$

15.
54, 17 \quad 54 = 3(17) + 3
3, 17 \quad 17 = 5(3) + 2
3, 2 and we see that gcd is 1.
1 = $-1(3) + 2(2)$ using $2 = 17 - 5(3)$ we get
1 = $-1(3) + 2(17 - 5(3)) = -11(3) + 2(17)$
1 = $-11(3) + 2(17)$ using $3 = 54 - 3(17)$ we get
1 = $-11(54 - 3(17)) + 2(17) = -11(54) + 35(17)$