Show all work.

No notes, no books, no calculators, no cell phones, no pagers, no electronic devices of any kind.

Name_____________________________________

Circle your Discussion Section:

DIS 303 12:05p T B235 VAN VLECK
DIS 304 12:05p R B235 VAN VLECK
DIS 307 2:25p T B139 VAN VLECK
DIS 308 2:25p R B309 VAN VLECK

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Solutions will be posted shortly after the exam: www.math.wisc.edu/~miller/m240
1. (4 pts) Construct a truth table for the compound proposition:

\((p \rightarrow q) \lor (\neg p \rightarrow q)\)

2. (4 pts) Use a truth table to verify:

\((p \rightarrow q) \equiv (\neg q \rightarrow \neg p)\)
3. (6 pts) Let \( P(x) \) be the statement \( x + 1 > x^2 \) and suppose that the universe of discourse consists of the integers. What are the truth values of the following?

1. \( P(0) \)
2. \( P(1) \)
3. \( P(-1) \)
4. \( \exists x \ P(x) \)
5. \( \forall x \ P(x) \)
6. \( \forall x \exists y \ ((y > x) \land P(y)) \)

4. (6 pts) Determine the truth value of each of the following if the universe of discourse for all variables consists of the positive integers \( \mathbb{N} = \{1, 2, 3, \ldots\} \).

1. \( \forall n \ \exists m \quad n^2 < m \)
2. \( \exists m \ \forall n \quad n^2 < m \)
3. \( \exists n \ \exists m \quad n^2 + m^2 = 5^2 \)
4. \( \exists n \ \exists m \quad n^2 + m^2 = 6^2 \)
5. \( \forall n \ \forall m \quad (n \leq m \lor m \leq n) \)
6. \( \forall n \ \forall m \quad (n < m \lor m < n) \)
5. (8 pts) Determine if the following arguments are correct. If it is correct, what rule of inference is being used. If it is not, what logical error occurs?

(a) If $n$ is an integer with $n \geq 2$, then $n^3 \geq 8$. Suppose $n < 2$. Then $n^3 < 8$.

(b) If $n$ is an integer with $n > 2$, then $n^3 > 8$. Suppose $n^3 \leq 8$. Then $n \leq 2$. 
6. (7 pts) How many different elements does \( A \times A \times A \) have if \( A \) has \( n \) elements?

7. (7 pts) What can we say about the sets \( A \) and \( B \) if \( A \oplus B = \emptyset \). The symbol \( \oplus \) denotes the symmetric difference.
8. (6 pts) Let $h(x) = \lceil x \rceil$. Find

1. $h^{-1}(\{2\})$

2. $h^{-1}(\{x : -1 \leq x \leq 1\})$

3. $h(\{x : -1 \leq x \leq 1\})$
9. (7 pts) Use the bubble sort to sort the list 3, 2, 4, 5, 1 showing the lists obtained at each step, i.e., after each time you do a comparison.
10. (8 pts) Find the least integer $n$ such that $f(x)$ is $O(x^n)$ where

$$f(x) = \frac{2x^5 + x^2 + 1}{3x^2 + 4x \ln(x)}$$
11. (8 pts) Show that if $2^n - 1$ is prime, then $n$ is prime.

Hint: $(x^n - 1) = (x - 1)(x^{n-1} + x^{n-2} + \cdots + x + 1)$
12. (7 pts) Convert the integer 11001111 from binary notation to decimal notation.

13. (7 pts) How much time does an algorithm using $2^{40}$ bit operations take if each bit operation takes $10^{-9}$ seconds?
14. (8 pts) Suppose that an integer $a$ is not divisible by the prime $p$. Show that no two of the integers:

$$a, 2a, 3a, \ldots, (p - 1)a$$

are congruent modulo $p$. 
15. (7 pts) Find $AB$ if

$$A = \begin{bmatrix} 1 & -1 & -2 \\ -1 & 2 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & -1 \\ -1 & 2 \\ 2 & 0 \end{bmatrix}$$
Answers

1. 1.1-27
This is a tautology.

2. 1.2-3
This is the contrapositive.

3. 1.3-11
TTFTFF

4. 1.4-27
TFTFTF

5. 1.5-13
(a) The logical form of this argument is:
\[ P \rightarrow Q \]
\[ \neg P \]
\[ \therefore \neg Q. \]
This is an incorrect inference even though it reaches a correct conclusion.

(b) The logical form of this argument is:
\[ P \rightarrow Q \]
\[ \neg Q \]
\[ \therefore \neg P. \]
This is a correct logical inference.

6. 1.6-25
\[ n^3. \]

7. 1.7-31
\[ A = B \]

8. 1.8-35
1. \((1, 2]\)
2. \((-2, 1]\)
3. \{\(-1, 0, 1]\}

9. 2.1-35
\[ 32451 \]
\[ 23451 \]
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\[ 23415 \]
10. 2.2-7

\( O(x^5) \)

11. 2.4-23

Suppose that \( n \) is not prime and let \( n = km \) for integers \( k, m \) with \( 1 < k, m < n \). Put \( x = 2^k \) and using the hint note that

\[
2^n - 1 = (2^k)^m - 1 = (x^m - 1) = (x - 1)(x^{m-1} + x^{m-2} + \cdots + x + 1)
\]

and so \( 2^n - 1 \) is not prime.

12. 2.5-3

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13. 2.3-11

\( 2^{40}10^{-9} \) seconds. A good estimate is to use \( 2^{10} = 1024 \approx 1000 \) so

\[
2^{40}10^{-9} = \frac{2^{40}}{10^9} = \frac{(2^{10})^4}{10^9} \approx \frac{(1000)^4}{10^9} = \frac{(10^3)^4}{10^9} = \frac{10^{12}}{10^9} = 10^3
\]

14. 2.6-17

Suppose for contradiction that there are \( i, j \) integers with \( 1 \leq i < j \leq p - 1 \) such that

\[ ia \equiv_p ja \]

Then

\[ 0 \equiv_p (j - i)a \]

and so \( p \) divides \( (j - i)a \). Since \( p \) is prime and does not divide \( a \) it must divide \( j - i \). But this is impossible because \( 1 \leq j - i < p \).

15. 2.7-3

\[
\begin{bmatrix}
-2 & -3 \\
-3 & 5
\end{bmatrix}
\]
The following program was used to pick the problems on this test. In some cases the problem is identical and in others it is just similar.

```python
#!/usr/ucb/python

import string
import sys
import random

f=open("hmwk1","r") # input file
lines=f.readlines()

random.seed("the three stooges")

for line in lines:
    s=string.split(line)
    if len(s)> 4:
        section=s.pop(0)
        print random.choice(s).rjust(2) + " " +string.lstrip(line)
```