You may use without proof any theorem proved in class, all of which are on the fourteen page handout; however when you do so, you should **state the result** you are using. You may also use any assigned exercise, but in this case you should **state the exercise and give a full proof** of the exercise in your solution.

1. Suppose $V$ is a vector space, $W$ is a subset of $V$, and $v_1, v_2, \ldots, v_n$ are elements of $V$.

   Define

   (a) $W$ is a subspace of $V$.
   (b) $\text{span}(\{v_1, v_2, \ldots, v_n\})$.
   (c) $v_1, v_2, \ldots, v_n$ are linearly independent.
   (d) $v_1, v_2, \ldots, v_n$ are linearly dependent.
   (e) $v_1, v_2, \ldots, v_n$ are a basis for $V$.

2. A. Let $u_1, u_2, u_3$ be elements of a vector space. Suppose

   $v_1 = u_1 - u_2$
   $v_2 = u_2 - u_3$
   $v_3 = u_3 - u_1$

   Prove that $v_1, v_2, v_3$ are linearly dependent.

   B. Suppose $u_1, u_2, u_3$ is a basis of the vector space $V$. Let

   $v_1 = u_1 + u_2$
   $v_2 = u_2 + u_3$
   $v_3 = u_1 + u_3$

   Prove that $v_1, v_2, v_3$ is a basis of $V$. 
3. A matrix $A_{n \times n}$ is Toplitz iff it is constant on all its ‘diagonals’. For example:

\[
A = \begin{bmatrix}
0 & 6 & -4 & 0 \\
-8 & 0 & 6 & -4 \\
-2 & -8 & 0 & 6 \\
6 & -2 & -8 & 0 \\
\end{bmatrix}
\]

So entry$_{i,j}(A) = 0$ if $i = j$ or $i - j = 0$, entry$_{i,j}(A) = -8$ if $i - j = 1$, entry$_{i,j}(A) = 6$ if $i - j = -1$, etc. More precisely an $n \times n$ matrix $A$ is Toplitz iff for all $i, j, i', j'$ between 1 and $n$

\[
\text{if } i - j = i' - j' \text{ then entry}_{i,j}(A) = entry_{i',j'}(A).
\]

A. Prove that $T_n = \{A \in \text{MAT}_{n \times n} : A \text{ is Toplitz } \}$ is a subspace of $\text{MAT}_{n \times n}$. If you would prefer you may assume $n = 4$, i.e., show $T_4$ is a subspace of the $4 \times 4$ matrices.

B. Find a formula for the dimension of $T_n$ in terms of $n$, i.e.,

\[
\dim(T_n) = ?
\]

Justify your formula by writing down a basis for $T_3$ and $T_4$.

4. Let $A$ be the following matrix:

\[
A = \begin{bmatrix}
0 & 1 \\
1 & 0 \\
\end{bmatrix}
\]

Let $W$ be defined as follows:

\[
W = \{B \in \text{MAT}_{2 \times 2} : AB = BA \}
\]

Find a basis for $W$ and prove that it is a basis.

5. Suppose $A$ is any $n \times n$ matrix. Prove there exists a polynomial

\[
p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_kx^k
\]

of degree $k$ with $1 \leq k \leq n^2$ such that

\[
p(A) = a_0I + a_1A + a_2A^2 + \cdots + a_kA^k = Z
\]

where $I$ is the $n \times n$ identity matrix and $Z$ is the $n \times n$ zero matrix.

Hint: What is the dimension of $\text{MAT}_{n,n}$ and what is the size of the set \{0, 1, 2, \ldots, n^2\}?
Answers

1a. \( u, v \in W \) implies \( u + v \in W \) and \( au \in W \).

1b. \( \text{span}\{u_1, \ldots, u_n\} = \{a_1u_1 + \cdots + a_nu_n : a_1, \ldots, a_n \text{ scalars}\} \)

1c. \( a_1v_1 + a_2v_2 + \cdots + a_nv_n = z \) implies \( a_1 = a_2 = \cdots = a_n = 0 \)

1d. not linearly independent.

1e. linearly independent and spans \( V \).

2A. \( v_1 + v_2 + v_3 = (u_1 - u_2) + (u_2 - u_3) + (u_3 - u_1) = z \).

2B. We show \( v_1, v_2, v_3 \) are linearly independent. Suppose

\[ av_1 + bv_2 + cv_3 = z. \]

Then

\[ a(u_1 + u_2) + b(u_2 + u_3) + c(u_1 + u_3) = z \text{ so } (a + c)u_1 + (a + b)u_2 + (b + c)u_3 = z. \]

Since \( u_1, u_2, u_3 \) are linearly independent we have \( a + c = 0, a + b = 0, \) and \( b + c = 0 \). Solving for \( a, b, c \) we get \( a = -c = b \) so \( a = b \). But \( a = -b \) also
so \( b = -b \) and so \( b = 0 \). Since \( a = -c = b \), we have \( a = 0 \) and \( c = 0 \). This shows \( v_1, v_2, v_3 \) are linearly independent. Since any \( n \) linearly independent vectors in a vector space of dimension \( n \) is a basis, it follows that \( v_1, v_2, v_3 \) is a basis.

Remark: Many students asserted without proof that

\[ \text{span}\{u_1, u_2, u_3\} = \{(a + c)u_1 + (a + b)u_2 + (b + c)u_3 : a, b, c \in \mathbb{R}\} \]

But if \( u_1, u_2, u_3 \) are linearly independent, then

\[ \text{span}\{u_1, u_2, u_3\} \neq \{(a - c)u_1 + (b - a)u_2 + (c - a)u_3 : a, b, c \in \mathbb{R}\} \]

(This follows from the fact the \( v_1, v_2, v_3 \) in part A are linearly dependent.) So something must be proved. A correct proof can be given by showing that for any \( a_1, a_2, a_3 \) there exists \( a, b, c \) so that \( a + b = a_1, b + c = a_2, \) and \( b + c = a_3 \).

3A. Suppose \( A, B \in T_n \). Suppose \( i - j = i' - j' \) and \( a, b \) are scalars. Then since \( A \) and \( B \) are Toplitz,

\[ \text{entry}_{i,j}(A) = \text{entry}_{i',j'}(A) \text{ and } \text{entry}_{i,j}(B) = \text{entry}_{i',j'}(B). \]
But then
\[
\text{entry}_{i,j}(aA + bB) = a\text{entry}_{i,j}(A) + b\text{entry}_{i,j}(B) = \\
\text{entry}_{i',j'}(A) + b\text{entry}_{i',j'}(B) = \text{entry}_{i',j'}(aA + bB)
\]
Hence \((aA + bB) \in T_n\).

3.B \(\text{dim}(T_n) = 2n - 1\).

\[
\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
1 & 0 & 0
\end{bmatrix}, \begin{bmatrix}
0 & 0 & 0 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{bmatrix}, \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}, \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{bmatrix}
\]
is a basis for \(T_3\). (Similarly for \(T_4\).) Each basis vector is determined by a diagonal and there are \(2n - 1\) diagonals. (Count the number of entries in a \(n \times n\) matrix which are in either the first row or first column. The \(-1\) is there because the \(1,1\) is counted twice.)

4.\[
\begin{bmatrix}
a & b \\
c & d
\end{bmatrix} \begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix} = \begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix} \begin{bmatrix}
a & b \\
c & d
\end{bmatrix}
\]
gives us
\[
\begin{bmatrix}
c & d \\
a & b
\end{bmatrix} = \begin{bmatrix}
b & a \\
d & c
\end{bmatrix}
\]
and therefore \(a = d\) and \(b = c\). So
\[
W = \{B \in \text{MAT}_{2,2} : AB = BA\} = \left\{ \begin{bmatrix}
a & b \\
b & a
\end{bmatrix} : a, b \in \mathbb{R} \right\}
\]
We show
\[
\begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}, \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\]
is a basis for \(W\). To see they are linearly independent, suppose
\[
a \begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix} + b \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix} = \begin{bmatrix}
a & b \\
b & a
\end{bmatrix} = \begin{bmatrix}
0 & 0 \\
0 & 0
\end{bmatrix}.
\]
But then \(a = b = 0\). To see that they span \(W\):
\[
\text{span} \left\{ \begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}, \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix} \right\} = \left\{ a \begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix} + b \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix} : a, b \in \mathbb{R} \right\} =
\]
4
\[
= \left\{ \begin{bmatrix} a & b \\ b & a \end{bmatrix} : a, b \in \mathbb{R} \right\} = W
\]

5. The dimension of the vector space MAT\(_{n,n}\) is \(n^2\). The **Main Theorem** proved in class is that if a vector space is spanned by \(m\) vectors, then any sequence of \(m + 1\) vectors is linearly dependent. The sequence of \(n \times n\) matrices

\[
I_{n \times n}, A, A^2, A^3, \ldots, A^{n^2}
\]

has length \(n^2 + 1\) and is therefore linearly dependent. So there exists a nontrivial linear combination

\[
a_0 I_{n \times n} + a_1 A + a_2 A^2 + \cdots + a_{n^2} A^{n^2} = Z_{n \times n}
\]

Let \(k\) be the largest such that \(a_k \neq 0\), clearly \(k > 0\) because \(a_0 I \neq Z\) if \(a_0 \neq 0\), and so

\[
p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_k x^k
\]

is a polynomial with degree \(k\) (\(1 \leq k \leq n^2\)) such that \(p(A) = Z\).

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**Grading Key:**

1. 4 pts each definition, 2 if partially right
2. 10 part A, 10 part B - 5 each for span, lin indep
3. 10 each part
4. 10 for a correct basis, 5 each for showing lin indep and spans
5. 2 each for dim(MAT\(_{n,n}\)) = \(n^2\) and size of \(\{0, 1, \ldots, n^2\}\) is \(n^2 + 1\).