Linear differential equations.

1. A mass $m_1$ is hung from a spring with spring constant $k_1$ is attached to a rigid support. A mass $m_2$ is hung from a second spring with spring constant $k_2$ is attached to the mass $m_2$. Let $y_1$ be the displacement from equilibrium of the mass $m_1$ and similarly $y_2$ the displacement from equilibrium of the mass $m_2$. Then using Hooke’s law and $F = ma$ we get that:

$$m_1y_1'' = -k_1 y_1 + k_2(y_2 - y_1)$$

$$m_2y_2'' = -k_2(y_2 - y_1).$$

Let $u_1 = y_1$, $u_2 = y_1'$, $u_3 = y_2$, and $u_4 = y_2'$. Find a matrix $A$ such that $U' = AU$.

2. Suppose $k_1 = 21$, $k_2 = 15$, $m_1 = 7$, $m_2 = 3$ in the appropriate units. Find the general solution.

3. Suppose $y_1(0) = 1$, $y_2(0) = -1$, and both have 0 initial velocities. Find the solution. Find the displacements and velocities of the masses at time $t = 2$. 

RULES: Do not discuss this exam with anybody except me. I will explain any definition or clarify the statement of any of these problems. My office is 403 Van Vleck, office hours MF 2:30-3:30 and W 4:30-5:30, or you can make an appointment. You may use minimat to do any calculation. The exam is due in class on Wed April 11. Exam 2 Part II will be in class on Mon April 16.
The method of least squares.

4. Prove that if $A$ is an $n \times m$ matrix of functions of $t$ and $B$ is an $m \times r$ matrix of functions of $t$, then

$$\frac{d}{dt}(AB) = (\frac{d}{dt}A)B + A(\frac{d}{dt}B).$$

5. Prove that if $U$ and $V$ are $n \times 1$ vectors of functions of $t$, then

$$\frac{d}{dt} \langle U, V \rangle = \langle \frac{d}{dt}U, V \rangle + \langle U, \frac{d}{dt}V \rangle.$$

6. Let $A_{n \times m}, B_{m \times 1}$ be matrices of reals. Prove that

$$\frac{\partial}{\partial x_i} ||AX - B||^2 = 2 \langle AX - B, Ae_i \rangle.$$

7. Prove that if $X$ minimizes $||AX - B||^2$, then $A^T(AX - B) = 0$.

8. It is good surveying practice to make more observations than are strictly necessary. For example, to determine the altitude of four points $x_1, x_2, x_3, x_4$, eight observations were taken:

- $x_1 = 2.947$
- $x_2 = 1.735$
- $x_3 = -1.449$
- $x_4 = 1.321$
- $x_1 - x_2 = 1.204$
- $x_1 - x_4 = 1.631$
- $x_2 - x_3 = 3.186$
- $x_3 - x_4 = -2.778$

Note that all of these observations contain error and there do not exists any solutions to all eight equations. Find $A_{8 \times 4}$ and $B_{8 \times 1}$ such that the most likely altitudes $X$ minimize $||AX - B||^2$. Find $X$. 

2
Linear programming or linear optimization.

Let \( A \) be \( n \times m \), \( B \) be \( m \times 1 \) and \( C = [c_1, c_2, \ldots, c_n] \) be \( 1 \times n \).

(\*) Maximize \( z = c_1x_1 + c_2x_2 + \cdots + c_nx_n \) subject to the constraints:
\( AX = B \) and \( X \geq 0 \).

\( X \geq 0 \) means that \( x_1 \geq 0, x_2 \geq 0, \ldots, \) and \( x_n \geq 0 \). \( X \) is called feasible iff \( AX = B \) and \( X \geq 0 \). A solution to (\*) is a vector \( X \) which is feasible and for which the value of \( z \) is as large as the value of \( z \) is for any feasible solution.
\( X \) is called an extreme point iff there exists \( m_1, m_2, \ldots, m_k \) such that \( X \) is the unique solution of \( AX = B, x_{m_1} = 0, x_{m_2} = 0, \ldots, \) and \( x_{m_k} = 0 \).

9. Give an example of (\*) where there are no feasible solutions. Give an example of (\*) where there are feasible solutions but no solutions. Give an example of (\*) where there are infinitely many solutions. Give an example of (\*) with an extreme point which is not feasible.

10. Let \( X_1 \) and \( X_2 \) be solutions of \( AX = B, x_{m_1} = 0, x_{m_2} = 0, \ldots, \) and \( x_{m_k} = 0 \). Define \( L[t] = X_1 + t(X_2 - X_1) \) where \( t \) is a scalar. Show that \( L[t] \) is a solution of \( AX = B, x_{m_1} = 0, x_{m_2} = 0, \ldots, \) and \( x_{m_k} = 0 \) for every \( t \).

11. Suppose in addition to above that all coordinates of \( X_1 \) other than \( m_1, m_2, \ldots, m_k \) are strictly positive. Show that there exists \( \epsilon > 0 \) such that \( L[t] \geq 0 \) for every \( t \) with \( -\epsilon < t < \epsilon \).

12. Define \( Z(t) = CL[t] \). Prove that for some scalars \( p \) and \( q \) that \( Z(t) = pt + q \). Prove that \( Z \) is maximized at \( t = 0 \) iff \( p = 0 \).

13. Given the problem (\*) suppose \( X_0 \) is any solution which has the maximum number of zero coordinates as any other solution. Prove that \( X_0 \) is an extreme point.
Graph theory.

Suppose that $A$ is an $n \times n$ matrix such that for any $i, j$ we have that $a_{i,j} = 0$ or $a_{i,j} = 1$. The matrix $A$ determines a graph $G_A$ on $n$ vertices $v_1, v_2, \ldots, v_n$ by the rule that $v_i$ and $v_j$ are connected by an edge iff $a_{i,j} = 1$. A path in a graph $G$ is a sequence of vertices $u_1, u_2, \ldots, u_k$ such that for each $i = 1, \ldots, k - 1$ there is an edge in $G$ which connects $u_i$ and $u_{i+1}$. A graph $G$ is connected iff any two vertices of $G$ can be connected by a path.

14. Give an example of a $4 \times 4$ matrix $A$ as above whose graph is connected. Give an example of a $4 \times 4$ matrix $A$ as above whose graph is not connected.

15. Prove that if $A^m = [c_{ij}]$, then $c_{ij}$ is the number of paths of length $m$ connecting $v_i$ to $v_j$.

16. Prove that the graph associated to such an $n \times n$ matrix $A$ is connected iff every entry of $I + A + A^2 + \cdots + A^{n-1}$ is positive.

Markov processes.

In the Land of OZ there are three states for the weather: nice, rain, and snow. The weather follows the following rules:

(a) There are never two nice days in a row.

(b) When it rains or snows, half the time it the same the next day.

(c) If the weather changes, the chances are equal for a change to each of the two other types of weather.

17. Draw a graph with vertices $N, R$, and $S$ and label the edges with the appropriate probabilities.

18. Find the $3 \times 3$ transition matrix $A$ corresponding to this graph.

19. Prove that the probability that it will rain exactly one week after it was nice is the $(1, 2)$ entry of the matrix $A^7$. Find this probability.

20. Find the probability that the weather will be nice, rain, or snow on a random day in the Land of OZ.
Determinants, etc.

The trace of an \( n \times n \) matrix \( A \) is defined by

\[
\text{trace}(A) = a_{11} + a_{22} + \cdots + a_{nn}.
\]

The characteristic polynomial of a matrix \( A \) is defined by \( p(x) = \det(xI - A) \).

21. Prove that if \( A \) and \( B \) are similar, then \( \text{trace}(A) = \text{trace}(B) \).

22. Prove that if \( A \) and \( B \) are similar, then \( \det(A) = \det(B) \).

23. Prove or disprove: if two matrices \( A \) and \( B \) have the same characteristic polynomial, then they are similar.

24. Prove or disprove: if two matrices \( A \) and \( B \) are similar, then they have the same characteristic polynomial.