There is a total of 150 points. Show all work.

1. (15 points) What is the determinant of \((AB)^T\)?

\[
A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 6 & -1 & 0 & 0 \\ 7 & 0 & 2 & 0 \\ 2 & 1 & 0 & 3 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 1 & 1 & 4 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 4 \end{bmatrix}
\]

2. (15 points) Let \(A\) be an \(m \times n\) matrix and \(X = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}\) an \(n \times 1\) column vector. Prove that

\[
AX = x_1\text{col}_1(A) + \ldots + x_n\text{col}_n(A).
\]

3. (15 points)

\[
A = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & -1 \end{bmatrix}
\]

Find an invertible matrix \(P\) and diagonal matrix \(D\) with \(A = PDP^{-1}\).

4. (15 points) Use Cramer’s rule to solve:

\[
\begin{align*}
x - y &= 2 \\
2x + y &= 3
\end{align*}
\]

5. (15 points) Suppose \(A\) is an invertible matrix and \(\lambda\) is an eigenvalue of \(A\).
   
   (a) Prove that \(\lambda \neq 0\).
   
   (b) Prove that \(\frac{1}{\lambda}\) is an eigenvalue of \(A^{-1}\).
6. (15 points) Suppose $L : V \rightarrow W$ is a linear transformation, the nullspace of $L$ has only zero vector in it ($\text{null}(L) = \{0\}$), and $v_1, \ldots, v_n$ are linearly independent vectors in $V$. Prove that $L(v_1), \ldots, L(v_n)$ are linearly independent.

7. (20 points)

$$A = \begin{bmatrix}
0 & -1 & 1 & 0 & -1 \\
0 & 1 & -1 & 0 & 1 \\
0 & 1 & -1 & 1 & -2
\end{bmatrix}$$

(a) Find a basis for the range space of $A$.
(b) Find a basis for the null space of $A$.

8. (15 points) Suppose $A$ is a $13 \times 12$ matrix, and $B$ is a $12 \times 13$ matrix, and $C = AB$. Prove that $C$ is not invertible.

9. (25 points) Let $\mathbb{R}^{4 \times 1}$ have the inner product defined by

$$\langle \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} \rangle = x_1y_1 + 2x_2y_2 + 2x_3y_3 + x_4y_4$$

and use this inner product in the following problem. Let

$$v_1 = \begin{bmatrix}
-1 \\
1 \\
1 \\
0
\end{bmatrix} \quad v_2 = \begin{bmatrix}
0 \\
-1 \\
1 \\
1
\end{bmatrix} \quad v_3 = \begin{bmatrix}
1 \\
1 \\
1 \\
1
\end{bmatrix}$$

Find a vector $w \in \text{span}\{v_1, v_2, v_3\}$ other than the zero vector such that $w$ is orthogonal to both $v_1$ and $v_2$. 