Vector Spaces

A vector space, $V$, is a set with two operations, vector addition (written $u + v$) and scalar multiplication (written $av$). The elements of $V$ will be denoted using $u, v, w$, etc. The formula ‘$u \in V$’ is short hand for ‘$u$ is an element of $V$’ or ‘$u$ in $V$’ or just ‘$u$ is a vector’. Vector spaces will be written using capital letters $V, W$, etc. Scalars are elements of some field $F$, for example, the real numbers, $\mathbb{R}$, or the complex numbers, $\mathbb{C}$. Scalars will be written using the letters $a, b, c$, etc.

Closure axioms:

1. If $u \in V$ and $v \in V$, then $u + v \in V$.
2. If $u \in V$ and $a$ a scalar, then $au \in V$.

Associative, commutative, distributive axioms:

1. For all $u, v, w \in V$ $(u + v) + w = u + (v + w)$.
2. For all $u, v \in V$ $u + v = v + u$.
3. For all scalars $a$ and $b$ and vectors $u \in V$ $(ab)u = a(bu)$.
4. For all scalars $a$ and $b$ and vectors $u \in V$ $(a + b)u = au + bu$.
5. For all scalars $a$ and vectors $u, v \in V$ $a(u + v) = au + av$.

Zero vector, additive inverse, identity axioms:

1. There exists a vector $\vec{0} \in V$ such that for all $u \in V$ $\vec{0} + u = u + \vec{0} = u$.
2. For every $u \in V$ there exists a vector $v \in V$ (for which we write $v = -u$) such that $u + v = v + u = \vec{0}$.
3. For every $u \in V$, $1u = u$. 
Any abstract set $V$ with two operations, vector addition and scalar multiplication which satisfy all the above axioms is a vector space.

Most author’s use either $0$ or $\vec{0}$ to denote the zero vector. Note that it is not the same as the zero element $0$ of the field.

**Exercise 1** Prove that $0u = \vec{0}$ for any $u \in V$ a vector space.

**Exercise 2** Prove that $(-1)u = -u$ for any $u \in V$ a vector space.

**Definition 3** For $W$ a subset of a vector space $V$ (written $W \subseteq V$) we say that $W$ is a **subspace** of $V$ iff

1. for every $u, v \in V$ if $u \in W$ and $v \in W$, then $u + v \in W$, and

2. for every $u \in V$ and scalar $a$ if $u \in W$, then $au \in W$.

**Theorem 4** If $W$ is a subspace of $V$, then $W$ is itself a vector space under the operations defined in $V$.

**proof:**

The closure axioms are easy since they are practically the same as the definition of subspace. The associative, commutative, distributive axioms are true in $W$ because they are true in $V$ and $W$ is a subset of $V$. The zero vector $\vec{0}$ is in $W$ because $0u = \vec{0}$ (exercise 1) so (assuming $W$ is nonempty) if anything is in $W$, then $\vec{0}$ is in $W$. Similarly $(-1)u = -u$ (exercise 2), so if $u \in W$, then also $-u \in W$. □

**Theorem 5** Suppose $W$ is a subset of $V$ (i.e., $W \subseteq V$). Then

1. $W$ is a subspace of $V$

   iff

2. for every $u, v \in W$ and scalars $a, b$ we have $au + bv \in W$.

**proof:**

(1) implies (2):

Assume $W$ is a subspace of $V$. Suppose $u, v \in W$ and $a, b$ are scalars. By the second axiom of subspaces we have that $au \in W$ and $bv \in W$. Letting $w_1 = au$ and $w_2 = bv$ we have that $w_1 \in W$ and $w_2 \in W$, therefore by the first axiom of subspaces we have that $w_1 + w_2 \in W$ and so $au + bv \in W$.  

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(2) implies (1):
Assume (2): for every \( u, v \in W \) and scalars \( a, b \) we have \( au + bv \in W \).
We must show the two axioms of a subspace hold for \( W \). Suppose \( u, v \in W \).
Then letting \( a = b = 1 \) we have that \( 1u + 1v \in W \), so \( 1u + 1v = u + v \in W \).
For the second axiom, suppose \( u \in W \) and \( a \) any scalar, then we have that \( au + 0u \in W \) by (2), but \( au = au + 0 = au + 0u \) so \( au \in W \). □

Definition 6 For \( u_1, \ldots, u_n \) elements of a vector space \( V \), define their span:

\[
\text{span}\{u_1, u_2, \ldots, u_n\} = \{a_1u_1 + a_2u_2 + \cdots + a_nu_n : a_1, a_2, \ldots, a_n \text{ scalars}\}
\]

Each of the vectors \( a_1u_1 + a_2u_2 + \cdots + a_nu_n \) is called a linear combination of the \( u \)'s so we could also say that the span is the set of all linear combinations.
If \( W = \text{span}\{u_1, u_2, \ldots, u_n\} \), we say that ‘\( W \) is spanned by \( u_1, u_2, \ldots, u_n \)’ or ‘\( u_1, u_2, \ldots, u_n \) span \( W \)’. The closure axioms of a vector space \( V \) guarantee that if \( u_1, u_2, \ldots, u_n \in V \), then \( \text{span}\{u_1, u_2, \ldots, u_n\} \subseteq V \). This is true because the second closure axiom says each \( a_iu_i \) is in \( V \), while the first axiom guarantees that their sum is in \( V \), e.g., if we write \( v_1 = a_1u_1 \), \( v_1, v_2 \in V \) implies \( v_1 + v_2 \in V \) and so \( v_1 + v_2, v_3 \in V \) implies \( v_1 + v_2 + v_3 = (v_1 + v_2) + v_3 \in V \), and so on.

Theorem 7 Suppose \( u_1, u_2, \ldots, u_n \) are elements of \( W \) which is a subspace of \( V \). Then \( \text{span}\{u_1, u_2, \ldots, u_n\} \subseteq W \).

proof:
Suppose \( v \in \text{span}\{u_1, u_2, \ldots, u_n\} \). Then for some scalars, \( a_1, \ldots, a_n \) we have that

\[
v = a_1u_1 + \cdots + a_nu_n.
\]

Since \( W \) is a subspace of \( V \) we have that \( a_iu_i \in W \) for each \( i \). Now let \( v_i = a_iu_i \) to simplify our writing. Since \( v_1 \in W \) and \( v_2 \in W \) we have by the first axiom of subspaces that \( v_1 + v_2 \in W \). Thus we have that the two vectors \( v_1 + v_2 \) and \( v_3 \) are elements of \( W \). This means their sum \( (v_1 + v_2) + v_3 \) is in \( W \). Continuing on like this we see that \( v_1 + v_2 + \cdots + v_k \in W \) for each \( k \) and so

\[
v = a_1u_1 + \cdots + a_nu_n = v_1 + v_2 + \cdots + v_n \in W
\]
as we needed to show. □
Theorem 8 Suppose $u_1, u_2, \ldots, u_n$ are elements of a vector space $V$. Then \( \text{span}\{u_1, u_2, \ldots, u_n\} \) is a subspace of $V$.

proof:
We verify each of the axioms of a subspace. Let 
\[ W = \text{span}\{u_1, u_2, \ldots, u_n\}. \]
Suppose $v, w$ are elements of $W$. Then since $W$ is the span of the $u$’s there exists scalars $c_1, \ldots, c_n$ and $d_1, \ldots, d_n$ such that 
\[ v = c_1 u_1 + \cdots + c_n u_n \quad \text{and} \quad w = d_1 u_1 + \cdots + d_n u_n. \]
But then 
\[ v + w = (c_1 + d_1) u_1 + (c_2 + d_2) u_2 + \cdots + (c_n + d_n) u_n \]
and so $v + u \in \text{span}\{u_1, u_2, \ldots, u_n\} = W$.
For the second axiom, suppose $v \in W$ and $a$ a scalar. Then for some scalars $c_1, \ldots, c_n$
\[ v = c_1 u_1 + \cdots + c_n u_n \]
but then 
\[ av = a(c_1 u_1 + \cdots + c_n u_n) = (ac_1) u_1 + \cdots + (ac_n) u_n \]
and so $av \in \text{span}\{u_1, u_2, \ldots, u_n\} = W$.
\[ \square \]

It follows from these last two theorems that \( \text{span}\{u_1, u_2, \ldots, u_n\} \) is the smallest subspace of $V$ which contains the vectors $u_1, u_2, \ldots, u_n$.

Theorem 9 Suppose $u \in \text{span}\{u_1, \ldots, u_n\}$ then 
\[ \text{span}\{u, u_1, \ldots, u_n\} = \text{span}\{u_1, \ldots, u_n\} \]
proof:

To show two sets $A$ and $B$ are equal, $A = B$, show that 
$A \subseteq B$ and $B \subseteq A$. 

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To show that \( A \subseteq B \), suppose that \( x \in A \) and then show \( x \in B \).

First we show \( \text{span}(\{u, u_1, \ldots, u_n\}) \subseteq \text{span}(\{u_1, \ldots, u_n\}) \). Since \( u \in \text{span}(\{u_1, \ldots, u_n\}) \) there exists scalars \( b_1, \ldots, b_n \) so that \( u = b_1 u_1 + \cdots + b_n u_n \). Now let \( w \) be any element of \( \text{span}(\{u, u_1, \ldots, u_n\}) \). This means there are scalars \( a, a_1, \ldots, a_n \) such that

\[
w = au + a_1 u_1 + \cdots + a_n u_n.
\]

But then substituting for \( u \):

\[
w = a(b_1 u_1 + \cdots + b_n u_n) + a_1 u_1 + \cdots + a_n u_n
= (ab_1 + a_1) u_1 + \cdots + (ab_n + a_n) u_n
\]

and so \( w \in \text{span}(\{u_1, \ldots, u_n\}) \) as was to be shown.

Second we show \( \text{span}(\{u_1, \ldots, u_n\}) \subseteq \text{span}(\{u, u_1, \ldots, u_n\}) \). This is easier. Suppose \( w \in \text{span}(\{u_1, \ldots, u_n\}) \). Then there are scalars \( c_1, \ldots, c_n \) so that

\[
w = c_1 u_1 + \cdots + c_n u_n
\]

but then

\[
w = 0u + c_1 u_1 + \cdots + c_n u_n
\]

so \( w \in \text{span}(\{u, u_1, \ldots, u_n\}) \) as was to be shown. \( \Box \)

**Definition 10** For \( v_1, v_2, \ldots, v_n \) vectors in a vector space \( V \) we say that they are linearly independent iff for any scalars \( a_1, a_2, \ldots, a_n \)

\[
a_1 v_1 + a_2 v_2 + \cdots + a_n v_n = \vec{0} \rightarrow a_1 = a_2 = \cdots = a_n = 0
\]

**Definition 11** We say \( v_1, v_2, \ldots, v_n \) are linearly dependent iff \( v_1, \ldots, v_n \) are not linearly independent.

**Definition 12** \( v_1, v_2, \ldots, v_n \) is a basis for the vector space \( V \) iff

1. \( v_1, v_2, \ldots, v_n \) are linearly independent and

2. \( V = \text{span}(\{v_1, v_2, \ldots, v_n\}) \).
Theorem 13  Let $A$ be any $n \times n$ matrix. Then $A$ is invertible iff the set of columns of $A$ is a basis for $\mathbb{R}^n$, i.e., $\text{col}_1(A), \text{col}_2(A), \ldots, \text{col}_n(A)$ is a basis for $\mathbb{R}^n$.

proof:
Before proving this result we first prove the following Lemma.

Lemma 14  Suppose $A$ is an $m \times n$ matrix and

$$B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

is $n \times 1$. Then

$$AB = A \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = b_1 \text{col}_1(A) + b_2 \text{col}_2(A) + \cdots + b_n \text{col}_n(A) = \sum_{k=1}^{n} b_k \text{col}_k(A)$$

proof:
Write the matrix $A$ as follows:

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & a_{2,3} & \cdots & a_{2,n} \\ a_{3,1} & a_{3,2} & a_{3,3} & \cdots & a_{3,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & a_{m,3} & \cdots & a_{m,n} \end{bmatrix}$$

i.e., $a_{i,j} = \text{entry}_{i,j}(A)$. Then
This proves the Lemma. □ Now to prove the Theorem we first assume that $A$ is invertible and show that the columns of $A$ are a basis for $\mathbb{R}^n$. To see that they are independent, suppose that

$$b_1 \text{col}_1(A) + \cdots + b_n \text{col}_n(A) = Z$$

where $Z_{n \times 1}$ is the zero vector. By the lemma

$$AB = Z$$

where $B$ is the column vector made from $b_1, \ldots, b_n$. Since $A$ is invertible we have that $B = A^{-1}Z = Z$ so $B = Z$ and so $b_i = 0$ for all $i$ with $1 \leq i \leq n$. This shows the the columns of $A$ are linearly independent. To see that they span $\mathbb{R}^n$, let $C$ be an arbitrary element of $\mathbb{R}^n$ so that

$$C = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}.$$
Since $A$ is invertible if we set $B = A^{-1}C$ then we know that $AB = C$ and by
the lemma we have that
\[ b_1 \text{col}_1(A) + \cdots + b_n \text{col}_n(A) = C \]
and so $C$ is in the span of the columns of $A$. This shows that if $A$ is in-
vertible then its columns are a basis. Next we prove the converse, using the
contrapositive.

The contrapositive of the implication:
\[ P \text{ implies } Q \]
is
\[ (\text{Not } Q) \text{ implies } (\text{Not } P) \]
They are logically equivalent.

Assume that $A$ is not invertible. Then as was shown the algorithm for
attempting to invert $A$ produces a $B_{n \times 1} \neq Z$ such that $AB = Z$. This means
by the Lemma that
\[ b_1 \text{col}_1(A) + \cdots + b_n \text{col}_n(A) = Z \]
and since $B \neq Z$ at least one $b_i \neq 0$. But this means that the columns of $A$
are linearly dependent and hence not a basis. This finishes the proof of the
Theorem. □

Theorem 15 $v_1, v_2, \ldots, v_n$ are linearly dependent iff there are scalars
\[ a_1, a_2, \ldots, a_n \]
such that $a_1 v_1 + a_2 v_2 + \cdots + a_n v_n = \vec{0}$ and for at least one $i$ we have $a_i \neq 0$.

proof:

Not For ALL $x$ Statement($x$)
is logically equivalent to
There exists $x$ such that Not Statement($x$)

So negating linear independence gives us:
There exists scalars $a_1, \ldots, a_n$ such that
Not $[ a_1 v_1 + \cdots + a_n = \vec{0}$ implies $a_1 = a_2 = \cdots = a_n = 0 ]$. 

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The negation of an implication

Not \([P \text{ implies } Q]\)

is logically equivalent to

\(P \text{ and Not } Q\).

So in this case we get

\[a_1 v_1 + \cdots + a_n = \vec{0} \text{ and Not } [a_1 = a_2 = \cdots = a_n = 0].\]

But Not \([a_1 = a_2 = \cdots = a_n = 0]\) is the same as saying ‘at least one of the \(a_i\) is not equal to 0.

\[\Box\]

**Exercise 16** Prove that \(v_1, \ldots, v_n\) are linearly dependent iff \(v_1 = \vec{0}\) or \(v_j \in \text{span}(\{v_1, \ldots, v_{j-1}\})\) for some \(j\) with \(1 < j \leq n\).

**Lemma 17** (Exchange) Suppose \(v_1, v_2, \ldots, v_{k+1}, w_1, \ldots, w_m\) are vectors in a vector space \(V\) and

1. \(v_1, v_2, \ldots, v_{k+1}\) are linearly independent, and

2. \(\text{span}(\{v_1, v_2, \ldots, v_k, w_1, \ldots, w_m\}) = V\).

Then for some \(i\) with \(1 \leq i \leq m\)

\[\text{span}(\{v_1, v_2, \ldots, v_{k+1}, w_1, w_2, \ldots, w_{i-1}, w_{i+1}, \ldots w_m\}) = V\]

(i.e., we have added \(v_{k+1}\) and removed \(w_i\).)

**proof:**

By (2) there are scalars \(a_i, b_j\) such that

\[v_{k+1} = a_1 v_1 + a_2 v_2 + \cdots + a_k v_k + b_1 w_1 + b_2 w_2 + \cdots + b_m w_m.\]

It must be that for some \(i\) with \(1 \leq i \leq m\) that \(b_i \neq 0\), because otherwise we would have that

\[v_{k+1} = a_1 v_1 + a_2 v_2 + \cdots + a_k v_k\]

and therefore

\[\vec{0} = a_1 v_1 + a_2 v_2 + \cdots + a_k v_k + (-1)v_{k+1}\]

contradicting their independence (1).

Therefore we have that

\[-b_i w_i = a_1 v_1 + \cdots + a_k v_k + (-1)v_{k+1} + \]

\[+ b_1 w_1 + \cdots + b_{i-1} w_{i-1} + b_{i+1} w_{i+1} + \cdots + b_m w_m\]
and since $b_i \neq 0$

$$w_i = \frac{a_1}{b_i}v_1 + \cdots + \frac{a_k}{b_i}v_k + \frac{(-1)^{k+1}}{b_i}v_{k+1} + \frac{b_i}{b_i}w_1 + \cdots + \frac{b_i}{b_i}w_{i-1} + \frac{b_i}{b_i}w_{i+1} + \cdots + \frac{b_m}{b_i}w_m.$$  

Hence

$$w_i \in \text{span}\{v_1, \ldots, v_{k+1}, w_1, \ldots, w_{i-1}, w_{i+1}, \ldots, w_m\}$$

and so by Theorem 9 we have

$$\text{span}\{v_1, v_2, \ldots, v_{k+1}, w_1, w_2, \ldots, w_{i-1}, w_{i+1}, \ldots, w_m\} = \text{span}\{v_1, v_2, \ldots, v_{k+1}, w_1, w_2, \ldots, w_m\}$$

and by (2) we have

$$\text{span}\{v_1, v_2, \ldots, v_{k+1}, w_1, w_2, \ldots, w_m\} = V$$

and so we are done. \(\square\)

**Theorem 18** *(Main Theorem)* If a vector space $V$ can be spanned by $n$ vectors, then any set of $n+1$ vectors in $V$ is linearly dependent.

**proof:**

Suppose for contradiction that $\text{span}\{u_1, \ldots, u_n\} = V$ and $v_1, \ldots, v_{n+1}$ are linearly independent.

Step 1. Apply the Exchange Lemma with $k = 0$ to obtain $i$ so that $\text{span}\{v_1, u_1, \ldots, u_{i-1}, u_{i+1}, \ldots u_n\} = V$.

Step 2. Rename (relabel? reorder?) the $u$’s so that

$$(w_1, \ldots, w_{n-1}) = (u_1, \ldots, u_{i-1}, u_{i+1}, \ldots u_n)$$

and apply the Exchange Lemma with $k = 1$ to obtain $i$ so that

$$\text{span}\{v_1, v_2, w_1, \ldots, w_{i-1}, w_{i+1}, \ldots w_{n-1}\} = V.$$  

Step k. Given $\{w_1, \ldots, w_{n-k}\} \subseteq \{u_1, \ldots, u_n\}$ such that

$$\text{span}\{v_1, v_2, \ldots, v_k, w_1, \ldots, w_{n-k}\} = V,$$  

and so we are done. \(\square\)
apply the Exchange Lemma to find \( i \) so that

\[
\text{span}(\{v_1, v_2, \ldots, v_k, v_{k+1}, w_1, \ldots, w_{i-1}, w_{i+1}, \ldots w_{n-k}\}) = V
\]

Last Step: Given that \( \text{span}(\{v_1, v_2, \ldots, v_{n-1}, w_1\}) = V \), apply the Exchange Lemma to get that

\[
\text{span}(\{v_1, v_2, \ldots, v_n\}) = V.
\]

But this is a contradiction, since \( v_{n+1} \in \text{span}(\{v_1, v_2, \ldots, v_n\}) \) implies that

\[
v_{n+1} = a_1v_1 + \cdots + a_nv_n
\]

for some scalars \( a_i \), but then

\[
\vec{0} = (-1)v_{n+1} + a_1v_1 + \cdots + a_nv_n
\]

and therefore \( v_1, \ldots, v_{n+1} \) would be linearly dependent, a contradiction. \( \square \)

**Definition 19** The **dimension** of a vector space \( V \) is \( n \), written \( \dim(V) = n \) iff \( V \) has a basis of size \( n \).

**Theorem 20** Any two bases for a vector space \( V \) have the same size.

**proof:**

Otherwise, if \( u_1, \ldots, u_m \) is a basis for \( V \) and \( v_1, \ldots, v_n \) is another basis for \( V \) and \( m < n \), then since the \( u \)'s span \( V \) it must be that the \( v \)'s are linearly dependent (by 18), contradicting that they are a basis. \( \square \)

**Theorem 21** If \( V \) is a vector space, \( \dim(V) = n \), and \( u_1, \ldots, u_n \in V \) are linearly independent, then \( u_1, \ldots, u_n \) are a basis for \( V \).

**proof:**

It is enough to show \( \text{span}(\{u_1, \ldots, u_n\}) = V \). By the main theorem (18) for any \( v \in V \) we know that \( u_1, u_2, \ldots, u_n, v \) are linearly dependent. Hence there are scalars \( a_1, a_2, \ldots, a_n, a \) (at least one of which is nonzero) such that

\[
a_1u_1 + a_2u_2 + \cdots + a_nu_n + av = \vec{0}.
\]

Since \( u_1, \ldots, u_n \) are linearly independent, it cannot be that \( a = 0 \) and so

\[
v = -\frac{1}{a}(a_1u_1 + a_2u_2 + \cdots + a_nu_n)
\]

and so \( v \in \text{span}(\{u_1, \ldots, u_n\}) \) and since \( v \) was an arbitrary element of \( V \) we have that \( \text{span}(\{u_1, \ldots, u_n\}) = V. \square \)
Theorem 22 Suppose $\dim(V) = n$ and
\[
\text{span}(\{u_1, \ldots, u_n\}) = V.
\]
Then $u_1, \ldots, u_n$ is a basis for $V$.

proof:
It is enough to prove that $u_1, \ldots, u_n$ are linearly independent.

In a proof by contradiction assume the negation of what you are trying to prove and then reason to a contradiction. It follows logically that what you are trying to prove must be true.

Suppose for contradiction that they are linearly dependent. Then there are scalars $a_1, \ldots, a_n$ such that
\[
a_1 u_1 + \cdots + a_n u_n = \vec{0}
\]
and for some $i$ we have $a_i \neq 0$. But then we have that
\[
u_i \in \text{span}(\{u_1, \ldots, u_{i-1}, u_{i+1}, \ldots, u_n\})
\]
and so by Theorem 9, we have that $V$ is spanned by $n - 1$ vectors. This would imply by the main theorem (18) that any set of $n$ vectors is linearly dependent, contradicting the fact the dimension of $V$ is $n$. □

Theorem 23 Every vector space has a basis.

proof:
This theorem is true in general but requires a more sophisticated proof for the infinite dimensional case. Here we prove it just for the case that our vector space $W$ is a subspace of a vector space $V$ with finite dimension. Suppose the dimension of $V$ is $n$.

If $W = \{\vec{0}\}$, then the dimension of $W$ is 0 and the empty set is a basis for it. Otherwise let $v_1 \in W$ be an arbitrary vector in $W$ not equal to the zero vector, $\vec{0}$. If $v_1$ spans $W$, then the dimension of $W$ is 1 and $v_1$ is a basis for it. Otherwise choose any $v_2 \in W$ such that $v_2 \notin \text{span}\{v_1\}$. Continue this procedure. That is given $v_1, \ldots, v_k \in W$, if $v_1, \ldots, v_k$ span $W$, then stop. Otherwise choose $v_{k+1} \in W$ arbitrary but not in the span of $v_1, \ldots, v_k$. By an exercise 16 $v_1, \ldots, v_k$ are linearly independent for every $k$. By the main theorem $k \leq n$ so this process must stop after $\leq n$ steps and when it stops we have found a basis for $W$. □
Exercise 24 Prove:
If \( \dim(V) = n \) and \( u_1, \ldots, u_m \in V \) are linearly independent, then we can extend this sequence to a basis of \( V \). That is, we can find \( u_{m+1}, u_{m+2}, \ldots, u_n \) so that \( u_1, \ldots, u_n \) is a basis for \( V \).

Exercise 25 If \( u_1, \ldots, u_n \) span a vector space \( V \), then there exists
\[
\{v_1, \ldots, v_m\} \subseteq \{u_1, \ldots, u_n\}
\]
such that \( v_1, \ldots, v_m \) is a basis for \( V \). In other words, any spanning set contains a basis.

Theorem 26 If \( W \) is a subspace of \( V \) and \( \dim(V) = n \), then \( \dim(W) = m \) for some \( m \leq n \). If \( m = n \) then \( W = V \).

proof:
Any basis for \( W \) is a set of \( m \) linearly independent vectors in \( V \). But the main theorem (18) implies that \( m \leq n \). Suppose \( m = n \), then let \( u_1, \ldots, u_n \) be a basis for \( W \). Since they are linearly independent vectors in \( V \), by Theorem 21 they must also be a basis for \( V \), and so
\[
W = \text{span}(\{u_1, \ldots, u_n\}) = V.
\]
\(\square\)