(1-23)
(A) Use Venn Diagrams to prove that
\[ A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \]

(B) Let \( A = \{1, 2, \ldots, n\} \). How many binary relations \( R \) on \( A \), (ie. \( R \subseteq A^2 \)), are there such that
1. (no conditions)
2. \( R \) is reflexive
3. \( (A, R) \) is a linear order
4. \( (A, R) \) is a graph, ie. \( R \) is irreflexive and symmetric
5. \( (A, R) \) is an equivalence relation with exactly two equivalence classes.

(1-25)
(A) Suppose \( f : A \to B \) and \( g : B \to C \) are functions. Prove
1. if \( f \) and \( g \) are 1-1, then \( g \circ f \) is 1-1.
2. if \( f \) and \( g \) are onto, then \( g \circ f \) is onto.
3. Show by examples that neither of the above implications reverse.

(B) Suppose \( A \) is a nonempty set. Prove the following are equivalent:
1. there is a 1-1 \( g : A \to \omega \)
2. there is an onto \( f : \omega \to A \).

(1-28)
(A) Prove that for every set \( X \) there is no map \( f : X \to P(X) \) which is onto.

(B) Let \( \mathbb{Q}[x] \) be the polynomials with rational coefficients, i.e.,
\[ \mathbb{Q}[x] = \{ p : \text{for some } n \in \omega, a_i \in \mathbb{Q} \ p = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n \} \]
Prove that \( \mathbb{Q}[x] \) is countable.
(C) Let $A \subseteq \mathbb{C}$ be the set of algebraic numbers. A complex number is algebraic iff it is the root of a nontrivial polynomial $p \in \mathbb{Q}[x]$. Prove that $A$ is countable.

(1-30)  
\hspace{1em} p.19 : 2,3,5

(2-1)  
\hspace{1em} p.27-29 : 5,9,10,14

(2-4)  
(A) Prove or disprove: For every WFF $\theta$ there exists a WFF $\theta^*$ which is logically equivalent to $\theta$ and does not contain the negation symbol, i.e., WFF which are strings of the symbols:

\{\lor, \land, \rightarrow, \leftrightarrow, (, A_1, A_2, A_3, \ldots)\}

(B) For each of the 16 binary logical connectives $\circ$, let $\text{WFF}_\circ$ be the set of well-formed formulas in the language $L = \{\circ, (, A_1, A_2, \ldots\}$. The connective $\circ$ is called adequate for propositional logic iff every WFF is logically equivalent to one in $\text{WFF}_\circ$. Determine (with proof) all adequate binary logical connectives.

(2-6)  
\hspace{1em} p.53-9 : It is not clear what $(A \leftrightarrow B \leftrightarrow C)$ means. Probably Enderton means $(A \leftrightarrow B) \leftrightarrow C$ or $(A \leftrightarrow (B \leftrightarrow C))$ which are logically equivalent. Every other mathematician would mean $((A \leftrightarrow B) \land (B \leftrightarrow C))$.
\hspace{1em} p.54-12

(2-8)  
\hspace{1em} prolog handout: 21,22,23

(2-11)  
(A). Suppose $\Sigma \subseteq WFF$ is complete and finitely satisfiable. Suppose $F \subseteq \Sigma$ is finite and $\theta$ is a WFF such that $F \models \theta$.
Prove that $\theta \in \Sigma$.

(2-13)  
(A). Suppose $\Lambda \subseteq WFF$ is complete and finitely satisfiable and $\theta$ and $\psi$ are logically equivalent WFFs. Prove that $\theta \in \Lambda$ iff $\psi \in \Lambda$.  

2
(2-15)
proplog handout: 24,25,26,27

(2-18)
proplog handout: 30,32,34

(2-20)
p.65-1,2,3

(2-22)
\[ U \text{ and } V \text{ are unary predicates, } x \text{ and } y \text{ distinct variables, and } \equiv \text{ means logically equivalent.} \]

Prove or disprove
(A) \((\exists x U(x)) \land (\exists x V(x)) \equiv \exists x(U(x) \land V(x))\)
(B) \((\forall x U(x)) \lor (\forall x V(x)) \equiv \forall x \forall y(U(x) \lor V(y))\)

(2-25)
p. 79 - 1,2,5

(2-27)
p. 100-104 : 9,16,18,27

(3-13)
p. 146 - 6,8,9

(3-20)
p. 145 - 3,7

(4-3)
p.100- 11,12,15

(4-8)
p.180- 1,4,5

Hint (1)
\((\mathbb{R}, \mathbb{Q}, <, ..) \models \forall x \forall y (x < y \rightarrow \exists q \mathbb{Q}(q) \land x < q < y)\)

Hint (4) If \( A \) is finite, then say \( A = \{a_1, a_2, \ldots, a_n\} \).
\((\mathbb{R}, A, ..) \models \forall x (A(x) \leftrightarrow (x = a_1 \lor x = a_2 \lor \cdots \lor x = a_n))\)
If $A$ is infinite, then $A$ is either unbounded or contains a limit point. Or you can use that $A$ contains an infinite sequence.

(4-12)

(A) In the language with one binary relation symbol and one unary relation (say $L = \{\leq, U\}$) prove that the following two structures are elementarily equivalent

$$(\mathbb{R}, \leq, \mathbb{Q}) \equiv (\mathbb{Q}, \leq, D_2)$$

where $D_2$ is the set of dyadic rational numbers:

$$D_2 = \left\{ \frac{m}{2^n} : m \in \mathbb{Z}, n = 1, 2, 3, \ldots \right\}$$

Prove that

$$(\mathbb{R}, \leq, \mathbb{Q}) \not\equiv (\mathbb{R}, \leq, \mathbb{Z})$$

(B) In the language of one binary relation let

$A = (\mathbb{R}^*, \approx)$ where $x \approx y$ iff $x$ and $y$ are infinitesimally close.

$B = (\mathbb{R}, \approx_\mathbb{Q})$ where $x \approx_\mathbb{Q} y$ iff $\exists q \in \mathbb{Q}$, $x = y + q$

$C = (\mathbb{Q}, \approx_\mathbb{Z})$ where $q \approx_\mathbb{Z} r$ iff $q - r \in \mathbb{Z}$

Prove that $A \equiv B \equiv C$

(C) In the same language let $D_k = (\mathbb{Z}, \equiv_k)$ for $k = 2, 3, \ldots$ where $m \equiv_k n$ iff $m - n$ is divisible by $k$ (i.e. $m = n \mod k$).

Prove that for every $k$, $D_k \not\equiv C$.

Prove that for every sentence $\theta$ if $C \models \theta$ then there exists $N$ such that for all $k \geq N$, $D_k \models \theta$.

Hint (A). Suppose the language of these structures is $\{\leq, U\}$ where $U$ is a unary predicate symbol. Let $\text{DLO}^*$ be the theory of dense linear orders without end points plus

$$\forall x \forall y (x < y) \rightarrow \exists u \exists v (U(u) \land \neg U(v) \land x < u < y \land x < v < y)$$

Use the Los-Vaught Test to prove that $\text{DLO}^*$ is a complete theory.

(B). Write down axioms $\Sigma$ which say that the binary relation is an equivalence relation, with infinitely many equivalence classes, and all equivalence classes are infinite. Prove the $\Sigma$ is complete by using the Los-Vaught Test.
(4-22)
Prove that the definable subsets of \((\omega, S, 0)\) are the finite and cofinite subsets of \(\omega\).

Hint: \((\omega, S, 0) \equiv (\omega, S, 0) + (\mathbb{Z}, S)\) or you can use elimination of quantifiers as in book.

(4-24)
Prove that \(\text{Th}(\omega, +, |)\) is undecidable where | is the binary predicate \(n|m\) iff \(n\) divides \(m\).

Hints: Show multiplication is definable in this structure. \(n^2 + 1\) is the least common multiple of \(n\) and \(n + 1\), \((a + b)^2 = a^2 + 2ab + b^2\)

(5-1)
(A) An integer \(x\) is square-free iff \(x \geq 2\) and no integer \(y \geq 2\) exists such that \(y^2\) divides \(x\). Let \(S(n)\) be the sum of the first \(n\) square-free integers. Prove that \(S : \omega \to \omega\) is primitive recursive.

(B) Prove there exists an integer \(n\) such that 5 divides \(n\), 7 divides \(n+1\), 9 divides \(n+2\), 11 divides \(n+3\), and 13 divides \(n+4\).