1. Let $H$ be the set of all integers which are either divisible by 3 or negative. Prove that $H$ is countable by constructing a map from $\mathbb{N} = \{1, 2, 3, \ldots\}$ onto $H$. (Be sure and prove that the map you construct is onto.)

2. Which of the following sets are countable?
   (a) $\mathbb{R}$
   (b) $\mathbb{Q}$
   (c) $\{0, 3, 5\}$
   (d) $\{\mathbb{R}, \mathbb{C}, \sqrt{2}\}$
   (e) the empty set
   (f) $\{x \in \mathbb{R} : x^2 = -1\}$
   (g) the set of points in the plane $\mathbb{R}^2$.
   (h) the set of all subsets of $\mathbb{N}$.
   (i) the set of all finite subsets of $\mathbb{N}$.
   (j) $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$

   (For this problem just answer countable or uncountable (ctble, unctble), you need not justify your answer.)

3. Give the truth table for the propositional sentence
   
   $$((A \rightarrow B) \lor C)$$

4. Let $\Sigma = \{A, (A \rightarrow B)\}$. For each of these prove or disprove:
   (a) $\Sigma \models B$
   (b) $\Sigma \models ((\neg A) \lor (\neg B))$

5. Find a sentence in disjunctive normal form which is logically equivalent to
   $$\neg(A \leftrightarrow B)$$
6. Prove or disprove:
Every well-formed formula of propositional logic is logically equivalent to one in which the only symbols are $\rightarrow$, $\neg$, parentheses, and propositional letters.

7. Suppose that $\Sigma$ is a set of formulas of propositional logic which is finitely satisfiable. Show that for any formula $\theta$ of propositional logic that either $\Sigma \cup \{\theta\}$ is finitely satisfiable or $\Sigma \cup \{\neg \theta\}$ is finitely satisfiable.
(Do not use the compactness theorem or any lemma used to prove compactness in your proof.)

8. A square in a graph $(V, E)$ is a set $\square = \{a, b, c, d\} \subseteq V$ of four distinct vertices such that $aEb$, $bEc$, $cEd$, $dEa$, $\neg aEc$, and $\neg bEd$. Let $n$ be a fixed positive integer.
Suppose that every finite subgraph of $V$ is the union of $n$ sets none of which contain a square. Show that $V$ is the union $n$ sets none of which contain a square.
(You may use the compactness theorem without proof.)

9. $R$ is a binary predicate symbol. Prove or disprove:
$(\forall y \exists x R(x, y)) \rightarrow (\exists x \forall y R(x, y))$ is a logical validity.

10. Let $\mathcal{L}_R$ be the language containing the binary relation symbol $R$. Write down a first order $\mathcal{L}_R$-sentence $\theta$ such that for any $\mathcal{L}_R$-structure $\mathfrak{A}$
$\mathfrak{A} \models \theta$ iff $\mathfrak{A}$ is a linear order with no greatest element.
Answers

1. Define $g(n) = 3n$ then $g$ maps $\mathbb{N}$ onto the positive multiples of 3. Define $h(n) = -n + 1$ then $h$ maps $\mathbb{N}$ onto the integers $\{0, -1, -2, \ldots\}$. Combine them into one map $f : \mathbb{N} \to H$ by the rule $f(2n) = g(n)$ and $f(2n-1) = h(n)$.

2. ucccccuccu

3. There is only one false line in the truth table. It is ABC - TFF.

4. a is true and b is false.

5. $(\neg A \land B) \lor (A \land \neg B)$

6. This is true. Note that $(\theta \lor \psi)$ is logically equivalent to $(\neg \theta) \rightarrow \psi$. Use induction to prove that every WFF equivalent to one using only $\rightarrow, \neg$.

7. (Do not use that $\Sigma$ is satisfiable in your proof. We only know this by the compactness theorem.) Suppose for contradiction that neither $\Sigma \cup \{\theta\}$ is finitely satisfiable nor $\Sigma \cup \{\neg \theta\}$ is finitely satisfiable. Then there must exists finite sets $\Sigma_0, \Sigma_1 \subseteq \Sigma$ such that $\Sigma_0 \cup \{\theta\}$ is not satisfiable and $\Sigma_1 \cup \{\neg \theta\}$ is not satisfiable.

But $\Sigma_0 \cup \Sigma_1 \subseteq \Sigma$ is finite. So by assumption it is satisfiable. Let $\nu$ be a truth evaluation which makes every $\psi$ in $\Sigma_0 \cup \Sigma_1$ true.

If $\nu(\theta) = T$, then $\Sigma_0 \cup \{\theta\}$ is satisfiable by $\nu$, which is a contradiction.

If $\nu(\theta) = F$, then $\nu(\neg \theta) = T$ and therefore $\Sigma_1 \cup \{\neg \theta\}$ is satisfiable by $\nu$, which is a contradiction.

Since either way leads to a contradiction the result is proved.

8. Let the set of propositional letters be the set of all $P^k_v$ for $v \in V$ and $k = 1, 2, \ldots, n$.

Define $\Sigma_1 = \{(P^1_v \lor P^2_v \lor \cdots \lor P^n_v) : v \in V\}$

Define $\Sigma_2 = \{\neg(P^k_a \land P^k_b \land P^k_c \land P^k_d) : \{a, b, c, d\} \text{ is a square in } V, 1 \leq k \leq n\}$

Define $\Sigma = \Sigma_1 \cup \Sigma_2$.
Claim. $\Sigma$ is finitely satisfiable.

proof: Given a finite $\Sigma_0 \subseteq \Sigma$. Let $V' \subseteq V$ be the set of all vertices mentioned in $\Sigma_0$. Since it is finite by assumption $V'$ is the union of $n$ square free sets, so let $V' = A_1 \cup A_2 \cup \ldots \cup A_n$ where each $A_i$ contains no square. Define a truth evaluation by $\nu(P_v^k) = T$ iff $v \in A_k$.

But then it is easy to check that $\nu(\psi) = T$ for each $\psi \in \Sigma_0$.

By the compactness theorem $\Sigma$ is satisfiable. So let $\nu$ be a truth evaluation such that $\nu(\Sigma) = T$. For each $k$ define the set $A_k$ by

$$A_k = \{v : \nu(P_v^k) = T\}$$

By the axioms in $\Sigma_1$ we see that for every $v \in V$ there exists a $k$ such that $v \in A_k$. By the axioms in $\Sigma_2$ we see that no $A_k$ contains a square. Hence $V$ is the union of $n$ sets, none of which contain a square.

9. This is not a logical validity. To show this exhibit a single specific structure in which it is false - don't natter on and on about this or that.

Counterexample: Let $A = \{0, 1\}$ and $R_A = \{(0, 0), (1, 1)\}$. Then

$$\mathfrak{A} \models (\forall y \exists x R(x, y))$$

since given $y$ taking $x = y$ always works. But

$$\mathfrak{A} \models \neg(\exists x \forall y R(x, y))$$

since neither $x = 0$ nor $x = 1$ works for all $y$.

10. Remember a sentence of first-order logic has no free variables.

$$\forall x \ R(x, x)$$
$$\forall x \forall y \ (R(x, y) \land R(y, x)) \rightarrow x = y$$
$$\forall x \forall y \forall z \ (R(x, y) \land R(y, z)) \rightarrow R(x, z)$$
$$\forall x \forall y \ R(x, y) \lor R(y, x)$$

These are the axioms of a linear order. To say that there is no greatest element: $\neg \exists x \forall y \ R(y, x)$ or $\forall x \exists y \ (R(x, y) \land x \neq y)$