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Introduction to Mathematical Logic

I have used these questions or some variations four times to teach a beginning graduate course in Mathematical Logic. I want to thank the many students who hopefully had some fun doing them, especially, Michael Benedikt, Tom Linton, Hans Mathew, Karl Peters, Mark Uscilka, Joan Hart, Stephen Mellendorf, Ganesan Ramalingam, Steven Schwalm, Garth Dickie, Garry Schumacher, Krung Sinapiromsaran, Stephen Young, Brent Hetherwick, Maciej Smuga-Otto, and Stephen Tanner.

Instructions

Do not read logic books during this semester, it is self-defeating. You will learn proofs you have figured out yourself and the more you have to discover yourself the better you will learn them. You will probably not learn much from your fellow student's presentations (although the one doing the presenting does). And you shouldn't! Those that have solved the problem should be sure that the presented solution is correct. If it doesn't look right it probably isn't. Don't leave this up to me, if I am the only one who objects I will stop doing it. For those that haven't solved the problem, you should regard the presented solution as a hint and go and write up for yourself a complete and correct solution. Also you might want to present it to one of your fellow students outside the classroom, if you can get one to listen to you.

The Moore Method

From P.R. Halmos[1]:

"What then is the secret–what is the best way to learn to solve problems? The answer is implied by the sentence I started with: solve problems. The method I advocate is sometimes known as the 'Moore method', because R.L. Moore developed and used it at the University of Texas. It is a method of teaching, a method of creating the problem-solving attitude in a student, that is a mixture of what Socrates taught us and the fiercely competitive spirit of the Olympic games."

From F.Burton Jones[2]:

"What Moore did: ... After stating the axioms and giving motivating examples to illustrate their meaning he would then state definitions and theorems. He simply read them from his book as the students copied them down. He would then instruct the class to find proofs of their own and also to construct examples to show that the hypotheses of the theorems could not be weakened, omitted, or partially omitted.

...

"When a student stated that he could prove Theorem x, he was asked to go to the blackboard and present the proof. Then the other students, especially those who hadn't been able to discover a proof, would make sure that the proof presented was correct and convincing. Moore sternly prevented heckling. This was seldom necessary because the whole atmosphere was one of a serious community effort to understand the argument."

From D.Taylor[3]:

"Criteria which characterize the Moore method of teaching include:

(1) The fundamental purpose: that of causing a student to develop his power at rational thought.

(2) Collecting the students in classes with common mathematical knowledge, striking from membership of a class any student whose knowledge is too advanced over others in the class.

[1] The teaching of problem solving, Amer. Math. Monthly, (1975)82, 466-470.

[2] The Moore method, Amer. Math. Monthly, (1977)84, 273-278.

[3] Creative teaching: heritage of R.L.Moore, University of Houston, 1972, QA 29 M6 T7, p149.

(3) Causing students to perform research at their level by confronting the class with impartially posed questions and conjectures which are at the limits of their capability.
(4) Allowing no collective effort on the part of the students inside or outside of class, and allowing the use of no source material.
(5) Calling on students for presentation of their efforts at settling questions raised, allowing a feeling of "ownership" of a theorem to develop.
(6) Fostering competition between students over the settling of questions raised.
(7) Developing skills of critical analysis among the class by burdening students therein with the assignment of "refereeing" an argument presented.
(8) Pacing the class to best develop the talent among its membership.
(9) Burdening the instructor with the obligation to not assist, yet respond to incorrect statements, or discussions arising from incorrect statements, with immediate examples or logically sound propositions to make clear the objection or understanding."

Taylor's (2) and (4) are a little too extreme for me. It is OK to collaborate with your fellow students as long as you give them credit. In fact, it is a good idea to try out your argument first by presenting it to fellow student. Avoid reading logic if you can, at least this semester, but if you do give a reference.

For more readings on the Moore method see:

Paul R. Halmos, What is Teaching?, Amer. Math. Monthly, 101 (1994), 848-854.

Donald R. Chalice, How to teach a class by the modified Moore method, Amer. Math. Monthly, 102 (1995), 317-321.

Quote From P.R. Halmos:

"A famous dictum of Pólya's about problem solving is that if you can't solve a problem, then there is an easier problem that you can't solve–find it!"

Propositional Logic and the Compactness Theorem

The *syntax* (grammar) of propositional logic is the following. The *logical symbols* are $\wedge, \vee, \neg, \rightarrow$, and $\iff$. The *nonlogical symbols* consist of an arbitrary nonempty set $\mathcal{P}$ that we assume is disjoint from the set of logical symbols to avoid confusion. The set $\mathcal{P}$ is referred to as the set of *atomic sentences* or as the set of *propositional letters*. For example, $\{P, Q, R\}$, $\{P_0, P_1, P_2, \ldots\}$, or $\{S_r : r \in \mathbb{R}\}$. The set of *propositional sentences* $\mathcal{S}$ is the smallest set of finite strings of symbols such that $\mathcal{P} \subseteq \mathcal{S}$, and if $\theta \in \mathcal{S}$ and $\psi \in \mathcal{S}$, then $\neg\theta \in \mathcal{S}, (\theta \wedge \psi) \in \mathcal{S}, (\theta \vee \psi) \in \mathcal{S}, (\theta \rightarrow \psi) \in \mathcal{S}$, and $(\theta \iff \psi) \in \mathcal{S}$.

The *semantics* (meaning) of propositional logic consists of truth evaluations. A *truth evaluation* is a function $e : \mathcal{S} \rightarrow \{T, F\}$, that is consistent with the following truth tables:

θ	ψ	$\neg\theta$	$(\theta \wedge \psi)$	$(\theta \vee \psi)$	$(\theta \rightarrow \psi)$	$(\theta \iff \psi)$
T	T	F	T	T	T	T
T	F	F	F	T	F	F
F	T	T	F	T	T	F
F	F	T	F	F	T	T

For example if $e(\theta) = T$ and $e(\psi) = F$, then $e(\theta \rightarrow \psi) = F$. Also $e(\neg\theta) = T$ iff $e(\theta) = F$. For example, if $\mathcal{P} = \{P_x : x \in \mathbb{R}\}$ and we define $e(P_x) = T$ if x is a rational and $e(P_x) = T$ if x is a irrational, then $e((P_2 \wedge \neg P_{\sqrt{2}})) = T$. However if we define $e'(P_x) = T$ iff x is an algebraic number, then $e'((P_2 \wedge \neg P_{\sqrt{2}})) = F$.

A sentence θ is called a *validity* iff for every truth evaluation e, $e(\theta) = T$. A sentence θ is called a *contradiction* iff for every truth evaluation e, $e(\theta) = F$.

We say that two sentences θ and ψ are *logically equivalent* iff for every truth evaluation e, $e(\theta) = e(\psi)$. A set of logical symbols is *adequate* for propositional logic iff every propositional sentence is logically equivalent to one whose only logical symbols are from the given set.

1.1 Define $\mathcal{S}_0 = \mathcal{P}$ the atomic sentences and define

$$\mathcal{S}_{n+1} = \mathcal{S}_n \cup \{\neg\theta : \theta \in \mathcal{S}_n\} \cup \{(\theta \# \psi) : \theta, \psi \in \mathcal{S}_n, \# \in \{\wedge, \vee, \rightarrow, \iff\}\}$$

Prove that $\mathcal{S} = \mathcal{S}_0 \cup \mathcal{S}_1 \cup \mathcal{S}_2 \cup \cdots$.

1.2 Prove that for any function $f : \mathcal{P} \to \{T, F\}$ there exists a unique truth evaluation $e : \mathcal{S} \to \{T, F\}$ such that $f = e \restriction \mathcal{P}$. The symbol $e \restriction \mathcal{P}$ stands for the restriction of the function e to $\mathcal{P}$.

1.3 Let θ and ψ be two propositional sentences. Show that θ and ψ are logically equivalent iff $(\theta \iff \psi)$ is a validity.

1.4 Suppose θ is a propositional validity, P and Q are two of the propositional letters occurring in θ, and ψ is the sentence obtained by replacing each occurrence of P in θ by Q. Prove that ψ is a validity.

1.5 Can you define $\vee$ using only $\to$? Can you define $\wedge$ using only $\to$?

1.6 Show that $\{\vee, \neg\}$ is an adequate set for propositional logic.

1.7 The definition of the logical connective *nor* ($\oplus$) is given by the following truth table:

θ	ψ	$(\theta \oplus \psi)$
T	T	F
T	F	F
F	T	F
F	F	T

Show that $\{\oplus\}$ is an adequate set for propositional logic.

1.8 (Sheffer) Find another binary connective that is adequate all by itself.

1.9 Show that $\{\neg\}$ is not adequate.

1.10 Show that $\{\vee\}$ is not adequate.

1.11 How many binary logical connectives are there? We assume two connectives are the same if they have the same truth table.

1.12 Show that there are exactly two binary logical connectives that are adequate all by themselves. Two logical connectives are the same iff they have the same truth table.

1.13 Suppose $\mathcal{P} = \{P_1, P_2, \ldots, P_n\}$. How many propositional sentences (up to logical equivalence) are there in this language?

1.14 Show that every propositional sentence is equivalent to a sentence in *disjunctive normal form*, i.e. a disjunction of conjunctions of atomic or the negation of atomic sentences:

$$\bigvee_{i=1}^{m} \Big(\bigwedge_{j=1}^{k_i} \theta_{ij} \Big)$$

where each θij is atomic or $\neg$atomic. The expression $\vee_{i=1}^{n} \psi_i$ abbreviates $(\psi_1 \vee (\psi_2 \vee (\cdots \vee (\psi_{n-1} \vee \psi_n))) \cdots)$.

In the following definitions and problems Σ is a set of propositional sentences in some fixed language and all sentences are assumed to be in this same fixed language. Σ is *realizable* iff there exists a truth evaluation e such that for all $\theta \in \Sigma$, $e(\theta) = T$. Σ is *finitely realizable* iff every finite subset of Σ is realizable. Σ is *complete* iff for every sentence θ in the language of Σ either θ is in Σ or $\neg\theta$ is in Σ.

1.15 Show that if Σ is finitely realizable and θ is any sentence then either $\Sigma \cup \{\theta\}$ is finitely realizable or $\Sigma \cup \{\neg\theta\}$ is finitely realizable.

1.16 Show that if Σ is finitely realizable and $(\theta \vee \psi)$ is in Σ, then either $\Sigma \cup \{\theta\}$ is finitely realizable or $\Sigma \cup \{\psi\}$ is finitely realizable.

1.17 Show that if Σ is finitely realizable and complete and if θ and $(\theta \to \psi)$ are both in Σ, then ψ is in Σ.

1.18 Show that if Σ is finitely realizable and complete, then Σ is realizable.

1.19 Suppose that the set of all sentences in our language is countable, e.g., $S = \{\theta_n : n = 0, 1, 2, \ldots\}$. Show that if Σ is finitely realizable, then there exists a complete finitely realizable Σ' with $\Sigma \subseteq \Sigma'$.

1.20 (**Compactness theorem for propositional logic**) Show that every finitely realizable Σ is realizable. You may assume there are only countably many sentences in the language.

A family of sets $\mathcal{C}$ is a *chain* iff for any X, Y in $\mathcal{C}$ either $X \subseteq Y$ or $Y \subseteq X$. The union of the family $\mathcal{A}$ is

$$\bigcup \mathcal{A} = \{b : \exists c \in \mathcal{A}, b \in c\}.$$

M is a *maximal* member of a family $\mathcal{A}$ iff $M \in \mathcal{A}$ and for every B if $B \in \mathcal{A}$ and $M \subseteq B$, then $M = B$. A family of sets $\mathcal{A}$ is closed under the unions of chains iff for every subfamily, $\mathcal{C}$, of $\mathcal{A}$ which is a chain the union of the chain, $\bigcup \mathcal{C}$, is also a member of $\mathcal{A}$.

Maximality Principle: Every family of sets closed under the unions of chains has a maximal member.

1.21 Show that the family of finitely realizable Σ is closed under unions of chains.

1.22 Show how to prove the compactness theorem without the assumption that there are only countably many sentences. (You may use the Maximality Principle.)

1.23 Suppose Σ is a set of sentences and θ is some sentence such that for every truth evaluation e if e makes all sentences in Σ true, then e makes θ true. Show that for some finite $\{\psi_1, \psi_2, \psi_3, \ldots \psi_n\} \subseteq \Sigma$ the sentence

$$(\psi_1 \wedge \psi_2 \wedge \psi_3 \wedge \cdots \wedge \psi_n) \to \theta$$

is a validity.

A *binary relation* R on a set A is a subset of $A \times A$. Often we write xRy instead of $\langle x, y\rangle \in R$. A binary relation $\leq$ on a set A is a *partial order* iff

a. (reflexive) $\forall a \in A\ a \leq a$;
b. (transitive) $\forall a, b, c \in A\ [(a \leq b \wedge b \leq c) \to a \leq c]$; and
c. (antisymmetric) $\forall a, b \in A\ [(a \leq b \wedge b \leq a) \to a = b]$.

Given a partial order $\leq$ we define the *strict order* $<$ by

$$x < y \iff (x \leq y \wedge x \neq y)$$

A binary relation $\leq$ on a set A is a *linear order* iff $\leq$ is a partial order and

d. (total) $\forall a, b \in A (a \leq b \vee b \leq a)$.

A binary relation R on a set A extends a binary relation S on A iff $S \subseteq R$.

1.24 Show that for every finite set A and partial order $\leq$ on A there exists a linear order $\leq^*$ on A extending $\leq$.

1.25 Let A be any set and let our set of atomic sentences $\mathcal{P}$ be:

$$\mathcal{P} = \{P_{ab} : a, b \in A\}$$

For any truth evaluation e define $\leq_e$ to be the binary relation on A defined by

$$a \leq_e b \text{ iff } e(P_{ab}) = T.$$

Construct a set of sentences Σ such that for every truth evaluation e,

e makes Σ true iff $\leq_e$ is a linear order on A.

1.26 Without assuming the set A is finite prove for every partial order $\leq$ on A there exists a linear order $\leq^*$ on A extending $\leq$.

In the next problems n is an arbitrary positive integer.

1.27 If $X \subseteq A$ and R is a binary relation on A then the restriction of R to X is the binary relation $S = R \cap (X \times X)$. For a partial order $\leq$ on A, a set $B \subseteq A$ is an $\leq$-chain iff the restriction of $\leq$ to B is a linear order. Show that given a partial order $\leq$ on A:

the set A is the union of less than n $\leq$-chains iff every finite subset of A is the union of less than n $\leq$-chains.

1.28 A partial order $\leq$ on a set A has *dimension* less than $n+1$ iff there exists n linear orders $\{\leq_1, \leq_2, \leq_3, \ldots, \leq_n\}$ on A (not necessarily distinct) such that:

$$\forall x, y \in A\ [x \leq y \text{ iff } (x \leq_i y \text{ for } i = 1, 2, \ldots, n)].$$

Show that a partial order $\leq$ on a set A has dimension less than $n+1$ iff for every finite X included in A the restriction of $\leq$ to X has dimension less than $n+1$.

1.29 A binary relation E (called the edges) on a set V (called the vertices) is a *graph* iff

a. (irreflexive) $\forall x \in V \neg xEx$; and

b. (symmetric) $\forall x, y \in V\ (xEy \to yEx)$.

We say x and y are adjacent iff xEy. (V', E') is a subgraph of (V, E) iff $V' \subseteq V$ and E' is the restriction of E to V'. For a graph (V, E) an n coloring is a map $c : V \to \{1, 2, \ldots, n\}$ satisfying $\forall x, y \in V (xEy \to c(x) \neq c(y))$, i.e. adjacent vertices have different colors. A graph (V, E) has *chromatic number* $\leq n$ iff there is a n coloring on its vertices. Show that a graph has chromatic number $\leq n$ iff every finite subgraph of it has chromatic number $\leq n$.

1.30 A triangle in a graph (V, E) is a set $\Delta = \{a, b, c\} \subseteq V$ such that aEb, bEc, and cEa. Suppose that every finite subset of V can be partitioned into n or fewer sets none of which contain a triangle. Show that V is the union of n sets none of which contain a triangle.

1.31 (Henkin) A *transversal* for a family of sets $\mathcal{F}$ is a one-to-one choice function. That is a one-to-one function f with domain $\mathcal{F}$ and for every $x \in \mathcal{F}$ $f(x) \in x$. Suppose that $\mathcal{F}$ is a family of finite sets such that for every finite $\mathcal{F}' \subseteq \mathcal{F}, \mathcal{F}'$

has a transversal. Show that $\mathcal{F}$ has a transversal. Is this result true if $\mathcal{F}$ contains infinite sets?

1.32 Let $\mathcal{F}$ be a family of subsets of a set X. We say that $\mathcal{C} \subseteq \mathcal{F}$ is an *exact cover* of $Y \subseteq X$ iff every element of Y is in a unique element of $\mathcal{C}$. Suppose that every element of X is in at most finitely many elements of $\mathcal{F}$. Show that there exists an exact cover $\mathcal{C} \subseteq \mathcal{F}$ of X iff for every finite $Y \subseteq X$ there exists $\mathcal{C} \subseteq \mathcal{F}$ an exact cover of Y. Is it necessary that every element of X is in at most finitely many elements of $\mathcal{F}$?

1.33 If $\mathcal{F}$ is a family of subsets of X and $Y \subseteq X$ then we say Y *splits* $\mathcal{F}$ iff for any $Z \in \mathcal{F}$, $Z \cap Y$ and $Z \setminus Y$ are both nonempty. Prove that if $\mathcal{F}$ is a family of finite subsets of X then $\mathcal{F}$ is split by some $Y \subseteq X$ iff every finite $\mathcal{F}' \subseteq \mathcal{F}$ is split by some $Y \subseteq X$. What if $\mathcal{F}$ is allowed to have infinite sets in it?

1.34 Given a set of students and set of classes, suppose each student wants one of a finite set of classes, and each class has a finite enrollment limit. Show that if each finite set of students can be accommodated, they all can be accommodated.

1.35 Show that the compactness theorem of propositional logic is equivalent to the statement that for any set I, the space 2^I, with the usual Tychonov product topology is compact, where $2 = \{0, 1\}$ has the discrete topology. (You should skip this problem if you do not know what a topology is.)

The Axioms of Set Theory

Here are some. The whole system is known as *ZF* for Zermelo-Fraenkel set theory. When the axiom of choice is included it is denoted *ZFC*. It was originally developed by Zermelo to make precise what he meant when he said that the well-ordering principal follows from the axiom of choice. Latter Fraenkel added the axiom of replacement. Another interesting system is GBN which is Gödel-Bernays-von Neumann set theory.

Empty Set:

$$\exists x \, \forall y (y \notin x)$$

The empty set is usually written $\emptyset$.

Extensionality:

$$\forall x \forall y (x = y \iff \forall z (z \in x \iff z \in y))$$

Hence there is only one empty set.

Pairing:

$$\forall x \forall y \exists z \forall u (u \in z \iff u = x \vee u = y)$$

We usually write $z = \{x, y\}$.

Union:

$$\forall x \, \exists y \, (\forall z (z \in y \iff (\exists u \, u \in x \wedge z \in u))$$

We usually write $y = \cup x$. $A \cup B$ abbreviates $\cup \{A, B\}$. $z \subseteq x$ is an abbreviation for $\forall u \, (u \in z \to u \in x)$.

Power Set:

$$\forall x \, \exists y \, \forall z (z \in y \iff z \subseteq x)$$

We usually write $y = P(x)$. For any set x , $x + 1 = x \cup \{x\}$.

Infinity:

$$\exists y \, (\emptyset \in y \wedge \forall x (x \in y \to x + 1 \in y))$$

The smallest such y is denoted ω, so $\omega = \{0, 1, 2, \ldots\}$.

Comprehension Scheme:

$$\forall z \exists y \forall x [x \in y \iff (x \in z \wedge \theta(x))]$$

The comprehension axiom is being invoked when we say given z let

$$y = \{x \in z : \theta(x)\}.$$

The formula θ may refer to z and to other sets, but not to y. In general given a formula $\theta(x)$ the family $\{x : \theta(x)\}$ is referred to as a *class*, it may not be a set. For example, the class of all sets is

$$\mathbf{V} = \{x : x = x\}.$$

Classes that are not sets are referred to as *proper classes*. Every set a is a class, since the formula "$x \in a$" defines it. The comprehension axioms say that the intersection of a class and a set is a set. We use **boldface** characters to refer to classes.

2.1 Define $X \cap Y$, $X \setminus Y$, and $\bigcap X$ and show they exist.

2.2 The *ordered pair* is defined by

$$\langle x, y\rangle = \{\{x\}, \{x, y\}\}.$$

Show it exists. Show the $\langle x, y\rangle = \langle u, v\rangle$ iff $x = u$ and $y = v$.

2.3 The *cartesian product* is defined by

$$X \times Y = \{\langle x, y\rangle : x \in X \text{ and } y \in Y\}.$$

Show it exists.

2.4 A function is identified with its graph. For any sets X and Y we let Y^X be the set of all functions with domain X and range Y. Show this set exists.

2.5 Given a function $f : A \mapsto B$ and set $C \subseteq A$ the *restriction* of f to C, written $f \restriction C$ is the function with domain C and equal to f everywhere in C. Show that it exists. $f''C$ is the set of all elements of B that are in the image of C. Show that it exists.

2.6 Prove ω exists (i.e. that there does exist a smallest such y). Prove for any formula $\theta(x)$ if $\theta(0)$ and $\forall x \in \omega(\theta(x) \to \theta(x + 1))$, then $\forall x \in \omega\, \theta(x)$.

2.7 Suppose $G : Z \to Z$. Show that for any $x \in Z$ there exists a unique $f : \omega \to Z$ such that $f(0) = x$ and for all $n \in \omega$ $f(n + 1) = G(f(n))$.

2.8 Let (V, E) be a graph. Informally, two vertices in any graph are *connected* iff (either they are the same or) there is a finite path using the edges of the graph connecting one to the other. Use the preceding problem to formally define and prove that the relation x is connected to y, written $x \sim y$, exists and is an equivalence relation on V. Equivalence classes of $\sim$ are called the *components* of the graph.

2.9 Let A and B be disjoint sets and $f : A \to B$ and $g : B \to A$ be one-to-one functions. Consider the *bipartite graph* V which has vertices $A \cup B$ and edges given by the union of the graphs of f and g, i.e., there is an edge between $a \in A$ and $b \in B$ iff either $f(a) = b$ or $g(b) = a$. Describe what the finite components of V must look like as a subgraph of V. Describe the infinite components of V.

2.10 Define $|X| = |Y|$ iff there is a one-to-one onto map from X to Y. We say X and Y have the same cardinality. Define $|X| \leq |Y|$ iff there is a one-to-one map from X to Y. Define $|X| < |Y|$ iff $|X| \leq |Y|$ and $|X| \neq |Y|$. (Cantor-Shröder-Bernstein) Show that if $|A| \leq |B|$ and $|B| \leq |A|$ then $|A| = |B|$.

2.11 Show that $|A| \leq |A|$. Show that if $|A| \leq |B|$ and $|B| \leq |C|$ then $|A| \leq |C|$.

2.12 Show that $|P(X)| = |\{0, 1\}^X|$.

2.13 (Cantor) Show that $|X| < |P(X)|$.

2.14 Show that the class of sets, $\mathbf{V}$, is not a set.

2.15 Show that $|A \times (B \times C)| = |(A \times B) \times C|$.

2.16 Show that $|A^{B \times C}| = |(A^B)^C|$.

2.17 Show that if there is a function $f : A \mapsto B$ that is onto, then $|B| \leq |A|$. [4]

A set is finite iff it can be put into one-to-one correspondence with an element of ω A set is *countable* iff it is either finite or of the same cardinality as ω. A set is *uncountable* iff it is not countable. $\mathbb{R}$ is the set of real numbers and we use $\mathfrak{c} = |\mathbb{R}|$ to denote its cardinality which is also called the cardinality of the continuum. Below, you may use whatever set theoretic definitions of the integers, rationals and real numbers that you know. For example, you may regard the reals as either Dedekind cuts in the rationals, equivalences

[4] This requires the axiom of choice. It is open if it is equivalent to AC.

classes of Cauchy sequences of rationals, infinitely long decimals, or ?points on a line.

2.18 Show that the set of integers $\mathbb{Z}$ is countable.

2.19 Show that the set of odd positive integers is countable.

2.20 Show that the set of points in the plane with integer coordinates is countable.

2.21 Show that the countable union of countable sets is countable. [5]

2.22 For any set X let $[X]^{<\omega}$ be the finite subsets of X. Show that the set of finite subsets of ω, which is written $[\omega]^{<\omega}$, is countable.

2.23 Show that if there are only countably many atomic sentences then the set of all propositional sentences is countable.

2.24 Show that the set of rationals $\mathbb{Q}$ is countable.

2.25 A number is *algebraic* iff it is the root of some polynomial with rational coefficients. Show that the set of algebraic numbers is countable.

2.26 Show that any nontrivial interval in $\mathbb{R}$ has cardinality $\mathfrak{c}$.

2.27 Show that $P(\omega)$ has cardinality $\mathfrak{c}$.

2.28 Show that the set of all infinite subsets of ω, which is written $[\omega]^{\omega}$, has cardinality $\mathfrak{c}$.

2.29 Show that the cardinality of $\mathbb{R} \times \mathbb{R}$ is $\mathfrak{c}$.

2.30 For any set X let $[X]^{\omega}$ be the countably infinite subsets of X. Show that $|\mathbb{R}^{\omega}| = |[\mathbb{R}]^{\omega}| = \mathfrak{c}$. [6]

2.31 Show that the cardinality of the set of open subsets of $\mathbb{R}$ is $\mathfrak{c}$.

2.32 Show that the set of all continuous functions from $\mathbb{R}$ to $\mathbb{R}$ has size $\mathfrak{c}$.

2.33 Show that ω^{ω} has cardinality $\mathfrak{c}$.

2.34 Show that the set of one-to-one, onto functions from ω to ω has cardinality $\mathfrak{c}$.

2.35 Show that there is a family $\mathcal{A}$ of subsets of $\mathbb{Q}$ such that $|\mathcal{A}| = \mathfrak{c}$ and for any two distinct $s, t \in \mathcal{A}$ the set $s \cap t$ is finite. $\mathcal{A}$ is called an *almost disjoint family*.

[5]Do you think you needed to use the Axiom of Choice?

[6]See previous footnote.

2.36 Show that there is a family $\mathcal{F}$ of functions from ω to ω such that $|\mathcal{F}| = \mathfrak{c}$ and for any two distinct $f, g \in \mathcal{F}$ the set $\{n \in \omega : f(n) = g(n)\}$ is finite. These functions are called *eventually different.*

Wellorderings

A linear order $(L, \leq)$ is a *wellorder* iff for every nonempty $X \subseteq L$ there exists $x \in X$ such that for every $y \in X$ $x \leq y$ (x is the minimal element of X). For an ordering $\leq$ we use $<$ to refer to the strict ordering, i.e $x < y$ iff $x \leq y$ and not $x \neq y$. We use $>$ to refer to the converse order, i.e. $x > y$ iff $y < x$.

3.1 Let $(L, \leq)$ be a well ordering. Let $(L \times L, \leq')$ be defined in one of the following ways:
a. $(x, y) \leq' (u, v)$ iff $x < u$ or ($x = u$ and $y \leq v$)
b. $(x, y) \leq' (u, v)$ iff $x \leq u$ and $y \leq v$
c. $(x, y) \leq' (u, v)$ iff $max\{x, y\} < max\{u, v\}$ or $[max\{x, y\} = max\{u, v\}$ and ($x < u$ or ($x = u$ and $y \leq v$))].
Which are well-orderings?

3.2 Prove: Let $(L, \leq)$ be any well-ordering and $f : L \to L$ an increasing function $(\forall x, y \in L\ (x < y \to f(x) < f(y)))$. Then for every x in L $x \leq f(x)$.

3.3 For two binary relations R on A and S on B we write $(A, R) \simeq (B, S)$ iff there exists a one-to-one onto map $f : A \to B$ such that

for every x, y in A (xRy iff $f(x)Sf(y)$).

Such a map is called an *isomorphism*. If $(L_1, \leq_1)$ and $(L_2, \leq_2)$ are well-orders and $(L_1, \leq_1) \simeq (L_2, \leq_2)$ then show the isomorphism is unique. Is this true for all linear orderings?

3.4 Let $(L, \leq)$ be a wellorder and for any $a \in L$ let $L_a = \{c \in L : c < a\}$. Show that $(L, \leq)$ is not isomorphic to $(L_a, \leq)$ for any $a \in L$.

3.5 (Cantor) Show that for any two wellorders exactly one of the following occurs: they are isomorphic, the first is isomorphic to an initial segment of the second, or the second is isomorphic to an initial segment of the first.

3.6 Let $(A, \leq)$ be a linear order such that

$$\forall X \subseteq A (X \simeq A \text{ or } (A \setminus X) \simeq A).$$

Show that A is a well order or an inverse well order.

3.7 Show that a linear order $(L, \leq)$ is a wellorder iff there does not exist an infinite sequence x_n for $n = 0, 1, 2, \ldots$ with $x_{n+1} < x_n$ for every n. Does this use AC?

Axiom of Choice

(AC) *Axiom of Choice*: For every family F of nonempty sets there exists a choice function, i.e. a function f with domain F such that for every x in F, $f(x) \in x$.

(WO) *Well-ordering Principle* : Every nonempty set can be well ordered.

(TL) *Tuckey's Lemma*: Every family of sets with finite character has a maximal element. A family of sets F has finite character iff for every set X, $X \in F$ iff for every finite $Y \subseteq X$, $Y \in F$. (MP) *Maximality Principle*: Every family

of sets closed under the unions of chains has a maximal member.

(ZL) *Zorn's Lemma*: Every family of sets contains a maximal chain.

4.1 Show that ZL implies MP.

4.2 Show that MP implies TL.

4.3 Show that TL implies AC.

4.4 (Zermelo) Show that AC implies WO.

4.5 Given a nonempty family $\mathcal{F}$ let $<$ be a strict well-ordering of $\mathcal{F}$. Say that a chain $\mathcal{C} \subseteq \mathcal{F}$ is greedy iff for every $a \in \mathcal{F}$ if

$$\{b \in \mathcal{C} : b < a\} \cup \{a\}$$

is a chain, then either $a \in \mathcal{C}$ or $b < a$ for every $b \in \mathcal{C}$. Show that the union of all greedy chains is a maximal chain. Conclude that WO implies ZL.

4.6 Given a nonempty set X let $*$ be a point not in X and let $Y = X \cup \{*\}$. Give Y the topology where the open sets are $\{\emptyset, Y, X, \{*\}\}$. Prove that Y is a compact topological space and X is a closed subspace of Y.

4.7 (Kelley) The product of a family of sets is the same as the set of all choice functions. Show that Tychonov's Theorem that the product of compact spaces is compact implies the Axiom of Choice.

Ordinals

A set X is *transitive* iff $\forall x \in X\,(x \subseteq X)$. A set α is an *ordinal* iff it is transitive and strictly well ordered by the membership relation (define $x \leq y$ iff $x \in y$ or $x = y$, then $(\alpha, \leq)$ is a wellordering). We also include the empty set as an ordinal. For ordinals α and β we write $\alpha < \beta$ for $\alpha \in \beta$. The first infinite ordinal is written ω. We usually write

$$0 = \emptyset, 1 = \{\emptyset\}, 2 = \{\emptyset, \{\emptyset\}\}, \ldots, n = \{0, 1, \ldots, n-1\}, \cdots, \omega = \{0, 1, 2, \ldots\}$$

5.1 Show: If α is an ordinal then so is $\alpha + 1$. (Remember $\alpha + 1 = \alpha \cup \{\alpha\}$.) Such ordinals are called *successor ordinals*. Ordinals that are not successors are called *limit ordinals*.

5.2 Show: If α is an ordinal and $\beta < \alpha$, then β is an ordinal and $\beta \subseteq \alpha$ and $\beta = \{\gamma \in \alpha : \gamma \in \beta\}$.

Axiom of Regularity:

$$\forall x\, x \neq \emptyset \rightarrow \exists y (y \in x \wedge \neg \exists z\, (z \in y \wedge z \in x))$$

Another way to say this is that the binary relation $R = \{(u, v) \in x \times x : u \in v\}$ has a minimal element, i.e., there exist z such that for every $y \in x$ it is not the case that zRy. Note: a minimal element is not the same as a least element.

5.3 Show α is an ordinal iff α is transitive and linearly ordered by the membership relation.

5.4 For any ordinals α and β show that $\alpha \cap \beta = \alpha$ or $\alpha \cap \beta = \beta$. Show any two ordinals are comparable, i.e., for any two distinct ordinals α and β either $\alpha \in \beta$ or $\beta \in \alpha$.

5.5 The union of set A of ordinals is an ordinal, and is $sup(A)$.

5.6 Show that the intersection of a nonempty set A of ordinals is the least element of A, written $inf(A)$. Hence any nonempty set of ordinals has a least element.

5.7 Prove transfinite induction: Suppose $\phi(0)$ and $\forall \alpha \in \mathbf{ORD}$ if $\forall \beta < \alpha\ \phi(\beta)$, then $\phi(\alpha)$. Then $\forall \alpha \in \mathbf{ORD}\ \phi(\alpha)$.

Replacement Scheme Axioms:

$$\forall a\ ([\forall x \in a \exists! y\, \psi(x, y)] \rightarrow \exists b \forall x \in a \exists y \in b\ \psi(x, y))$$

The formula ψ may refer to a and to other sets but not to b. Replacement says that for any function that is a class the image of a set is a set. If $\mathbf{F}$ is a function, then for any set a there exists a set b such that for every $x \in a$ there exists a $y \in b$ such that $\mathbf{F}(x) = y$.

5.8 (von Neumann) Let $(L, \leq)$ be any well-ordering. Show that the following is a set:

$$\{(x, \alpha) : x \in L, \alpha \in \mathbf{ORD}, \text{ and } (L_x, \leq) \simeq \alpha\}.$$

Show that every well ordered set is isomorphic to a unique ordinal.

Let $\mathbf{ORD}$ denote the class of all ordinals.

Transfinite Recursion:

If $\mathbf{F}$ is any function defined on all sets then there exists a unique function $\mathbf{G}$ with domain $\mathbf{ORD}$ such that for every α in $\mathbf{ORD}$ $\mathbf{G}(\alpha) = \mathbf{F}(\mathbf{G} \restriction \alpha)$.

This is also referred to as a transfinite construction of $\mathbf{G}$.

5.9 Suppose $\mathbf{F} : \mathbf{V} \to \mathbf{V}$, i.e., a class function. Define g (an ordinary set function) to be a good guess iff $\text{dom}(g) = \alpha \in \mathbf{ORD}$, $g(0) = \mathbf{F}(\emptyset)$, and $g(\beta) = \mathbf{F}(g \restriction \beta)$ for every $\beta < \alpha$. Show that if g is a good guess, then $g \restriction \beta$ is good guess for any $\beta < \alpha$.

5.10 Show that if g and g' are good guesses with the same domain, then $g = g'$.

5.11 Show that for every $\alpha \in \mathbf{ORD}$ there exists a (necessarily unique) good guess g with domain α.

5.12 (Fraenkel) Prove transfinite recursion.

5.13 Explain the proof of WO implies ZL in terms of a transfinite construction.

For an example, consider the definition of V_α for every ordinal α. Let $V_0 = \emptyset$, $V_{\alpha+1} = P(V_\alpha)$ (power set) for successor ordinals, and for limit ordinals $V_\lambda = \cup_{\beta<\lambda} V_\beta$. Thus if we define $\mathbf{F}$ as follows:

$$\mathbf{F}(x) = \begin{cases} P(x(\alpha)) & \text{if } x \text{ is a function with domain } \alpha + 1 \in \mathbf{ORD} \\ \bigcup_{\alpha<\lambda} x(\alpha) & \text{if } x \text{ is a function with domain limit ordinal } \lambda \\ P(\emptyset) & \text{if } x = \emptyset \\ -\pi & \text{otherwise} \end{cases}$$

then $\mathbf{G}(\alpha) = V_\alpha$.

5.14 Show that if $\alpha \leq \beta$, then $V_\alpha \subseteq V_\beta$ and if $\alpha < \beta$, then $V_\alpha \in V_\beta$. Show that each V_α is transitive.

5.15 Show that every set is included in a transitive set. Show that for every transitive set x there exists an ordinal α such that $x \in V_\alpha$, i.e., $\mathbf{V} = \cup_{\alpha \in \mathbf{ORD}} V_\alpha$.

Ordinal arithmetic: (α, β are ordinals, and λ is a limit ordinal.)

Addition:
$\alpha + 0 = \alpha$
$\alpha + (\beta + 1) = (\alpha + \beta) + 1$
$\alpha + \lambda = sup\{\alpha + \beta : \beta < \lambda\}$

Multiplication:
$\alpha 0 = 0$
$\alpha(\beta + 1) = (\alpha\beta) + \alpha$
$\alpha\lambda = sup\{\alpha\beta : \beta < \lambda\}$

Exponentiation:
$\alpha^0 = 1$
$\alpha^{\beta+1} = \alpha^\beta \alpha$
$\alpha^\lambda = sup\{\alpha^\beta : \beta < \lambda\}$

So for example the addition function $+ : \mathbf{ORD} \times \mathbf{ORD} \mapsto \mathbf{ORD}$ exists by transfinite recursion. For each $\alpha \in \mathbf{ORD}$ define a function $\mathbf{F}_\alpha$ on all sets as follows:

$\mathbf{F}_\alpha(g) = g(\beta) + 1$ if g is a map with domain an ordinal $\beta + 1$,

$\mathbf{F}_\alpha(g) = sup\{g(\gamma) : \gamma < \lambda\}$ if g is a map with domain a limit ordinal λ, and

$\mathbf{F}_\alpha(g) = \alpha$ otherwise.

Hence for each α we have a unique $\mathbf{G}_\alpha : \mathbf{ORD} \mapsto \mathbf{ORD}$ which will exactly be $\mathbf{G}_\alpha(\beta) = \alpha + \beta$. Since each $\mathbf{G}_\alpha$ is unique we have defined the function $+$ on all pairs of ordinals.

Note that more intuitively $\alpha + \beta$ is the unique ordinal isomorphic to the well-order $(\{0\} \times \alpha) \cup (\{1\} \times \beta)$ ordered lexicographically. Similarly $\alpha \cdot \beta$ is the unique ordinal isomorphic to the well-order $\beta \times \alpha$ ordered lexicographically. Exponentiation is much harder to describe.

5.16 Show that $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$.

5.17 Assume α, β, and γ are ordinals, which of the following are always true?

$$
\begin{array}{rcl}
\alpha+\beta & = & \beta+\alpha \\
\alpha+(\beta+\gamma) & = & (\alpha+\beta)+\gamma \\
\alpha\beta & = & \beta\alpha \\
\alpha(\beta\gamma) & = & (\alpha\beta)\gamma \\
\alpha(\beta+\gamma) & = & (\alpha\beta)+(\alpha\gamma) \\
(\alpha+\beta)\gamma & = & (\alpha\gamma)+(\beta\gamma) \\
\alpha^{\beta\gamma} & = & (\alpha^{\beta})^{\gamma} \\
\alpha^{\beta}\alpha^{\gamma} & = & \alpha^{\beta+\gamma} \\
(\alpha\beta)^{\gamma}=\alpha^{\gamma}\beta^{\gamma} & & \\
\alpha+\beta=\alpha+\gamma & \rightarrow & \beta=\gamma \\
\beta<\gamma & \iff & \alpha+\beta<\alpha+\gamma \\
\beta+\alpha=\gamma+\alpha & \rightarrow & \beta=\gamma \\
(\alpha>0\ \wedge\ \alpha\beta=\alpha\gamma) & \rightarrow & \beta=\gamma \\
(\alpha>1\ \wedge\ \beta<\gamma) & \rightarrow & \alpha^{\beta}<\alpha^{\gamma}
\end{array}
$$

5.18 For any ordinals α and $\beta > 0$ show there exists unique ordinals γ and δ such that $\alpha = \beta\gamma + \delta$ and $\delta < \beta$.

5.19 Is the previous problem true for $\alpha = \gamma\beta + \delta$?

5.20 Show for any $\beta > 0$ there exists unique ordinals $\gamma_1, \ldots, \gamma_n, d_1, \ldots, d_n$ such that $\gamma_1 > \gamma_2 > \cdots > \gamma_n$; $0 < d_1, d_2, \ldots, d_n < \omega$ and

$$\beta = \omega^{\gamma_1}d_1 + \omega^{\gamma_2}d_2 + \ldots + \omega^{\gamma_n}d_n$$

This is called *Cantor normal form.*

5.21 Sort the following set of five ordinals:

$$
\begin{array}{ccc}
(\omega^{\omega})(\omega+\omega) & (\omega+\omega)(\omega^{\omega}) & \omega^{\omega}\omega+\omega^{\omega}\omega \\
\omega\omega^{\omega}+\omega\omega^{\omega} & \omega^{\omega}\omega+\omega\omega^{\omega} &
\end{array}
$$

5.22 An ordinal α is *indecomposable* iff it satisfies any of the following.

a) $\exists\beta\ \alpha = \omega^{\beta}$

b) $\forall\beta\forall\gamma$ if $\alpha = \beta + \gamma$ then $\alpha = \beta$ or $\alpha = \gamma$
c) $\forall X \subseteq \alpha$ $[(X, <) \simeq (\alpha, <)$ or $(\alpha \setminus X, <) \simeq (\alpha, <)]$
d) $\forall\beta < \alpha$ $\beta + \alpha = \alpha$
Show they are all equivalent.

5.23 (Goodstein) The complete expansion of a positive integer in a base is gotten by writing everything possible including exponents and exponents of exponents etc. as powers of the base. For example the number 36 written in complete base two is:

$$2^{(2^2+1)} + 2^2$$

The same number in complete base 3 is:

$$3^3 + 3^2$$

Let a_n be a sequence described as follows. Given a_k calculate a_{k+1} by writing a_k in base k then substitute k+1 for every k, then subtract one. For example:

$$a_2 = 36 = 2^{(2^2+1)} + 2^2 \to a_3 = 3^{\left(3^3+1\right)} + 3^3 - 1 = 2.2876\ldots \times 10^{13}$$

or for example if $a_6 = 6^4 + 2 \cdot 6^3 = 1728$, then

$$a_7 = (7^4 + 2 \cdot 7^3) - 1 = 7^4 + 7^3 + 6 \cdot 7^2 + 6 \cdot 7^1 + 6 = 3086$$

Show that given any pair of positive integers n and m, if we let $a_n = m$ then for some $k > n$ we get $a_k = 0$.

5.24 Let S be a countable set of ordinals. Show that

$$\{\beta : \exists\langle\alpha_n : n \in \omega\rangle \in S^\omega \;\; \beta = \Sigma_{n<\omega}\alpha_n\}$$

is countable. $(\Sigma_{n<\omega}\alpha_n = sup\{\Sigma_{n<m}\alpha_n : m < \omega\})$

Cardinal Arithmetic

An ordinal κ is a *cardinal* iff for every $\alpha < \kappa$, $|\alpha| < |\kappa|$. The cardinality of a set A is the least cardinal κ, $|A| = |\kappa|$ and we write $|A| = \kappa$. The α^{th} uncountable cardinal is written either $\aleph_\alpha$ or ω_α.

6.1 Show that for any ordinal α the cardinal $\aleph_\alpha$ exists.

6.2 Is there a cardinal such that $\aleph_\alpha = \alpha$?

For cardinals κ and γ we define $\kappa\gamma$ to be the cardinality of the cross product and $\kappa + \gamma$ to be the cardinality of the union of A and B where A and B are disjoint and $|A| = \kappa$ and $|B| = \gamma$.

6.3 Let κ be an infinite cardinal. Define the *lexicographical order* $\leq_l$ on $\kappa \times \kappa$ by $(x, y) \leq_l (u, v)$ iff $x < u$ or $(x = u$ and $y \leq v)$. Define $\leq'$ on $\kappa \times \kappa$ by $(x, y) \leq' (u, v)$ iff $max\{x, y\} < max\{u, v\}$ or $[(max\{x, y\} = max\{u, v\}$ and $(x, y) \leq_l (u, v)]$. Show that $(\kappa, \leq) \simeq (\kappa \times \kappa, \leq')$.

6.4 Show that for infinite cardinals κ and γ, $\kappa + \gamma = \kappa\gamma = max\{\kappa, \gamma\}$.

6.5 Show that for any infinite cardinal κ the union of κ many sets of cardinality κ has cardinality κ.

The *cofinality* of an infinite limit ordinal β, $cf(\beta)$, is the least $\alpha \leq \beta$ such that there is a map $f : \alpha \to \beta$ whose range is unbounded in β.

6.6 For λ a limit ordinal show the following are all equivalent:

a. α is the minimum ordinal such that $\exists X \subseteq \lambda$ unbounded in λ such that $(X, \leq) \simeq (\alpha, \leq)$.

b. α is the minimum ordinal such that $\exists f : \alpha \to \lambda$ such that f is one-to-one, order preserving, and the range of f is cofinal (unbounded) in λ.

c. $cf(\lambda) = \alpha$

6.7 For κ an infinite cardinal show that $cf(\kappa) = \alpha$ iff α is the minimum cardinal such that κ is the union of α many sets of cardinality less than κ.

6.8 Let α and β be limit ordinals and suppose $f : \alpha \to \beta$ is strictly increasing and cofinal in β. Show $cf(\alpha) = cf(\beta)$.

6.9 κ is *regular* iff $cf(\kappa) = \kappa$. κ is *singular* iff $cf(\kappa) < \kappa$. Show that for any limit ordinal β, $cf(\beta)$ is a regular cardinal.

6.10 κ^+ is the least cardinal greater than κ. Show that for any infinite cardinal κ , κ^+ is a regular cardinal.

6.11 For α a limit ordinal show that $cf(\aleph_\alpha) = cf(\alpha)$.

κ^γ is the set of all functions from γ to κ, but we often use it to denote its own cardinality. $\kappa^{<\gamma}$ is $\cup\{\kappa^\alpha : \alpha < \gamma\}$, but we often use it to denote its own cardinality, i.e., $\kappa^{<\gamma} = |\kappa^{<\gamma}|$. **Note in the section all exponentiation is cardinal exponentiation.**

6.12 Show that $|\kappa^{<\omega}| = |[\kappa]^{<\omega}| = \kappa$ for any infinite cardinal κ. $[\kappa]^{<\omega}$ is the set of finite subsets of κ.

6.13 Show that $\aleph_\omega < \aleph_\omega^\omega$. Show that for any cardinal κ, $\kappa < \kappa^{cf(\kappa)}$.

6.14 (König) Show that $cf(2^\kappa) > \kappa$.

6.15 Show $(\forall n \in \omega\ 2^{\aleph_n} = \aleph_{n+1}) \to 2^{\aleph_\omega} = (\aleph_\omega)^\omega$.

6.16 Show that $(2^{<\kappa})^{cf(\kappa)} = 2^\kappa$.

6.17 Show $(\forall n \in \omega\ 2^{\aleph_n} = \aleph_{\omega+17}) \to 2^{\aleph_\omega} = \aleph_{\omega+17}$.

6.18 Let κ be the least cardinal such that $2^\kappa > 2^\omega$. Show that κ is regular.

6.19 Prove that for every infinite regular cardinal κ, there is a cardinal λ such that $\aleph_\lambda = \lambda$ and λ has cofinality κ.

6.20 Show that $cf(2^{<\kappa}) = cf(\kappa)$ or $cf(2^{<\kappa}) > \kappa$.

6.21 Show that if $\omega \le \lambda \le \kappa$ then $(\kappa^+)^\lambda = max\{\kappa^\lambda, \kappa^+\}$.

6.22 For any set X *Hartog's ordinal* $h(X)$ is defined by:

$$h(X) = sup\{\alpha \in ORD : \exists \,\text{onto}\, f : X \to \alpha\}$$

Show without AC that h is well defined and with AC $h(X) = |X|^+$. Show that AC is equivalent to the statement that for every two sets X and Y either $|X| \le |Y|$ or $|Y| \le |X|$.

6.23 Given sets $A_\alpha \subseteq \kappa$ for each $\alpha < \kappa$ each of cardinality κ, show there exists $X \subseteq \kappa$ such that

$$\forall \alpha < \kappa \ \ |A_\alpha \cap X| = |A_\alpha \setminus X| = \kappa$$

6.24 Show there exists $X \subseteq \mathbb{R}$ which contains the rationals and the only automorphism of $(X, \leq)$ is the identity, i.e., any order preserving bijection from X to X is the identity.

6.25 Show there exists $X \subseteq \mathbb{R}^2$ such that for every $x \in X$ and positive $r \in \mathbb{R}$ there is a unique $y \in X$ with $d(x, y) = r$ where d is Euclidean distance.

6.26 (Sierpiński) Show that there exists $X \subseteq \mathbb{R}^2$ such that for every line L in the plane, $|L \cap X| = 2$.

6.27 (Jech) Without using AC show that ω_2 is not the countable union of countable sets.

First Order Logic and the Compactness Theorem

Syntax

We begin with the syntax of first order logic. The *logical symbols* are $\vee, \neg, \exists, =$ and for each $n \in \omega$ a variable symbol x_n. There are also grammatical symbols such as parentheses and commas that we use to parse things correctly but have no meaning. For clarity we usually use x, y, z, u, v, etc. to refer to arbitrary variables. The *nonlogical symbols* consist of a given set L that may include operation symbols, predicate symbols, and constant symbols. The case where L is empty is referred to as the *language of pure equality*. Each symbol $s \in L$ has a nonnegative integer $\#(s)$ called its *arity* assigned to it. If $\#(s) = 0$, then s is a constant symbol. If f is an operation symbol and $\#(f) = n$ then f is an n-ary operation symbol. Similarly if R is a predicate symbol and $\#(R) = n$ then R is an n-ary predicate symbol. In addition we always have that "=" is a logical binary predicate symbol.

For the *theory of groups* the appropriate language is $L = \{e, \cdot, ^{-1}\}$ where "e" is a constant symbol, so $\#(e) = 0$, "$\cdot$" is a binary operation symbol, so $\#(\cdot) = 2$, and "$^{-1}$" is a unary operation symbol, so $\#(^{-1}) = 1$. For the *theory of partially ordered sets* we have that $L = \{\leq\}$ where $\leq$ is a binary relation symbol, so $\#(\leq) = 2$.

Our next goal is to define what it means to be a formula of first order logic. Let L be a fixed language. An expression is a finite string of symbols that are either logical symbols or symbols from L.

The set of *terms* of L is the smallest set of expressions that contain the variables and constant symbols of L (if any), and is closed under the formation rule: if $t_1, t_2, \ldots, t_n$ are terms of L and f is an n-ary operation symbol of L, then $t = f(t_1, t_2, \ldots, t_n)$ is a term of L. If L has no function symbols then the only terms of L are the variables and constant symbols. So for example if c is a constant symbol, f is a 3-ary operation symbol, g is a binary operation symbol, and h is a unary operation symbol, then

$$h(f(g(x, h(y)), y, h(c)))$$

is a term.

The set of *atomic formulas* of L is the set of all expressions of the form $R(t_1, t_2, \ldots, t_n)$, where $t_1, t_2, \ldots, t_n$ are terms of L and R is a n-ary predicate symbol of L. Since we always have equality as a binary relation we always have atomic formulas of the form $t_1 = t_2$.

The set of *formulas* of L is the smallest set of expressions that includes the atomic formulas and is closed under the formation rule: if θ and ψ are L formulas and x is any variable, then

- $(\theta \vee \psi)$,
- $\neg\theta$, and
- $\exists x\, \theta$

are L formulas. We think of other logical connectives as being abbreviations, e.g.,

- $(\theta \wedge \psi)$ abbreviates $\neg(\neg\theta \vee \neg\psi)$,
- $(\theta \rightarrow \psi)$ abbreviates $(\neg\theta \vee \psi)$,
- $\forall x\, \theta$ abbreviates $\neg\exists x\, \neg\theta$, and so forth.

We often add and sometimes drop parentheses to improve readability. Also we write $x \neq y$ for the formally correct but harder to read $\neg x = y$.

It is common practice to write symbols not only in *prefix* form as above but also in *postfix* and *infix* forms. For example in our example of group theory instead of writing the term $\cdot(x, y)$ we usually write it in infix form $x \cdot y$, and $^{-1}(x)$ is usually written in postfix form x^{-1}. Similarly in the language of partially ordered sets we usually write $x \leq y$ instead of the prefix form $\leq (x, y)$. Binary relations such as partial orders and equivalence relations are most often written in infix form. We regard the more natural forms we write as abbreviations of the more formally correct prefix notation.

Next we want to describe the syntactical concept of *substitution*. To do so we must first describe what it means for an occurrence of a variable x in a formula θ to be *free*. If an occurrence of a variable x in a formula θ is not free it is said to be *bound*. Example:

$$(\exists x\ x = y \vee x = f(y))$$

Both occurrences of y are free, the first two occurrences of x are bound, and the last occurrence of x is free. In the formula:

$$\exists x\ (x = y \vee x = f(y))$$

all three occurrences of x are bound.

Formally we proceed as follows. All occurrences of variables in an atomic formula are free. The free occurrences of x in $\neg\theta$ are exactly the free occurrences of x in θ. The free occurrences of x in $(\theta \vee \psi)$ are exactly the free occurrences of x in θ and in ψ. If x and y are distinct variables, then the free occurrences of x in $\exists y\, \theta$ are exactly the free occurrences of x in θ. And finally no occurrence of x in $\exists x\, \theta$ is free. This gives the inductive definition of free and bound variables.

We show that x might occur freely in θ by writing $\theta(x)$. If c is a constant symbol the formula $\theta(c)$ is gotten by substituting c for all free occurrences (if any) of x in θ. For example: if $\theta(x)$ is

$$\exists y\ (y = x \wedge \forall x\ x = y),$$

then $\theta(c)$ is

$$\exists y\ (y = c \wedge \forall x\ x = y).$$

We usually write $\theta(x_1, x_2, \ldots, x_n)$ to indicate that the free variables of θ are amongst the $x_1, x_2, \ldots, x_n$. A formula is called a sentence if no variable occurs freely in it.

Semantics

Our next goal is to describe the semantics of first order logic. A *structure* $\mathfrak{A}$ for the language L is a pair consisting of a set A called the universe of $\mathfrak{A}$ and an assignment or interpretation function from the nonlogical symbols of L to individuals, relations, and functions on A. Thus

- for each constant symbol c in L we have an assignment $c^{\mathfrak{A}} \in A$,
- for each n-ary operation symbol f in L we have a function $f^{\mathfrak{A}} : A^n \to A$, and
- for each n-ary predicate symbol R we have a relation $R^{\mathfrak{A}} \subseteq A^n$.

The symbol $=$ is always interpreted as the binary relation of equality, which is why we consider it a logical symbol, i.e., for any structure $\mathfrak{A}$ we have $=^{\mathfrak{A}}$ is $\{(x, x) : x \in A\}$. We use the word structure and *model* interchangeably.

For example, suppose L is the language of group theory. One structure for this theory is

$$\mathfrak{Q} = (\mathbb{Q}, +, -x, 0)$$

where

- the universe is the rationals,
- $\cdot^{\mathfrak{Q}}$ is ordinary addition of rationals,
- ${}^{-1^{\mathfrak{Q}}}$ is the function which takes each rational r to $-r$, and
- $e^{\mathfrak{Q}} = 0$.

Another structure in this language is

$$\mathfrak{R} = (\mathbb{R}^{+}, \times, \frac{1}{x}, 1)$$

where

- the universe is the set of positive real numbers,
- $\cdot^{\mathfrak{R}}$ is multiplication $\times$,
- ${}^{-1^{\mathfrak{R}}}$ is the function which takes x to $\frac{1}{x}$, and
- $e^{\mathfrak{R}} = 1$.

Another example is the group S_n of permutations. Here $\cdot^{S_n}$ is composition of functions, ${}^{-1^{S_n}}$ is the functional which takes each permutation to its inverse, and e^{S_n} is the identity permutation. Of course there are many examples of structures in this language which are not groups.

For another example, the language of partially ordered sets is $L = \{\leq\}$ where $\leq$ is a binary relation symbol. The following are all L-structures which happen to be partial orders:

- $(\mathbb{R}, \{(x, y) \in \mathbb{R}^2 : x \leq y\})$,
- $(\mathbb{Q}, \{(x, y) \in \mathbb{Q}^2 : x \geq y\})$, and
- $(\mathbb{N}, \{(x, y) \in \mathbb{N}^2 : x \text{ divides } y\})$.

For any nonempty set A and $R \subseteq A^2$, (A, R) is an L-structure. If in addition the relation R is transitive, reflexive, and antisymmetric, then (A, R) is a partial order.

$$\mathfrak{A} \models \theta$$

Next we define what it means for an L structure $\mathfrak{A}$ to model or satisfy an L sentence θ (written $\mathfrak{A} \models \theta$). For example,

$$(\mathbb{Q}, +, 0) \models \forall x \exists y\; x \cdot y = e,$$

because for all $p \in \mathbb{Q}$ there exists $q \in \mathbb{Q}$ such that $p + q = 0$.

Usually it is not the case that every element of a model has a constant symbol which names it. But suppose this just happened to be the case. Let's suppose that for ever $a \in A$ there is a constant symbol c_a in the language L so that $c_a^{\mathfrak{A}} = c$. The interpretation function can be extended to the variable free terms of L by the rule:

$$(f(t_1, t_2, \ldots, t_n))^{\mathfrak{A}} = f^{\mathfrak{A}}(t_1^{\mathfrak{A}}, t_2^{\mathfrak{A}}, \ldots, t_n^{\mathfrak{A}})$$

Hence for each variable free term t we get an interpretation $t^{\mathfrak{A}} \in A$. For example, if $L = \{S, c\}$ where S is a unary operation symbol and c is a constant symbol, and $\mathfrak{Z}$ is the L-structure with universe $\mathbb{Z}$ and where $S^{\mathfrak{Z}}(x) = x + 1$ and $c^{\mathfrak{Z}} = 0$, then $S(S(S(S(c))))^{\mathfrak{Z}} = 4$.

Our definition of $\models$ is by induction on the *logical complexity* of the sentence θ, i.e. the number of logical symbols in θ.

1. $\mathfrak{A} \models R(t_1, \ldots, t_n)$ iff $(t_1^{\mathfrak{A}}, \ldots, t_n^{\mathfrak{A}}) \in R^{\mathfrak{A}}$.

2. $\mathfrak{A} \models \neg\theta$ iff not $\mathfrak{A} \models \theta$.

3. $\mathfrak{A} \models (\theta \vee \psi)$ iff $\mathfrak{A} \models \theta$ or $\mathfrak{A} \models \psi$.

4. $\mathfrak{A} \models \exists x\, \theta(x)$ iff there exists a b in the universe A such that $\mathfrak{A} \models \theta(c_b)$.

Now we would like to define $\mathfrak{A} \models \theta$ for arbitrary languages L and L-structures $\mathfrak{A}$. Let $L_{\mathfrak{A}} = L \cup \{c_a : a \in A\}$ where each c_a is a new constant symbol. Let $(\mathfrak{A}, a)_{a \in A}$ be the $L_{\mathfrak{A}}$ structure gotten by augmenting the structure $\mathfrak{A}$ by interpreting each symbol c_a as the element a.

If θ is an L-sentence and $\mathfrak{A}$ is and L-structure, then we define $\mathfrak{A} \models \theta$ iff θ is true in the augmented structure, i.e., $(\mathfrak{A}, a)_{a \in A}$.

If $L_1 \subseteq L_2$ and $\mathfrak{A}$ is a L_2 structure, then the reduct of $\mathfrak{A}$ to L_1, written $\mathfrak{A} \restriction L_1$, is the L_1 structure with the same universe as $\mathfrak{A}$ and same relations, operations, and constants as $\mathfrak{A}$ for the symbols of L_1.

Lemma 1 *Let $L_1 \subseteq L_2$ and $\mathfrak{A}$ be an L_2 structure. Then for any θ an L_1 sentence,*

$$\mathfrak{A} \models \theta \textit{ iff } \mathfrak{A} \restriction L_1 \models \theta$$

proof:

We prove by induction on the number of logical symbols in the sentence that for any $L_{1\mathfrak{A}}$ sentence θ:

$$(\mathfrak{A}, a)_{a \in A} \models \theta \text{ iff } (\mathfrak{A}, a)_{a \in A} \restriction L_{1\mathfrak{A}} \models \theta$$

Let $\mathfrak{A}_2 = (\mathfrak{A}, a)_{a \in A}$ and $\mathfrak{A}_1 = (\mathfrak{A}, a)_{a \in A} \restriction L_{1\mathfrak{A}}$.

Atomic sentences: By induction on the size of the term, for any $L_{1\mathfrak{A}}$ variable free term t we have that $t^{\mathfrak{A}_1} = t^{\mathfrak{A}_2}$.

For any n-ary relation symbol R in L_1 we have $R^{\mathfrak{A}_1} = R^{\mathfrak{A}_2}$ (since $\mathfrak{A}_1$ is a reduct of $\mathfrak{A}_2$). Hence for any atomic $L_{1\mathfrak{A}}$-sentence $R(t_1, \ldots, t_n)$,

$$\mathfrak{A}_1 \models R(t_1, \ldots, t_n)$$

$$\text{iff}$$

$$\langle t_1^{\mathfrak{A}_1}, \ldots t_n^{\mathfrak{A}_1} \rangle \in R^{\mathfrak{A}_1}$$

$$\text{iff}$$

$$\langle t_1^{\mathfrak{A}_2}, \ldots t_n^{\mathfrak{A}_2} \rangle \in R^{\mathfrak{A}_2}$$

$$\text{iff}$$

$$\mathfrak{A}_2 \models R(t_1, \ldots, t_n).$$

Negation:

$$\mathfrak{A}_1 \models \neg\theta$$

$$\text{iff}$$

$$\text{not } \mathfrak{A}_1 \models \theta$$

$$\text{iff (by induction)}$$

$$\text{not } \mathfrak{A}_2 \models \theta$$

$$\text{iff}$$

$$\mathfrak{A}_2 \models \neg\theta.$$

Disjunction:

$$\mathfrak{A}_1 \models (\theta \vee \rho)$$

$$\text{iff}$$

$$\mathfrak{A}_1 \models \theta \text{ or } \mathfrak{A}_1 \models \rho$$

$$\text{iff (by induction)}$$

$$\mathfrak{A}_2 \models \theta \text{ or } \mathfrak{A}_2 \models \rho$$

$$\text{iff}$$

$$\mathfrak{A}_2 \models (\theta \vee \rho).$$

Existential quantifier:

$$\mathfrak{A}_1 \models \exists x \theta(x)$$

$$\text{iff}$$

$$\text{there exists } a \in A \text{ such that } \mathfrak{A}_1 \models \theta(c_a)$$

$$\text{iff}$$

$$\text{there exists } a \in A \text{ such that } \mathfrak{A}_2 \models \theta(c_a)$$

$$\text{iff}$$

$$\mathfrak{A}_2 \models \exists x \theta(x).$$

□

Compactness Theorem

The compactness theorem (for countable languages) was proved by Kurt Gödel in 1930. Malcev extended it to uncountable languages in 1936. The proof we give here was found by Henkin in 1949.

We say that a set of L sentences Σ is *finitely satisfiable* iff every finite subset of Σ has a model. Σ is *complete* iff for every L sentence θ either θ is in Σ or $\neg\theta$ is in Σ.

Lemma 2 *For every finitely satisfiable set of L sentences Σ there is a complete finitely satisfiable set of L sentences $\Sigma' \supseteq \Sigma$.*

proof:

Let $B = \{Q : Q \supseteq \Sigma \text{ is finitely satisfiable}\}$. B is closed under unions of chains, because if $C \subseteq B$ is a chain, and $F \subseteq \cup C$ is finite then there exists $Q \in C$ with $F \subseteq Q$, hence F has a model. By the maximality principal, there exists $\Sigma' \in B$ maximal. But for every L sentence θ either $\Sigma' \cup \{\theta\}$ is finitely satisfiable or $\Sigma' \cup \{\neg\theta\}$ is finitely satisfiable. Otherwise there exists finite $F_0, F_1 \subseteq \Sigma'$ such that $F_0 \cup \{\theta\}$ has no model and $F_1 \cup \{\neg\theta\}$ has no model. But $F_0 \cup F_1$ has a model $\mathfrak{A}$ since Σ is finitely satisfiable. Either $\mathfrak{A} \models \theta$ or $\mathfrak{A} \models \neg\theta$. This is a contradiction.

□

Lemma 3 *If Σ is a finitely satisfiable set of L sentences, and $\theta(x)$ is an L formula with one free variable x, and c a new constant symbol (not in L), then $\Sigma \cup \{(\exists x\, \theta(x)) \to \theta(c)\}$ is finitely satisfiable in the language $L \cup \{c\}$.*

proof:

This new sentence is called a *Henkin sentence* and c is called the *Henkin constant*. Suppose it is not finitely satisfiable. Then there exists $F \subseteq \Sigma$ finite such that $F \cup \{(\exists x\, \theta(x)) \to \theta(c)\}$ has no model. Let $\mathfrak{A}$ be an L-structure modeling F. Since the constant c is not in the language L we are free to interpret it any way we like. If $\mathfrak{A} \models \exists x\, \theta(x)$ choose $c \in A$ so that $(\mathfrak{A}, c) \models \theta(c)$, otherwise choose $c \in A$ arbitrarily. In either case $(\mathfrak{A}, c)$ models F and the Henkin sentence.

□

We say that a set of L sentences Σ is *Henkin* iff for every L formula $\theta(x)$ with one free variable x, there is a constant symbol c in L such that $(\exists x\, \theta(x)) \to \theta(c) \in \Sigma$.

Lemma 4 *If Σ is a finitely satisfiable set of L sentences, then there exists $\Sigma' \supseteq \Sigma$ with $L' \supseteq L$ and Σ' a finitely satisfiable Henkin set of L' sentences.*

proof:

For any set of Σ of L sentences, let

$$\Sigma^* = \Sigma \cup \{(\exists x\, \theta(x)) \to \theta(c_\theta) : \theta(x) \text{ an } L \text{ formula with one free variable}\}$$

The language of Σ^* contains a new constant symbol c_θ for each L formula $\theta(x)$. Σ^* is finitely satisfiable, since any finite subset of it is contained in a set of the form

$$F \cup \{(\exists x\, \theta_1(x)) \to \theta(c_{\theta_1}), \ldots, (\exists x\, \theta_n(x)) \to \theta(c_{\theta_n})\}$$

where $F \subseteq \Sigma$ is finite. To prove this set has a model use induction on n and note that from the point of view of

$$\Sigma \cup \{(\exists x\, \theta_1(x)) \to \theta(c_{\theta_1}), \ldots, (\exists x\, \theta_{n-1}(x)) \to \theta(c_{\theta_{n-1}})\}$$

c_{θ_n} is a new constant symbol, so we can apply the last lemma.

Now let $\Sigma_0 = \Sigma$ and let $\Sigma_{m+1} = \Sigma_m^*$. Then

$$\Sigma' = \cup_{m<\omega} \Sigma_m$$

is Henkin. It is also finitely satisfiable, since it is the union of a chain of finitely satisfiable sets.

□

If Σ is a set of L sentences, then the *canonical structure* $\mathfrak{A}$ built from Σ is the following.

- Let X be the set of all variable free terms of L.
- For $t_1, t_2 \in X$ define $t_1 \sim t_2$ iff $(t_1 = t_2) \in \Sigma$.
- Assuming that $\sim$ is an equivalence relation let $[t]$ be the equivalence class of $t \in X$.
- The universe of the canonical model $\mathfrak{A}$ is the set of equivalence classes of $\sim$.

- For any constant symbol c we define

$$c^{\mathfrak{A}} = [c].$$

- For any n-ary operation symbol f we define

$$f^{\mathfrak{A}}([t_1], [t_2], \ldots, [t_n]) = [f(t_1, t_2, \ldots, t_n)].$$

- For any n-ary relation symbol R we define

$$([t_1], [t_2], \ldots, [t_n]) \in R^{\mathfrak{A}} \text{ iff } R(t_1, t_2, \ldots, t_n) \in \Sigma.$$

Lemma 5 *If Σ is a finitely satisfiable complete Henkin set of L sentences, then the canonical model $\mathfrak{A}$ built from Σ is well defined and for every L sentence θ,*

$$\mathfrak{A} \models \theta \textit{ iff } \theta \in \Sigma.$$

proof:

First we show that $\sim$ is an equivalence relation. Suppose t, t_1, t_2, t_3 are variable free terms.

$t \sim t$: If $t = t \notin \Sigma$ then, since Σ is complete we have that $\neg t = t \in \Sigma$. But clearly $\neg t = t$ has no models and so Σ is not finitely satisfiable.

$t_1 \sim t_2$ implies $t_2 \sim t_1$: If not, by completeness of Σ we must have that $t_1 = t_2$ and $\neg t_2 = t_1$ are both in Σ. But then Σ is not finitely satisfiable.

($t_1 \sim t_2$ and $t_2 \sim t_3$) implies $t_1 \sim t_3$: If not, by completeness of Σ we must have that $t_1 = t_2$, $t_2 = t_3$, and $\neg t_1 = t_3$ are all in Σ. But then Σ is not finitely satisfiable.

So $\sim$ is an equivalence relation. Next we show that it is a congruence relation.

Suppose $t_1, \ldots, t_n, t'_1, \ldots, t'_n$ are variable free terms and f is an n-ary operation symbol.

If $t_1 \sim t'_1, \ldots, t_n \sim t'_n$ then $f(t_1, \ldots, t_n) \sim f(t'_1, \ldots, t'_n)$.

This amounts to saying if

$$\{t_1 = t'_1, \ldots, t_n = t'_n\} \subseteq \Sigma,$$

then

$$f(t_1, \ldots, t_n) = f(t'_1, \ldots, t'_n) \in \Sigma.$$

But again since Σ is complete we would have

$$f(t_1, \ldots, t_n) \neq f(t'_1, \ldots, t'_n) \in \Sigma$$

but

$$\{t_1 = t'_1, \ldots, t_n = t'_n, f(t_1, \ldots, t_n) \neq f(t'_1, \ldots, t'_n)\}$$

has no models and so Σ wouldn't be finitely satisfiable.

By a similar argument: Suppose $t_1, \ldots, t_n, t'_1, \ldots, t'_n$ are variable free terms and R is an n-ary operation symbol.

If $t_1 \sim t'_1, \ldots, t_n \sim t'_n$ then $R(t_1, \ldots, t_n) \in \Sigma$ iff $R(t'_1, \ldots, t'_n) \in \Sigma$.

This shows the canonical model is well defined.

Now we prove by induction on the number of logical symbols that for any L sentence θ,

$$\mathfrak{A} \models \theta \iff \theta \in \Sigma.$$

The atomic formula case is by definition.

$\neg$:

$$\mathfrak{A} \models \neg\theta$$

iff

not $\mathfrak{A} \models \theta$

iff(by induction)

not $\theta \in \Sigma$

iff(by completeness)

$\neg\theta \in \Sigma$.

$\vee$:

$$\mathfrak{A} \models (\theta \vee \rho)$$

iff

$\mathfrak{A} \models \theta$ or $\mathfrak{A} \models \rho$

iff(by induction)

$\theta \in \Sigma$ or $\rho \in \Sigma$

iff

$$(\theta \vee \rho) \in \Sigma.$$

This last "iff" uses completeness and finite satisfiability of Σ. If $(\theta \vee \rho) \notin \Sigma$ then by completeness $\neg(\theta \vee \rho) \in \Sigma$ but $\{\theta, \rho, \neg(\theta \vee \rho)\}$ has no model. Conversely if $\theta \notin \Sigma$ and $\rho \notin \Sigma$, then by completeness $\neg\theta \in \Sigma$ and $\neg\rho \in \Sigma$, but $\{(\theta \vee \rho), \neg\theta, \neg\rho\}$ has no model.

$\exists$:

$\mathfrak{A} \models \exists x\theta(x)$ implies there exists $a \in A$ such that $\mathfrak{A} \models \theta(a)$. This implies (by induction) $\theta(a) \in \Sigma$. Which in turn implies $\exists x\theta(x) \in \Sigma$, since otherwise $\neg\exists x\theta(x) \in \Sigma$ but $\{\neg\exists x\theta(x), \theta(a)\}$ has no model. Hence $\mathfrak{A} \models \exists x\theta(x)$ implies $\exists x\theta(x) \in \Sigma$.

For the other direction suppose $\exists x\theta(x) \in \Sigma$. Then since Σ is Henkin for some constant symbol c we have $(\exists x\theta(x)) \to \theta(c) \in \Sigma$. Using completeness and finite satisfiability we must have $\theta(c) \in \Sigma$. By induction $\mathfrak{A} \models \theta(c)$ hence $\mathfrak{A} \models \exists x\theta(x)$. Hence $\exists x\theta(x) \in \Sigma$ implies $\mathfrak{A} \models \exists x\theta(x)$.

□

Compactness Theorem. For any language L and set of L sentences Σ, if every finite subset of Σ has a model, then Σ has a model.

proof:

First Henkinize Σ, then complete it. Take its canonical model. Then reduct it back to an L-structure.

□

Problems

7.1 We say that T a set of L sentences is an L *theory* iff there exists a set Σ of L sentences such that T is the set of all L sentences true in every model of Σ. In this case we say that Σ is an *axiomatization* of T or that Σ *axiomatizes* the theory T. Prove that any theory axiomatizes itself. Give an axiomatization of the theory of partially ordered sets. The theory of groups is just the set of all sentences of group theory which are true in every group. Give an axiomatization of the theory of abelian groups.

7.2 Suppose that T is a theory with a model. Show that T is complete iff for every sentence θ in the language of T either every model of T is a model of θ or every model of T is a model of $\neg\theta$.

7.3 A theory T is *finitely axiomatizable* iff there is a finite Σ which axiomatizes T. Let T be the set of sentences in the language of pure equality which are true in every infinite structure. Prove that T is a theory by finding axioms for it. Show that no finite set of these axioms axiomatizes T.

7.4 The theory of $\mathfrak{A}$, $Th(\mathfrak{A})$, is defined as follows:

$$Th(\mathfrak{A}) = \{\theta : \theta \text{ is an } L \text{ sentence and } \mathfrak{A} \models \theta\}$$

Prove that $Th(\mathfrak{A})$ is a complete theory.

7.5 $\mathfrak{A}$ and $\mathfrak{B}$ are *elementarily equivalent* (written $\mathfrak{A} \equiv \mathfrak{B}$) iff $Th(\mathfrak{A}) = Th(\mathfrak{B})$. Show that $\mathfrak{A} \equiv \mathfrak{B}$ iff for every sentence θ if $\mathfrak{A} \models \theta$, then $\mathfrak{B} \models \theta$.

7.6 Suppose T is a theory with a model. Show the following are equivalent:

- T is complete
- any two models of T are elementarily equivalent
- $T = Th(\mathfrak{A})$ for some model of T
- $T = Th(\mathfrak{A})$ for all models of T

7.7 Show that for any set of sentences Σ and sentence θ, every model of Σ is a model of θ iff there exists a finite $\Sigma' \subseteq \Sigma$ such that every model of Σ' is a model of θ.

7.8 Suppose that T is a finitely axiomatizable theory and Σ is any axiomatization of T. Show that some finite $\Sigma' \subseteq \Sigma$ axiomatizes T.

7.9 (Separation) Let $\mathcal{M}(T)$ be the class of all models of T. Suppose T and T' are theories in a language L and $\mathcal{M}(T) \cap \mathcal{M}(T') = \emptyset$. Show that for some L sentence θ,

$$\mathcal{M}(T) \subseteq \mathcal{M}(\theta) \text{ and } \mathcal{M}(\theta) \cap \mathcal{M}(T') = \emptyset.$$

7.10 Let L be a first order language and suppose $T_i, i \in I$ are theories in L such that every L structure is a model of exactly one of the T_i's. If I is finite does it then follow that each of the T_i's is finitely axiomatizable? What about infinite I?

7.11 Show that any theory with arbitrarily large finite models must have an infinite model.

7.12 Suppose that T is an L-theory with an infinite model. Let X be any nonempty set and $\{c_x : x \in X\}$ be new constant symbols not appearing in L. Prove that the set of sentences $T \cup \{c_x \neq c_y : x, y \in X, x \neq y\}$ has a model. Prove that any theory with an infinite model has an uncountable model.

7.13 Let the language of fields be $\{+, \cdot, 0, 1\}$. Give an axiomatization for fields of characteristic 0. Show that this theory is not finitely axiomatizable. Show that for any sentence θ true in all fields of characteristic 0 there is a k such that θ is true in all fields of characteristic $> k$.

7.14 The cardinality of a model is the number of elements in its universe. Give an example of a theory T such that for all n,

T has a model of cardinality n iff n is even.

Can you find a finitely axiomatizable T?

7.15 Is there a finitely axiomatizable theory with only infinite models? What if the language consists of a single unary operation symbol? What about the languages which contain only unary relation symbols?

7.16 Suppose that T is any theory in a language which includes a binary relation symbol $\leq$ such that for every model $\mathfrak{A}$ of T $\leq^{\mathfrak{A}}$ is a linear order. Show that if T has an infinite model then T has a model $\mathfrak{B}$ such that there is an order embedding of the rationals into $\leq^{\mathfrak{B}}$, i.e., there is a function $f : \mathbb{Q} \to B$ such that $p \leq q$ iff $f(p) \leq^{\mathfrak{B}} f(q)$ for every $p, q \in \mathbb{Q}$.

7.17 Let $\mathfrak{A} = (A, \approx)$ be the equivalence relation with exactly one equivalence class of cardinality n for each $n = 1, 2, \ldots$ and no infinite equivalence classes. Show that there exists $\mathfrak{B} \equiv \mathfrak{A}$ which has infinitely many infinite equivalence classes.

7.18 A theory is *consistent* iff it has a model, i.e., it is realizable. Let T be a finitely axiomatizable theory with only a countable number of complete consistent extensions (ie $T' \supseteq T$) in the language of T. Prove that one of these complete consistent extensions is finitely axiomatizable.

7.19 For each of the following prove or give a counterexample:

1. Let T_n for $n \in \omega$ be finite sets of sentences and S be a finite set of sentences. Assume for all $n \in \omega$ that $T_n \subseteq T_{n+1}$, and there is a model of T_n which is not a model of S. Then there is a model of $\cup_{n\in\omega} T_n$ which is not a model of S.

2. Let S_n and T_n for $n \in \omega$ be finite sets of sentences. Assume that for all $n \in \omega, S_n \subseteq S_{n+1}, T_n \subseteq T_{n+1}$, and there is a model of T_n which is not a model of S_n. Then there is a model of $\cup_{n\in\omega} T_n$ which is not a model of $\cup_{n\in\omega} S_n$.

7.20 Let $T_0 \subseteq T_1 \subseteq T_2 \subseteq \ldots$ be a sequence of L theories such that for each $n \in \omega$ there exists a model of T_n which is not a model of T_{n+1}. Prove that $\cup_{n\in\omega} T_n$ is not finitely axiomatizable. If L is finite, prove that $\cup_{n\in\omega} T_n$ has an infinite model.

Lowenheim-Skolem Theorems

The first version of the Lowenheim-Skolem Theorem was proved in 1915. The final version that is presented here was developed by Tarski in the 1950's.

Lemma 6 *The number of L formulas is $|L| + \aleph_0$.*

proof:

There are only countably many logical symbols. Hence if $\kappa = |L| + \aleph_0$ then every formula may be regarded as an element of $\kappa^{<\omega}$ and we know $|\kappa^{<\omega}| = \kappa$.

□

Theorem *For any theory T in a language L if T has a model, then it has one of cardinality less than or equal to $|L| + \aleph_0$.*

proof:

The canonical model of the completion of Henkinization of T has cardinality $\leq |L| + \aleph_0$. It is enough to see that the language of the Henkinization of T has cardinality $\leq |L| + \aleph_0$, since the canonical models universe is the set of equivalence classes of variable free terms. The Henkinization language is $\cup_{n<\omega} L_n$ where $L_0 = L$ and L_{n+1} is L_n plus one more constant symbol for each formula of L_n. But by Lemma 6 there are $|L_n| + \aleph_0$ formulas of L_n. It follows that if $\kappa = |L| + \aleph_0$, then $|L_n| = \kappa$ for all n and so $|\cup_{n<\omega} L_n| = \kappa$.

□

$\mathfrak{A} \subseteq \mathfrak{B}$ means that $\mathfrak{A}$ is a *substructure* of $\mathfrak{B}$; equivalently $\mathfrak{B}$ is an *extension* or *superstructure* of $\mathfrak{A}$. This means that both structures are in the same language L, $A \subseteq B$, for every n-ary relation symbol R of L, $R^{\mathfrak{A}} = R^{\mathfrak{B}} \cap A^n$, for every n-ary function symbol f of L, $f^{\mathfrak{A}} = f^{\mathfrak{B}} \restriction A^n$, and for every constant symbol c of L, $c^{\mathfrak{A}} = c^{\mathfrak{B}}$.

$\mathfrak{A} \preceq \mathfrak{B}$ means that $\mathfrak{A}$ is an *elementary substructure* of $\mathfrak{B}$; equivalently $\mathfrak{B}$ is an *elementary extension* of $\mathfrak{A}$. This means that $\mathfrak{A} \subseteq \mathfrak{B}$ and for every formula $\theta(x_1, x_2, \ldots, x_n)$ of the language L and for every $a_1, a_2, \ldots, a_n \in A$ we have

$$(\mathfrak{A}, a)_{a\in A} \models \theta(c_{a_1}, c_{a_2}, \ldots, c_{a_n}) \text{ iff } (\mathfrak{B}, a)_{a\in A} \models \theta(c_{a_1}, c_{a_2}, \ldots, c_{a_n})$$

To ease the notational complexity we will write $\mathfrak{A} \models \theta(a_1, \ldots, a_n)$ instead of $(\mathfrak{A}, a)_{a\in A} \models \theta(c_{a_1}, c_{a_2}, \ldots, c_{a_n})$. It should be kept in mind that the language L may have no constant symbols in it.

Example. Let $\mathfrak{A} = (\omega, <)$ and let $\mathfrak{B} = (Evens, <)$. Then $\mathfrak{B} \subseteq \mathfrak{A}$ but $\mathfrak{B}$ is not an elementary substructure of $\mathfrak{A}$. This is because $\mathfrak{A} \models \exists x\, 0 < x < 2$ but $\mathfrak{B} \models \neg\exists x\, 0 < x < 2$.

Lemma 7 *(Tarski-Vaught criterion) Suppose $\mathfrak{A} \subseteq \mathfrak{B}$ are L structures and for every L formula $\theta(x, y_1, y_2, \ldots, y_n)$, for all $a_1, a_2, \ldots, a_n \in A$, and $b \in B$*

$$\mathfrak{B} \models \theta(b, a_1, a_2, \ldots, a_n) \text{ implies there exists } a \in A \;\; \mathfrak{B} \models \theta(a, a_1, a_2, \ldots, a_n).$$

Then $\mathfrak{A} \preceq \mathfrak{B}$.

proof:

The proof is by induction on the number of logical symbols in the formula θ. The atomic formula case is trivial because $\mathfrak{A}$ is a substructure of $\mathfrak{B}$.

$\neg$: $\mathfrak{A} \models \neg\theta$ iff not $\mathfrak{A} \models \theta$ iff(by induction) not $\mathfrak{B} \models \theta$ iff $\mathfrak{B} \models \neg\theta$.

$\vee$: $\mathfrak{A} \models (\theta \vee \rho)$ iff $\mathfrak{A} \models \theta$ or $\mathfrak{A} \models \rho$ iff(by induction) $\mathfrak{B} \models \theta$ or $\mathfrak{B} \models \rho$ iff $\mathfrak{B} \models (\theta \vee \rho)$.

$\exists$: $\mathfrak{A} \models \exists x\theta(x, a_1, \ldots, a_n)$ implies there exists $a \in A$ such that $\mathfrak{A} \models \theta(a, a_1, \ldots, a_n)$ which implies (by induction) $\mathfrak{B} \models \theta(a, a_1, \ldots, a_n)$. For the other direction we use the criterion. $\mathfrak{B} \models \exists x\theta(x, a_1, \ldots, a_n)$ implies there exists $a \in A$ such that $\mathfrak{B} \models \theta(a, a_1, \ldots, a_n)$. Hence by induction $\mathfrak{A} \models \theta(a, a_1, \ldots, a_n)$ and so $\mathfrak{A} \models \exists x\theta(x, a_1, \ldots, a_n)$.

$\square$

Lemma 8 *Suppose $X \subseteq B$ and $|X| = \kappa \geq |L| + \aleph_0$ where $\mathfrak{B}$ is an L structure. Then there exists $X^* \supseteq X, |X^*| = \kappa$, and for every formula $\theta(x, y_1, y_2, \ldots, y_n)$, for all $a_1, a_2, \ldots, a_n \in X$, and $b \in B$*

$$\mathfrak{B} \models \theta(b, a_1, a_2, \ldots, a_n) \text{ implies } \exists a \in X^* \;\; \mathfrak{B} \models \theta(a, a_1, a_2, \ldots, a_n)$$

proof:

Fix $\leq$ a wellordering of B. For any L-formula $\theta(x, y_1, \ldots, y_n)$ and

$$a_1, \ldots, a_n \in B,$$

define $a_{\theta(x,a_1,\ldots,a_n)} \in B$ to be the $\leq$ least element b of B such that $\mathfrak{B} \models \theta(b, a_1, \ldots, a_n)$ if one exists otherwise let it be arbitrary. Now let $X_0 = X$, $L_0 = L$, and for any $m < \omega$ let $X_{m+1} =$

$$\{a_{\theta(x,a_1,\ldots,a_n)} : \theta(x, y_1, \ldots, y_n) \text{ is an } L_m \text{ formula and } \{a_1, \ldots, a_n\} \subseteq X_m\}$$

and let $L_{m+1} \supseteq L_m$ be the language with all these new constant symbols adjoined. Clearly if X_m and L_m have cardinality κ then so do X_{m+1} and L_{m+1}, since $|\kappa^{<\omega}| = \kappa$. Let $X^* = \cup_{m<\omega} X_m$, then it has cardinality κ since it is the countable union of sets of cardinality κ. For every formula $\theta(x, a_1, a_2, \ldots, a_n)$ there exist some $m < \omega$ such that $\{a_1, a_2, \ldots, a_n\} \subseteq X_m$ and so the criterion for $\theta(x, a_1, a_2, \ldots, a_n)$ is satisfied at stage $m + 1$.
$\square$

Definition: $|\mathfrak{A}|$ is the cardinality of the universe A of $\mathfrak{A}$.

Downward Lowenheim-Skolem Theorem. Suppose $\mathfrak{B}$ is an infinite structure in the language L, κ is a cardinal such that $\aleph_0 + |L| \leq \kappa \leq |\mathfrak{B}|$, and $X \subseteq B$ such that $|X| \leq \kappa$. Then there is a structure $\mathfrak{A}$ such that $\mathfrak{A} \preceq \mathfrak{B}$, $X \subseteq A$, and $|\mathfrak{A}| = \kappa$.

proof:
By the lemma there exists $X^* \supseteq X$ of cardinality κ satisfying the criterion. But note that the criterion implies that X^* is closed under the operations of $\mathfrak{B}$. (Just look at the sentence $\exists x \;\; x = f(a_1, \ldots, a_n)$.) Consequently there is a substructure $\mathfrak{A}$ of $\mathfrak{B}$ with universe $A = X^*$. By the Tarski-Vaught criterion $\mathfrak{A} \preceq \mathfrak{B}$.
$\square$

$\mathfrak{A} \simeq \mathfrak{B}$ means that $\mathfrak{A}$ and $\mathfrak{B}$ are *isomorphic*, that is, there is a bijection $j : A \to B$ such that for every n-ary relation symbol R and for every $a_1, a_2, \ldots, a_n \in A$,

$$\langle a_1, a_2, \ldots, a_n \rangle \in R^{\mathfrak{A}} \text{ iff } \langle j(a_1), j(a_2), \ldots, j(a_n) \rangle \in R^{\mathfrak{B}}$$

and for every n-ary function symbol f and for every $a_1, a_2, \ldots, a_n \in A$,

$$j(f^{\mathfrak{A}}(a_1, a_2, \ldots, a_n)) = f^{\mathfrak{B}}(j(a_1), j(a_2), \ldots, j(a_n))$$

(for n=0 this means that for every constant symbol c, $j(c^{\mathfrak{A}}) = c^{\mathfrak{B}}$.)

Lemma 9 *Suppose j is an isomorphism between the L structures $\mathfrak{A}$ and $\mathfrak{B}$. Then for any L formula $\theta(x_1, x_2, \ldots, x_n)$ and any $a_1, a_2, \ldots, a_n \in A$,*

$$\mathfrak{A} \models \theta(a_1, a_2, \ldots, a_n) \textit{ iff } \mathfrak{B} \models \theta(j(a_1), j(a_2), \ldots, j(a_n))$$

proof:

First show by induction that for every L-term $\tau(x_1, \ldots, x_n)$ and sequence $a_1, \ldots, a_n \in A$ that

$$j(\tau^{\mathfrak{A}}(a_1, \ldots, a_n)) = \tau^{\mathfrak{B}}(j(a_1), \ldots, j(a_n)).$$

The proof of the lemma is by induction on the logical complexity of θ. For atomic formula it follows from the definition of isomorphism. The propositional steps are easy and the quantifier step is handled by using that j is onto.

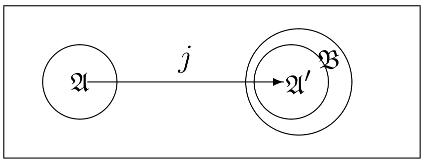
□

A map $j : \mathfrak{A} \to \mathfrak{B}$ is an *elementary embedding* iff it is an isomorphism of $\mathfrak{A}$ with an elementary substructure of $\mathfrak{B}$. We write $\mathfrak{A} \preceq^j \mathfrak{B}$.

Lemma 10 *A map $j : \mathfrak{A} \to \mathfrak{B}$ is an elementary embedding iff for any formula $\theta(x_1, x_2, \ldots, x_n)$ and any $a_1, a_2, \ldots, a_n \in A$,*

$$\mathfrak{A} \models \theta(a_1, a_2, \ldots, a_n) \text{ iff } \mathfrak{B} \models \theta(j(a_1), j(a_2), \ldots, j(a_n))$$

proof:



Just unravel the definitions.

□

Lemma 11 *Suppose that $\mathfrak{A} \preceq^j \mathfrak{B}$, then there exists a structure $\mathfrak{B}'$ isomorphic to $\mathfrak{B}$ such that $\mathfrak{A} \preceq \mathfrak{B}'$. Furthermore j is the restriction of this isomorphism to A.*

proof:

Let B' be a superset of A such that the map j can be extended to a bijection $j : B' \to B$, (which we also will call j). Now define $f^{\mathfrak{B}'}$ and $R^{\mathfrak{B}'}$ in such away as to make j an isomorphism. This means that

$$R^{\mathfrak{B}'} = \{(b_1, \ldots, b_n) : (j(b_1), \ldots, j(b_n)) \in R^{\mathfrak{B}}\}$$

and

$$f^{\mathfrak{B}'}(b_1, \ldots, b_n) = j^{-1}(f^{\mathfrak{B}}(j(b_1), \ldots, j(b_n))$$

for all $b_1, \ldots, b_n \in B'$. Now check that $\mathfrak{B}'$ works.
□

The *elementary diagram* of $\mathfrak{A}$ is defined as follows:

$$D(\mathfrak{A}) = \{\theta : \theta \text{ is an } L_{\mathfrak{A}} \text{ sentence and } (\mathfrak{A}, a)_{a \in A} \models \theta\}$$

This means that $D(\mathfrak{A})$ is the theory of $\mathfrak{A}$ with constants adjoined for each element of the universe.

Lemma 12 *If $\mathfrak{A}$ is an L structure and $\mathfrak{B}$ is an $L_{\mathfrak{A}}$ structure such that $\mathfrak{B} \models D(\mathfrak{A})$, then there is a j such that $\mathfrak{A} \preceq^j \mathfrak{B} \restriction L$.*

proof:

Define $j : A \to B$ by $j(a) = c_a^{\mathfrak{B}}$.
□

Upward Lowenheim-Skolem Theorem. For any infinite structure $\mathfrak{A}$ in the language L and cardinal κ such that $|\mathfrak{A}| + |L| \leq \kappa$, there is a structure $\mathfrak{B}$ such that $\mathfrak{A} \preceq \mathfrak{B}$ and $|\mathfrak{B}|=\kappa$.

proof:

Let $\Sigma = D(\mathfrak{A}) \cup \{c_\alpha \neq c_\beta : \alpha, \beta \in \kappa, \alpha \neq \beta\}$ where the c_α are new constant symbols.

Σ is finitely satisfiable. To see this let $F \subseteq \Sigma$ be finite. Then there exists a finite $G \subseteq \kappa$ such that $F \subseteq D(\mathfrak{A}) \cup \{c_\alpha \neq c_\beta : \alpha, \beta \in G, \alpha \neq \beta\}$. Since the model $\mathfrak{A}$ is infinite we can choose distinct elements of $a_\alpha \in A$ for $\alpha \in G$ and then $(\mathfrak{A}, a_\alpha)_{\alpha \in G} \models F$.

It follows from the compactness theorem that Σ has a model. Since the language of Σ has cardinality κ by the downward Lowenheim Skolem theorem Σ has a model $\mathfrak{C}$ of cardinality κ. By the lemma there exists j such that $\mathfrak{A} \preceq^j \mathfrak{C} \restriction L$. By the other lemma $\mathfrak{C} \restriction L$ is isomorphic to a model $\mathfrak{B}$ such that $\mathfrak{A} \preceq \mathfrak{B}$.
□

Problems

8.1 If an L theory T has an infinite model, then for any $\kappa \geq \aleph_0 + |L|$, T has a model of cardinality κ.

8.2 Let T be a consistent theory in a countable language. Suppose that T has only countably many countable models up to isomorphism. Show that for some sentence θ in the language of T that $T \cup \{\theta\}$ axiomatizes a complete consistent theory.

8.3 Show that if $\mathfrak{A} \simeq \mathfrak{B}$, then $\mathfrak{A} \equiv \mathfrak{B}$. Is the converse true? Show that if $\mathfrak{A} \preceq \mathfrak{B}$, then $\mathfrak{A} \equiv \mathfrak{B}$. Is the converse true?

8.4 Assume that $\mathfrak{A} \subseteq \mathfrak{B}$. Show that $\mathfrak{A} \preceq \mathfrak{B}$ iff for any n and $a_1, a_2, \ldots, a_n \in A$ we have

$$(\mathfrak{A}, a_1, a_2, \ldots, a_n) \equiv (\mathfrak{B}, a_1, a_2, \ldots, a_n).$$

8.5 A subset X of a model $\mathfrak{A}$ is *definable* iff for some formula $\theta(x)$ in the language of $\mathfrak{A}$

$$X = \{b \in A : \mathfrak{A} \models \theta(b)\}.$$

Show that if $U \subseteq A$ is definable then for any isomorphism $j : \mathfrak{A} \to \mathfrak{A}$ we have $U = \{j(x) : x \in U\}$. Show that the only definable subsets of $(\mathbb{Z}, \leq)$ are the empty set and $\mathbb{Z}$ the set of all integers.

8.6 A theory T is *categorical* in power κ iff every two models of T of cardinality κ are isomorphic. Suppose that T is a consistent theory that has only infinite models and for some infinite cardinal $\kappa \geq |L|$, T is κ categorical. Show that T must be complete. This is called the *?Los-Vaught test*. Is the assumption of no finite models necessary?

8.7 Let the language of T_s be $\{S, c\}$ and

1. $\theta \equiv \forall x \forall y\, (S(x) = S(y) \to x = y)$
2. $\rho \equiv \forall x\, (x \neq c \iff \exists y S(y) = x)$
3. $\psi_n \equiv \forall x\, S^n(x) \neq x$, where $S^n(x)$ abbreviates $S(S(\cdots S(x) \cdots))$ where we have n S's.
4. Let T_s be the theory axiomatized by $\{\theta, \rho\} \cup \{\psi_n : n = 1, 2, \ldots\}$.

Show that $\mathfrak{N}_{Sc} = (\omega, Sc, 0)$ is a model of T_s, where Sc is the successor function $(Sc(n) = n + 1)$. Show that T_s is not finitely axiomatizable.

8.8 Is T_s categorical in some power? Show that T_s is complete.

8.9 Let $\mathfrak{N}^*_{Sc}$ be the model consisting of $(\omega, Sc, 0)$ plus a disjoint copy of $(\mathbb{Z}, Sc)$. Show that if a subset of $\mathfrak{N}^*_{Sc}$ is definable then it must either contain all of the $\mathbb{Z}$ part or none of it. A set is *cofinite* iff it is the complement of a finite set. Show that a subset of $\mathfrak{N}_{Sc}$ is definable iff it is finite or cofinite.

8.10 Show that $\leq$ is not definable in $\mathfrak{N}_{Sc}$.

8.11 Axiomatize the theory of algebraically closed fields of characteristic k. Show that this theory is categorical in every uncountable power. Hence this theory is complete. Show that the reals are not a definable subset of $(\mathbb{C}, +, \cdot, 0, 1)$ where $\mathbb{C}$ is the complex numbers.

8.12 Let $\mathfrak{A}$ be a countable structure in a countable language that includes a unary predicate symbol U such that $U^{\mathfrak{A}}$ is infinite. Show that there is a countable $\mathfrak{B} \succeq \mathfrak{A}$ such that $U^{\mathfrak{B}} \neq U^{\mathfrak{A}}$.

8.13 Axiomatize the theory of dense linear orders without endpoints (DLO). That is linear orders without greatest or least element such that between any two distinct elements lies a third. For example, the rationals or the reals under their usual order are dense linear orders. (Cantor) Show that any two countable dense linear orders are isomorphic. Show that $(\mathbb{Q}, \leq) \preceq (\mathbb{R}, \leq)$ where $\mathbb{Q}$ is the rationals and $\mathbb{R}$ is the reals. Is this true if we add to our structures $+$ and $\cdot$?

8.14 Can a theory have exactly $\aleph_0$ nonisomorphic models? Can such a theory be in a finite language?

8.15 Show that for any wellordering $\mathfrak{B} = (B, \leq)$ there is a countable wellordering elementarily equivalent to it.

8.16 Prove or disprove: $(\mathfrak{A} \preceq \mathfrak{C} \wedge \mathfrak{B} \preceq \mathfrak{C} \wedge \mathfrak{A} \subseteq \mathfrak{B}) \rightarrow \mathfrak{A} \preceq \mathfrak{B}$.

8.17 Prove or disprove: $(\mathfrak{A} \preceq \mathfrak{B} \wedge \mathfrak{A} \preceq \mathfrak{C} \wedge \mathfrak{B} \subseteq \mathfrak{C}) \rightarrow \mathfrak{B} \preceq \mathfrak{C}$.

8.18 Prove or disprove: $(\mathfrak{A} \subseteq \mathfrak{B} \wedge \mathfrak{A} \simeq \mathfrak{B}) \rightarrow \mathfrak{A} \preceq \mathfrak{B}$.

8.19 Prove or disprove: $(\mathfrak{A} \preceq \mathfrak{B} \preceq \mathfrak{C}) \rightarrow \mathfrak{A} \preceq \mathfrak{C}$.

8.20 Prove or disprove: if $\mathfrak{A}$ is isomorphic to a substructure of $\mathfrak{B}$ and $\mathfrak{B}$ is isomorphic to a substructure of $\mathfrak{A}$, then $\mathfrak{A} \simeq \mathfrak{B}$.

8.21 Prove or disprove: $(\exists j \;\; \mathfrak{A} \preceq^j \mathfrak{B} \wedge \exists k \;\; \mathfrak{B} \preceq^k \mathfrak{A}) \rightarrow \mathfrak{A} \simeq \mathfrak{B}$.

8.22 The *atomic diagram* of an L structure $\mathfrak{A}$ (written $D_0(\mathfrak{A})$) is the set of quantifier free sentences in $D(\mathfrak{A})$. Show that for any L structure $\mathfrak{A}$ and $L_{\mathfrak{A}}$ structure $\mathfrak{B}$ that $\mathfrak{B} \models D_0(\mathfrak{A})$ iff $\mathfrak{A}$ is isomorphic to a substructure of $\mathfrak{B} \upharpoonright L$.

8.23 Let $\mathfrak{A}$ and $\mathfrak{B}$ be two infinite linear orderings. Show that $\mathfrak{B}$ is isomorphic to a substructure of some elementary extension of $\mathfrak{A}$.

8.24 Let $\mathfrak{A} \preceq \mathfrak{B}$ and $\mathfrak{A} \subseteq \mathfrak{A}'$ and $A' \cap B = A$. Show that there exists $\mathfrak{B}' \supseteq \mathfrak{B}$ such that $\mathfrak{A}' \preceq \mathfrak{B}'$.

8.25 Find a structure $\mathfrak{A}$ in a countable language such that $\mathfrak{A}$ has exactly ω_1 elementary substructures and every proper elementary substructure is countable.

8.26 Suppose that $\mathfrak{A}$ is a finite model and $\mathfrak{A} \equiv \mathfrak{B}$. Show that $\mathfrak{A} \simeq \mathfrak{B}$. Warning: the language of $\mathfrak{A}$ may be infinite.

8.27 Suppose

$$\mathfrak{A}_0 \preceq \mathfrak{A}_1 \preceq \mathfrak{A}_2 \preceq \dots$$

and let $\mathfrak{B} = \cup_{n<\omega}\mathfrak{A}_n$. Show that for each $n < \omega$, $\mathfrak{A}_n \preceq \mathfrak{B}$. This is called Tarski's *Elementary Chain Lemma.*

8.28 A family F of structures is directed iff any two structures of F are elementary substructures of some other structure in F. Show that every structure in a directed family is an elementary substructure of the union.

8.29 Suppose T is a theory in a countable language which has an infinite model. Show that T has a countable model that is not finitely generated. A structure $\mathfrak{A}$ is *finitely generated* iff there is a finite set $F \subseteq A$ such that no proper substructure of $\mathfrak{A}$ contains F.

8.30 (DCLO) Axiomatize the theory of discrete linear orders without endpoints. That is linear orders without greatest or least elements and any element has an immediate successor and predecessor, for example the integers $\mathbb{Z}$ (negative and positive) under their usual ordering. Let $\mathfrak{A}$ be any model of DCLO. Define for $x, y \in A$ $x \approx y$ iff there are only finitely many elements of A between x and y. Show this is an equivalence relation and each equivalence class is isomorphic to $\mathbb{Z}$, call such a class a $\mathbb{Z}$ chain. Describe any model of DCLO. Show that for any countable model of DCLO and any two $\mathbb{Z}$ chains in the model there is an a countable elementary extension with a $\mathbb{Z}$ chain in between. Show that any countable model of DCLO is an elementary substructure of a model with countably many chains ordered like the rationals, i.e. $\mathbb{Z} \times \mathbb{Q}$. Show that DCLO is a complete theory.

8.31 Let $\mathfrak{N}$ be the model $(\omega, +, \cdot, \leq, 0, 1)$. If $\mathfrak{N}^*$ is any proper elementary extension of $\mathfrak{N}$ we refer to it as a *nonstandard model* of arithmetic. The elements of

$N^* \setminus N$ are the infinite integers of $\mathfrak{N}^*$. Show that the *twin prime conjecture* is true, i.e. there are infinitely many p such that p and $p+2$ are prime iff in every nonstandard model of arithmetic $\mathfrak{N}^*$ there is an infinite p such that $\mathfrak{N}^* \models$ "p and $p+2$ are prime".

8.32 Let $\mathfrak{N}^*$ be any countable nonstandard model of arithmetic. Show that

$$(N^*, \leq^*) \simeq (\omega + (\mathbb{Z} \times \mathbb{Q}), \leq)$$

8.33 Show that for any set X of primes that there exist a countable model $M \equiv \mathfrak{N}$ and an $x \in M$ such that for every prime number p

$$p \in X \text{ iff } M \models \exists y \; x = py.$$

Show that $\mathfrak{N}$ has $\mathfrak{c}$ many pairwise nonisomorphic countable elementary extensions.

8.34 Let $\mathfrak{N}_{full}$ be the model with universe ω and having a relation symbol $\underline{R}$ for every relation R of any arity on ω, operation symbol $\underline{f}$ for every operation on ω, and a constant symbol $\underline{n}$ for each element of ω. This is known as *full arithmetic*. Show that the language of full arithmetic has cardinality $\mathfrak{c}$. Show that every proper elementary extension of full arithmetic has cardinality $\geq \mathfrak{c}$.

8.35 Let $g(n)$ be the number of distinct prime divisors or n, with $g(0) = g(1) = 0$. Prove that in every nonstandard model $\mathfrak{N}^*$ of full arithmetic, there is an element b such that in $\mathfrak{N}^*$,

$$b > g(b) > g(g(b)) > g(g(g(b))) > \cdots$$

8.36 Find a set of sentences T in an uncountable language such that T has arbitrarily large finite models but no countable infinite model.

8.37 Find a theory T in an uncountable language that has no finite models and has exactly one countable model up to isomorphism, but is not complete.

Turing Machines and Recursive Functions

A *Turing machine* is a partial function m such that for some finite sets A and S the domain of m is a subset of $S \times A$ and range of m is a subset of $S \times A \times \{l, r\}$.

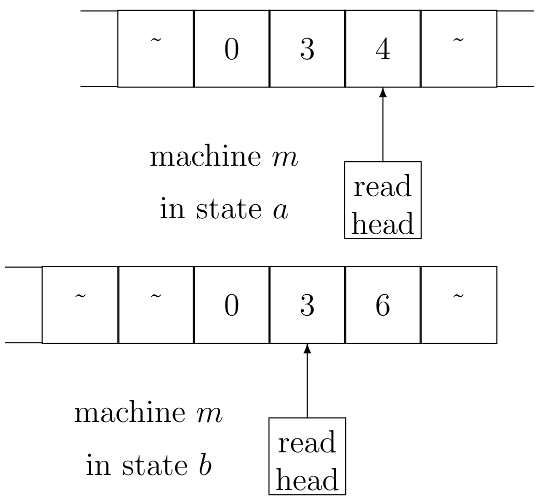
$$\text{partial } \; m : S \times A \to S \times A \times \{l, r\}$$

We call A the alphabet and S the states.

For example, suppose S is the set of letters $\{a, b, c, \ldots, z\}$ and A is the set of all integers less than seventeen, then

$$m(a, 4) = (b, 6, l)$$

would mean that when the machine m is in state a reading the symbol 4 it will go into state b, erase the symbol 4 and write the symbol 6 on the tape square where 4 was, and then move left one square.



If $(a, 4)$ is not in the domain of m, then the machine halts. This is the only way of stopping a calculation.

Let $A^{<\omega}$ be the set of all finite strings from the alphabet A. The Turing machine m gives rise to a partial function M from $A^{<\omega}$ to $A^{<\omega}$ as follows.

$$\text{partial } \; M : A^{<\omega} \to A^{<\omega}$$

We suppose that A always contains the blank space symbol:

$$\tilde{}$$

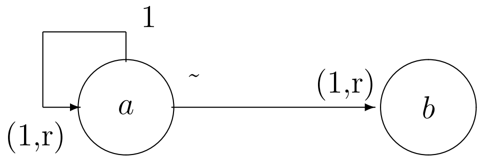
and S contains the starting state a. Given any word w from $A^{<\omega}$ we imagine a tape with w written on it and blank symbols everywhere else. We start the machine in state a and reading the leftmost symbol of w. A configuration consists of what is written on the tape, which square of tape is being read, and the state the machine is in. Successive configurations are obtained according to rules determined by m, namely if the machine is in state q reading symbol s and $m(q,s) = (q', s', d)$ then the next configuration has the same tape except the square we were reading now has the symbol s' on it, the new state is q', and the square being read is one to the left if $d = l$ and one to the right if $d = r$. If (q,s) is not in the domain of m, then the computation halts and $M(w) = v$ where v is what is written on the tape when the machine halts.

Suppose B is a finite alphabet that does not contain the blank space symbol (˜). Then a partial function

$$\text{partial } \ f : B^{<\omega} \to B^{<\omega}$$

is a *partial recursive function* iff there is a Turing machine m with an alphabet $A \supseteq B$ such that $f = M \upharpoonright B^{<\omega}$. A partial recursive function is *recursive* iff it is total. A function $f : \omega \to \omega$ is recursive iff it is recursive when considered as a map from $B^{<\omega}$ to $B^{<\omega}$ where $B = \{1\}$. Words in B can be regarded as numbers written in base one, hence we identify the number x with x ones written on the tape.

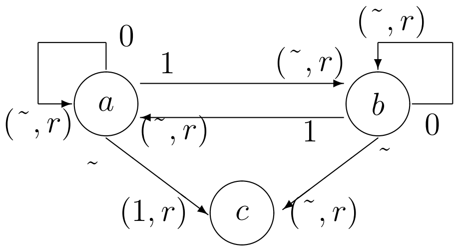
For example, the identity function is recursive, since it is computed by the empty machine. The successor function is recursive since it is computed by the machine:



$$m(a,1) = (a,1,r)$$
$$m(a,\tilde{\ }) = (b,1,r)$$

In the diagram on the left, states are represented by circles. The arrows represent the *state transition function* m. For example, the horizontal arrow represents the fact that when m is in state a and reads (˜), it writes 1, moves right, and goes into state b.

The set of strings of zeros and ones with an even number of ones is recursive. Its characteristic function (parity checker) can be computed by the following machine:



$$
\begin{aligned}
m(a,0) &= (a, \tilde{}, r)\\
m(a,1) &= (b, \tilde{}, r)\\
m(b,0) &= (b, \tilde{}, r)\\
m(b,1) &= (a, \tilde{}, r)\\
m(a, \tilde{}) &= (c, 1, r)\\
m(b, \tilde{}) &= (c, \tilde{}, r)
\end{aligned}
$$

The following problems are concerned with recursive functions and predicates on ω.

9.1 Show that any constant function is recursive.

9.2 A binary function $f : \omega \times \omega \to \omega$ is recursive iff there is a machine such that for any $x, y \in \omega$ if we input x ones and y ones separated by ",", then the machine eventually halts with $f(x,y)$ ones on the tape. Show that $f(x,y) = x + y$ is recursive.

9.3 Show that $g(x,y) = xy$ is recursive.

9.4 Let $x \dot{-} y = max\{0, x - y\}$. Show that $p(x) = x \dot{-} 1$ is recursive. Show that $q(x,y) = x \dot{-} y$ is recursive.

9.5 Suppose $f(x)$ and $g(x)$ are recursive. Show that $f(g(x))$ is recursive.

9.6 Formalize a notion of multitape Turing machine. Show that we get the same set of recursive functions.

9.7 Show that we get the same set of recursive functions even if we restrict our notion of computation to allow only tapes that are infinite in one direction.

9.8 Show that the family of recursive functions is closed under arbitrary compositions, for example $f(g(x,y), h(x,z), z)$. More generally, if $f(y_1, \ldots, y_m)$, $g_1(x_1, \ldots, x_n), \ldots$, and $g_m(x_1, \ldots, x_n)$ are all recursive, then so is

$$f(g_1(x_1, \ldots, x_n), \ldots, g_m(x_1, \ldots, x_n)).$$

9.9 Define

$$sgn(n) = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{otherwise} \end{cases}$$

Show it is recursive.

9.10 A set is recursive iff its characteristic function is. Show that the binary relation $x = y$ is recursive. Show that the binary relation $x \leq y$ is recursive.

9.11 Show that if $A \subseteq \omega$ is recursive then so is $\omega \setminus A$. Show that if A and B are recursive, then so are $A \cap B$ and $A \cup B$.

9.12 Suppose g(x) and h(x) are recursive and A is a recursive set. Show that f is recursive where:

$$f(x) = \begin{cases} g(x) & \text{if } x \in A \\ h(x) & \text{if } x \notin A \end{cases}$$

9.13 Show that the set of even numbers is recursive. Show that the set of primes is recursive.

9.14 Show that $e(x, y) = x^y$ is recursive. Show that $f(x) = x!$ is recursive.

9.15 Suppose that $h(z)$ and $g(x, y, z)$ are recursive. Define f by recursion,

- $f(0, z) = h(z)$
- $f(n + 1, z) = g(n, z, f(n, z))$.

Show that f is recursive.

9.16 We say that a set $A \subseteq \omega$ is *recursively enumerable* (re) iff it is the range of a total recursive function or the empty set. Show that a set is re iff it is the domain of a partial recursive function.

9.17 Show that every recursive set is re. Show that a set is recursive iff it and its complement are re.

9.18 Show that if f is an increasing total recursive function then the range of f is recursive.

9.19 Suppose that $f : \omega \to \omega$ and $g : \omega \to \omega$ are recursive functions such that $f(m) < g(n)$ whenever $m < n$. Prove that either the range of f or the range of g (or both) is recursive.

9.20 Let $f(n)$ be the n^{th} digit after the "." in the decimal expansion of e. ($f(1) = 7$, $f(2) = 1$, $f(3) = 8$). Prove that the function f is computable.

9.21 Show that there does not exist a total recursive function $f(n, m)$ such that for every total recursive function $g(m)$ there is an n such that $f(n, m) = g(m)$ for every m.

Church's Thesis

One day in the 1930's Alonzo Church said "Say, fellas, I think that every function that is effectively computable is recursive. I think we have captured this intuitive notion by this formal definition. From now on why don't we just assume if we describe an effective procedure it is possible to write done a Turing machine that does it. This will save a lot of time verifying silly details."[7] This philosophical position is known as *Church's Thesis*.

Here is an excerpt in support of Church's Thesis from Alan M. Turing[8]. Note that Turing uses the word computer for the person that is performing some effective procedure.

"Computing is normally done writing certain symbols on paper. We may suppose this is divided into squares like a child's arithmetic book. In elementary arithmetic the two-dimensional character of the paper is sometimes used. But such a use is always avoidable, and I think that it will be agreed that the two-dimensional character of paper is not essential for computation. I assume then that the computation is carried out on one-dimensional paper, i.e., on a tape divided into squares. I shall also suppose that the number of symbols which may be printed is finite. If we were to allow an infinity of symbols, then there would be symbols differing to an arbitrarily small extent. The effect of this restriction of the number of symbols is not very serious. It is always possible to use sequences of symbols in the place of a single symbol. Thus an Arabic numeral 17 or 999999999999999999 is normally treated as a single symbol. Similarly in any European language words are treated as single symbols (Chinese, however, attempts to have an infinity of symbols). The differences from our point of view between the single and compound symbols is that the compound symbols, if they are too lengthy, cannot be observed at one glance. This is in accordance with experience. We cannot tell at one glance whether 9999999999999999999999999 and 99999999999999999999999999 are the same.

"The behavior of the computer at any moment is determined by the symbols which he is observing, and his 'state of mind' at that moment. We may suppose that there is a bound B to the number of symbols or squares

[7] Actually these were probably not his exact words, in particular, he was interested in some bizarre notion known as the lambda calculus.

[8] "On computable numbers, with an application to the Entscheidungsproblem", Proceedings of the London Mathematical Society, 2-32(1936), 230-265.

which the computer can observe at one moment. If he wishes to observe more, he must use successive observations. We will also suppose that the number of states of mind which need be taken into account is finite. The reasons for this are of the same character as those which restrict the number of symbols. If we admitted an infinity of states of mind, some of them will be 'arbitrarily close' and will be confused. Again, the restriction is not one which seriously affects computation, since the use of more complicated states of mind can be avoided by writing more symbols on the tape.

"Let us imagine the operations performed by the computer to be split up into 'simple operations' which are so elementary that it is not easy to imagine them further divided. Every such operation consists of some change of the physical system consisting of the computer and his tape. We know the state of the system if we know the sequence of symbols on the tape, which of these are observed by the computer (possibly with a special order), and the state of mind of the computer. We may suppose that in a simple operation not more than one symbol is altered. Any other changes can be split up into simple changes of this kind. The situation in regard to squares whose symbols may be altered in this way is the same as in regard to the observed squares. We may, therefore, without loss of generality, assume that the squares whose symbols are changed are always 'observed' squares.

"Besides these changes of symbols, the simple operations must include changes of distribution of observed squares. The new observed squares must be immediately recognizable by the computer. I think it is reasonable to suppose that they can only be squares whose distance from the closest of the immediately previously observed squares does not exceed a certain fixed amount....

"The operation actually performed is determined, as has been suggested above, by the state of mind of the computer and the observed symbols. In particular, they determine the state of mind of the computer after the operation."

Other evidence for Church's thesis is the fact that all other ways people have come up with to formalize the notion of recursive functions (eg RAM machines, register machines, generalized recursive functions, etc) can be shown to define the same set of functions.

In his paper Turing also proved the following remarkable theorem.

Universal Turing Machine Theorem *There is a partial recursive function $f(n, m)$ such that for every partial recursive function $g(m)$ there is an n such that for every m, $f(n, m) = g(m)$.*

Equality here means either both sides are defined and equal or both sides are undefined.

proof:

Given the integer n we first decode it as a sequence of integers by taking its prime factorization, $n = 2^{k_1} 3^{k_2} \cdots p_m^{k_m}$ (p_m is the m^{th} prime number). Then we regard each integer k_j as some character on the typewriter (if k_j too big we ignore it). If the message coded by n is a straight forward description of a Turing machine, then we carry out the computation this machine would do when presented with input m. If this simulated computation halts with output k, then we halt with output k. If it doesn't halt, then neither does our simulation. If n does not in a straight forward way code the description of a Turing machine, then we go into an infinite loop, i.e., we just never halt.

$\square$

9.22 Let f be the universal function above and let $K = \{n : \langle n, n\rangle \in dom(f)\}$. Show that K is re but not recursive.

9.23 The family of re sets can be uniformly enumerated by $\langle W_e : e \in \omega\rangle$ where $W_e = \{n : (e, n) \in dom(f)\}$. Show there exists a recursive function $d : \omega \to \omega$ such that for any $e \in \omega$ if $K \cap W_e = \emptyset$, then $d(e) \notin K \cup W_e$. This d effectively witnesses that the complement of K is not re.

Completeness theorem

The completeness theorem was proved by Kurt Gödel in 1929. To state the theorem we must formally define the notion of proof. This is not because it is good to give formal proofs, but rather so that we can prove mathematical theorems about the concept of proof.

There are many systems for formalizing proofs in first order logic. One of the oldest is the Hilbert-Ackermann style natural deduction system. Natural systems seek to mimic commonly used informal methods of proofs. Another such system is the Beth semantic tableau method where a proof looks like a tree of formulas. This system is often used to teach undergraduates how to do formal proofs.[9]

Other systems, such as Gentzen style sequence calculus, were invented with another purpose in mind, namely to analyze the proof theoretic strength of various versions of arithmetic.

Some proof systems were invented to be easy for a computer to use. Robinson's resolution method is popular with artificial intelligence people who try to get the computer to prove mathematical theorems. On the other hand it is hard for a human being to read a proof in this style.

The proof system we will use is constructed precisely to make the completeness theorem easy to prove.

Definition of *proof*: If Σ is a set of sentences and θ is any sentence, then Σ proves θ (and we write $\Sigma \vdash \theta$) iff there exists a finite sequence $\theta_1, \ldots, \theta_n$ of sentences such that $\theta_n = \theta$ and for each i with $1 \leq i \leq n$

1. θ_i is an element of Σ, or
2. θ_i a logical axiom, or
3. θ_i can be obtained by a logical rule from the earlier θ_j for $j < i$.

Logical axioms and rules depend of course on the system being used.

Completeness Theorem: For any set of sentences Σ and sentence θ we have that:

[9] Corazza, Keisler, Kunen, Millar, Miller, Robbin, **Mathematical Logic and Computability**.

$\Sigma \vdash \theta$ iff every model of Σ is a model of θ.

MM proof system: In the MM proof system (for Mickey Mouse) the logical axioms and logical rules are the following. We let all validities be logical axioms, i.e., if a sentence θ is true in every model, then θ is a logical axiom. In the MM system there is only one logical rule: *Modus Ponens.* This is the rule that from $(\psi \to \rho)$ and ψ we can infer ρ. So in a proof if some $\theta_i = (\psi \to \rho)$ and some $\theta_j = \psi$ and $k > i, j$, then we can apply Modus Ponens to get $\theta_k = \rho$.

proof of completeness: The easy direction of the completeness theorem is often called the *soundness theorem*: if $\Sigma \vdash \theta$ then every model of Σ is a model of θ. To check that MM is sound is an easy induction on the length of the proof: If $\theta_1, \ldots, \theta_n$ is a sequence of sentences such that each θ_i is either an element of Σ, a logical validity, or can be obtained by Modus Ponens from the earlier θ_j for $j < i$; then by induction on $j \leq n$ every model of Σ is a model of θ_j.

Now we prove the other direction of the completeness theorem. So suppose that every model of Σ is a model of θ, we need to show that $\Sigma \vdash \theta$. Since every model of Σ is a model of θ the set of sentences $\Sigma \cup \{\neg\theta\}$ has no models. By the compactness theorem there exists $\{\theta_1, \ldots, \theta_n\} \subseteq \Sigma$ such that $\{\theta_1, \ldots, \theta_n, \neg\theta\}$ has no model. This means that

$$(\theta_1 \to (\theta_2 \to (\cdots \to (\theta_n \to \theta) \cdots)))$$

is a validity and hence it is a logical axiom of system MM. Now it is easy to show that $\Sigma \vdash \theta$: just write down $\theta_1, \ldots, \theta_n$, and this logical axiom, and then apply Modus Ponens n times.
□

Corollary 1 of the Completeness Theorem:
If Σ axiomatizes a theory T, then $T = \{\theta : \Sigma \vdash \theta\}$.

A set of sentences Σ is *inconsistent* iff there exists a propositional contradiction # such that $\Sigma \vdash \#$. All contradictions are logically equivalent so we write them #. Note that if Σ is inconsistent, then Σ proves every sentence, because if # is a contradiction, then $(\# \to \theta)$ is a propositional validity, so if $\Sigma \vdash \#$, then $\Sigma \vdash \theta$. A set of sentences Σ is *consistent* iff it is not inconsistent.

Corollary 2 of the Completeness Theorem:

Every consistent set of sentences has a model.

The poor MM system went to the Wizard of OZ and said, "I want to be more like all the other proof systems."

And the Wizard replied, "You've got just about everything any other proof system has and more. The completeness theorem is easy to prove in your system. You have very few logical rules and logical axioms. You lack only one thing. It is too hard for mere mortals to gaze at a proof in your system and tell whether it really is a proof. The difficulty comes from taking all logical validities as your logical axioms."

The Wizard went on to give MM a subset Val of logical validities that is recursive and has the property that every logical validity can be proved using only Modus Ponens from Val.

Let L be the largest countable language you can think of, i.e., for each $n, m < \omega$ an operation symbol $f_{n,m}$ of arity m and a relation symbol $R_{n,m}$ of arity m. Note that we think of the $f_{n,0}$ as constant symbols and the $R_{n,0}$ as propositional symbols. The symbols of L can be written in a finite alphabet if we imagine that each $n < \omega$ is a string of the symbols $\{0, 1, 2, \ldots, 9\}$. Since there are only finitely many logical symbols and variables are written x_n for $n < \omega$, the following definition makes sense.

A set Σ of formulas in the language L is recursive iff there exists an effective procedure which will decide whether θ is in Σ. That is, there exists a machine which when given any string θ of symbols in this finite alphabet will eventually halt and say "yes" or "no" depending on whether θ is in Σ. A set Σ of formulas is recursively enumerable iff there exists an effective procedure that will list all the formulas in Σ.

10.1 Show that the set of sentences of L is recursive.

10.2 Show that the set of propositional sentences of L that are validities is recursive.

10.3 An L formula θ is in *prenex normal form* iff

$$\theta = Q_1 x_1 \, Q_2 x_2 \cdots Q_n \, x_n \psi$$

where ψ is quantifier free and each Q_i is either $\forall$ or $\exists$ (i.e., all quantifiers occur up front). Show there exists a recursive map p from L-formulas to prenex normal form L-formulas such that θ and $p(\theta)$ are logically equivalent for every θ. Two formulas $\theta(x_1, \dots, x_n), \psi(x_1, \dots, x_n)$ are logically equivalent iff $\forall x_1 \cdots \forall x_n(\theta(x_1, \dots, x_n) \iff \psi(x_1, \dots, x_n))$ is a logical validity.

10.4 Let f is a new operation symbol not in appearing in $\theta(x, y)$. Show $\forall x \exists y \theta(x, y)$ has a model iff $\forall x \theta(x, f(x))$ has a model.

10.5 (Skolemization) An L formula θ is *universal* iff it is in prenex normal form and all the quantifiers are universal (Q_i is $\forall$ for each i). Show there exists a recursive map s from L sentences to universal L sentences such that

$$\theta \text{ has a model iff } s(\theta) \text{ has a model}$$

for every θ an L sentence.

10.6 Similarly an L formula is *existential* iff it is in prenex normal form and all the quantifiers are existential (Q_i is $\exists$ for each i). Show there exists a recursive map e from L sentences to existential L sentences such that

$$\theta \text{ is a validity iff } e(\theta) \text{ is a validity}$$

for every θ an L sentence.

10.7 (Herbrand) Suppose we have an existential L sentence θ of the form

$$\exists x_1 \dots \exists x_n \, \psi(x_1, \dots, x_n)$$

where ψ is quantifier free. Show that θ is a validity iff for some $m < \omega$ there exists a sequence

$$\vec{\tau_1}, \vec{\tau_2}, \dots \vec{\tau_m}$$

such that each $\vec{\tau_i}$ is a n-tuple of variable free terms of L and

$$[\psi(\vec{\tau_1}) \vee \psi(\vec{\tau_2}) \vee \cdots \vee \psi(\vec{\tau_m})] \text{ is a validity.}$$

10.8 Show that the set of quantifier free sentences of L that are validities is recursive.

10.9 Prove that the set of logical validities of L is recursively enumerable.

10.10 Find a recursive set of logical validities Val in the language L such that every logical validity in L can be proved using only Modus Ponens and logical axioms from Val.

In the following problems assume all languages are recursive subsets of L.

10.11 Show that for any recursive set Q of sentences the set of all θ such that $Q \vdash \theta$ is recursively enumerable.

10.12 A theory is said to be *recursively axiomatizable* iff it can be axiomatized by a recursive set of sentences. (Craig) Show that any recursively enumerable theory is recursively axiomatizable.

10.13 A theory axiomatized by Σ is *decidable* iff the set $\{\theta : \Sigma \vdash \theta\}$ is recursive. Show that any complete recursively axiomatizable theory is decidable. Show that the theory of $(\omega, Sc, 0)$ is decidable. Show that the theory of dense linear orderings with no end points is decidable. Show that the theory of $(\mathbb{R}, E)$ is decidable, where $E = \{(x, y) : (x - y) \in \mathbb{Q}\}$. Show that the $Th(\mathbb{C}, +, \cdot)$ is decidable.

10.14 Show that any undecidable recursively axiomatizable theory is incomplete.

10.15 Let T be a theory in a finite language without any infinite models. Show that T is decidable.

10.16 Suppose that T is a recursively axiomatizable theory. Show that T is decidable iff the set of sentences θ such that $T \cup \{\theta\}$ has a model is recursively enumerable.

10.17 Show that the set of validities in the language of pure equality is decidable.

10.18 Show that the theory of one equivalence relation is decidable.

10.19 Let T be a consistent recursively axiomatizable theory. Call a formula $\theta(x)$ in the language of T strongly finite iff in every model $\mathfrak{A}$ of T, only a finite number of elements satisfy $\theta(x)$. Prove that the set of strongly finite formulas is recursively enumerable.

10.20 Suppose T is a decidable theory and θ is a sentence in the language of T. Show that $T \cup \{\theta\}$ is decidable. Hence finite extensions of decidable theories are decidable.

10.21 Suppose T is a consistent decidable theory. Show there exists a complete consistent decidable $T' \supseteq T$ in the same language as T.

10.22 Suppose T is a consistent decidable theory in a language L. Suppose that $L' = L \cup \{c\}$ is the bigger language with one new constant symbol. Prove that the L' theory axiomatized by T is decidable.

10.23 Suppose T is a consistent decidable theory. Show there exists a consistent Henkin $T' \supseteq T$ which is decidable.

10.24 Suppose T is a consistent decidable theory. Show that T has a *recursive model* (i.e., a model whose universe is a recursive subset of ω and such that all relations and operations are recursive).

Undecidable Theories

The *incompleteness theorem* was proved by Kurt Gödel in 1930. It was improved by Barkley Rosser in 1936. A vague statement of it is that any theory which can be recursively axiomatized and is strong enough to prove elementary facts about multiplication and addition on the natural numbers must be incomplete. Here we will develop a version of this theorem based on a small fragment of the theory of finite sets.

HF is the set of all hereditarily finite sets. A set is *hereditarily finite* iff the set is finite, all elements of it are finite, all elements of elements of it are finite, etc, etc. The formal definition is that $HF = \cup_{n<\omega} V_n$ where $V_0 = \emptyset$ and V_{n+1} is the set of all subsets of V_n.

11.1 Show that if $x, y \in HF$ then $\langle x, y\rangle \in HF$. Show that if $X, Y \in HF$ then $X \times Y \in HF$. Let Y^X be the set of all functions with domain X and range Y. Show that if $X, Y \in HF$ then $Y^X \in HF$.

11.2 Prove that HF is transitive.

The Δ_0 formulas are the smallest set of formulas containing $x \in y$, $x = y$ (i.e., all atomic formulas), closed under $\vee$ and $\neg$ and bounded quantifiers $\forall x \in y$ and $\exists x \in y$. By this we mean that if θ is a Δ_0 formula and x and y are any two variables, then $\exists x \in y\ \theta$ and $\forall x \in y\ \theta$ are Δ_0 formulas. The formal definition of $\exists x \in y\ \theta$ is $\exists x(x \in y \wedge \theta)$ and $\forall x \in y\ \theta$ written formally is $\forall x(x \notin y \vee \theta)$.

A formula is Σ_1 iff it has the form:

$$\exists x_1 \exists x_2 \dots \exists x_n \phi$$

where ϕ is a Δ_0 formula. A subset X of HF is Δ_0 iff there exists a formula $\theta(x)$ (which may have constant symbols for elements of HF) and $X = \{a \in HF : (HF, \in) \models \theta(a)\}$. Similarly for Σ_1 subsets. A subset of HF is Δ_1 iff it and its complement are Σ_1.

11.3 Show that ω is a definable subset of $(HF, \in)$. Show that ω is a Δ_1 subset of $(HF, \in)$.

11.4 $A^{<\omega}$ is the set of all functions with domain some $n \in \omega$ and range contained in A. Show for any $A \in HF$ that $A^{<\omega}$ is Δ_1.

11.5 Suppose partial $m : S \times A \to S \times A \times D$ is a partial function in HF and $B \subseteq A$ and $f : B^{<\omega} \to B^{<\omega}$ is the associated partial recursive function. Show there exists a Σ_1 formula $\theta(x, y)$ such that for every $u, v \in B^{<\omega}$

$$f(u) = v \text{ iff } (HF, \in) \models \theta(u, v)$$

11.6 Show that a set $X \subseteq B^{<\omega}$ is recursively enumerable iff it is a Σ_1 subset of HF.

11.7 Show that for any $B \in HF$ every recursive subset of $B^{<\omega}$ is a Δ_1 subset of HF. Show that every Δ_1 subset of $B^{<\omega}$ is recursive.

11.8 Assume that every recursively enumerable subset of ω is a definable subset of HF. Show that the complete theory $Th(HF, \in)$ is undecidable. Show that every recursively axiomatizable $T \subseteq Th(HF, \in)$ is incomplete.

11.9 Consider the following theory of finite sets *FIN*. The language of FIN has only one binary relation E which we like to write using infix notation. There are four axioms:

Empty Set. $\exists x\, \forall y \neg(yEx)$
Extensionality. $\forall x \forall y (x = y \iff \forall z (zEx \iff zEy))$
Pairing. $\forall x \forall y \exists z \forall u (uEz \iff (u = x \vee u = y))$
Union. $\forall x\, \exists y\, (\forall z (zEy \iff (\exists u\, uEx \wedge zEu))$

Prove that $(HF, \in)$ is a model of FIN.

11.10 The theory FIN has no constant symbols, however the empty set is unique and can be defined by $\theta(x) = \forall y \neg(yEx)$. Show there exists an effective mapping from HF into Δ_0 formulas (without parameters and having one free variable x), say $a \to \theta_a(x)$, such that

$$(HF, \in) \models \theta_a(a)$$

and

$$\text{FIN } \vdash \exists! x\; \theta_a(x).$$

11.11 If M and N are models of FIN we say that $M \subseteq_e N$ iff M is a substructure of N and N is an *end extension* of M, this means that for any $a \in M$ if $N \models bEa$ then $b \in M$. Show that for every model N of FIN there exists $M \subseteq_e N$ such that M is isomorphic to $(HF, \in)$.

11.12 Let $\theta(u)$ be any Σ_1 formula. Show that for any $u \in HF$:

$$(HF, \in) \models \theta(u) \text{ iff FIN} \vdash \theta(u)$$

Since FIN has no constant symbols we really mean $\exists x(\theta_u(x) \wedge \theta(x))$ where we have written $\theta(u)$.

11.13 Show that FIN is not a decidable theory, i.e., the set

$$\{\theta : \text{FIN} \vdash \theta\}$$

is not recursive.

11.14 (Church) By considering validities of the form $\wedge \text{FIN} \rightarrow \theta$, show that the set of validities in the language of one binary relation is not recursive.

11.15 Let T be any consistent recursively axiomatizable theory in one binary relation that is consistent with FIN (eg ZFC). Show that T is incomplete.

11.16 (Mostowski, Robinson, Tarski) Call a theory T *strongly undecidable* iff T is finitely axiomatizable and every theory T' in the language of T that is consistent with T is undecidable. Show that FIN is strongly undecidable.

11.17 We say that a model $\mathfrak{A}$ is *interpretable* in $\mathfrak{B}$ iff the universe of $\mathfrak{A}$ and the relations and operations of $\mathfrak{A}$ are definable in $\mathfrak{B}$. They may have completely different languages, say L_a and L_b. Assume these languages are finite. Show there exists a recursive map r from L_a sentences to L_b such that

$$\mathfrak{A} \models \theta \text{ iff } \mathfrak{B} \models r(\theta)$$

for any L_a sentence θ. This technique is called relativization of quantifiers.

11.18 Call a structure *strongly undecidable* iff it models some strongly undecidable theory. Suppose $\mathfrak{B}$ is a structure in which a strongly undecidable structure $\mathfrak{A}$ is definable. Show that $\mathfrak{B}$ is strongly undecidable.

11.19 Show the theory of graphs is undecidable, i.e., the set of sentences true in every graph is not recursive. Hint: Let (A, R) be any binary relation and find a graph (V, E) and a pair of formulas $U(x)$ and $Q(x, y)$ in the language of graph theory such that $A = \{v \in V : (V, E) \models U(v)\}$ and $R = \{(u, v) : (V, E) \models Q(u, v)\}$.

11.20 Show that the theory of partially ordered sets is undecidable.

11.21 Show that the theory of two unary operation symbols is undecidable.

11.22 Show that the structure $(\omega, +, \cdot, 0, 1, \leq, x^y)$ is strongly undecidable. Hint: Use only positive integers. Define xEy iff there exists a prime p such that p^x divides y but p^{x+1} does not. Say that z is transitive iff for any x, y if xEy and yEz then xEz. Define the $x \approx y$ iff for all z zEx iff zEy. Define x to be minimal iff for all $y \approx x$ we have $x \leq y$. Let H be the set of all x such there exists a transitive z with xEz and every y such that yEz is minimal. Show that there exists an isomorphism from (H, E) to $(HF, \in)$.

The next goal is show that the structure $(\omega, +, \cdot)$ is strongly undecidable. To do this we need a little number theory.

11.23 Prove the *Chinese Remainder* Theorem. Given any finite sequence of pairwise relatively prime integers $\langle p_0, \ldots, p_{n-1} \rangle$ and given any finite sequence of integers $\langle x_0, \ldots, x_{n-1} \rangle$ there exists an integer x such that $x_i = x \bmod p_i$ for all $i < n$. Two integers are *relatively prime* iff their greatest common divisor is one.

11.24 Prove that $p_i = 1 + (i + 1)(n!)$ for $i < n$ are pairwise relatively prime.

11.25 Show there exists a formula $\theta(i, u, x, y)$ in the language of $(\omega, +, \cdot)$ such that for every $\langle x_0, \ldots, x_{n-1} \rangle \in \omega^{<\omega}$ there exists $x, y \in \omega$ such that for each $i < n$ x_i is the unique u such $(\omega, +, \cdot) \models \theta(i, u, x, y)$.

11.26 Show that the structure $(\omega, +, \cdot)$ is strongly undecidable.

11.27 Show that the structure $(\mathbb{Z}, +, \cdot)$ is strongly undecidable. Hint:Lagrange proved that every positive integer is the sum of four squares.

11.28 Prove that the theory of rings is undecidable.

11.29 Show that the structure $(\mathbb{Z}, +, x \text{ divides } y)$ is strongly undecidable.

11.30 (Tarski) Show that the theory of groups is undecidable. Hint: Let G be the group of all bijections of $\mathbb{Z}$ into itself where the group operation is composition. Let $s \in G$ be defined by $s(x) = x + 1$. Embed $\mathbb{Z}$ into G by mapping i to s^i. Show that for any $t \in G$ that t commutes with s iff $t = s^i$ for some integer i. Show that i divides j iff every element of G that commutes with s^i commutes with s^j. Show therefore that $(\mathbb{Z}, +, i \text{ divides } j)$ can be defined in G.

11.31 Julia Robinson showed that $\mathbb{Z}$ is explicitly definable in the structure $(\mathbb{Q}, +, \cdot)$. Prove that the theory of fields is undecidable.

11.32 Show that $\mathbb{Z}$ is not definable in $(\mathbb{C}, +, \cdot)$.

11.33 (Julia Robinson) Show that plus is definable in $(\omega, \cdot, Sc)$. Hence the structure $(\omega, \cdot, Sc)$ is strongly undecidable. Hint: Let $a + b = c$ and multiply out $(ac + 1)(bc + 1)$.

11.34 Prove or give a counterexample: Let S be a decidable set of sentences in propositional logic using the propositional letters $\{P_n : n < \omega\}$. Then

$$\{\theta : S \vdash \theta\}$$

is decidable.

11.35 Let T be a recursively axiomatizable theory with no decidable consistent complete extensions. Show that there are $\mathfrak{c}$ many distinct complete extensions of T.

11.36 Let T be a consistent recursively axiomatizable extension of FIN. Let $A \subseteq \omega$ be a nonrecursive set of integers. Show there exists a model $\mathfrak{A}$ of T that is an end extension of $(HF, \in)$ and such that for no formula $\theta(x)$ is

$$A = \{n < \omega : \mathfrak{A} \models \theta(n)\}.$$

11.37 Prove that the following set is recursively enumerable but not recursive:

$$\{\theta : \theta \text{ is an } L \text{ sentence with a finite model}\}$$

11.38 Another way to see that the structure $(\omega, +, \cdot, 0, 1, \leq)$ is strongly undecidable is to use Robinson's theory Q. The axioms of Q are

1. $0 \neq 1$
2. $\forall x(x + 0 = x)$
3. $\forall x \forall y(x + (y + 1) = (x + y) + 1$
4. $\forall x(x \cdot 0 = 0)$
5. $\forall x \forall y(x \cdot (y + 1) = x \cdot y + 1)$
6. $\forall x \forall y(x \leq y \iff \exists z(x + z = y))$

Then the Δ_0 are the smallest family of formulas containing the atomic formulas, closed under $\vee, \neg$ and bounded quantification, i.e., $\exists x \leq y$ and $\forall x \leq y$. Show that a set $A \subseteq \omega$ is recursively enumerable iff there exists a Σ_1 formula $\theta(x)$ such that for $n < \omega$

$$n \in A \text{ iff } Q \vdash \theta(n)$$

11.39 The theory of *Peano arithmetic*, *PA*, is axiomatized by Q plus the induction axioms. For each formula $\theta(x, \vec{y})$ the following is an axiom of PA:

$$\forall \vec{y}[(\theta(0, \vec{y}) \wedge \forall x(\theta(x, \vec{y}) \rightarrow \theta(x+1, \vec{y}))) \rightarrow \forall x \theta(x, \vec{y})]$$

The analogue of PA in set theory is ZFCFIN. *ZFCFIN* is ZFC minus the axiom of infinity plus the negation of the axiom of infinity. PA and ZFCFIN have exactly the same strength. There is a correspondence between models and theorems of these two theories. Formalize these statements and prove them.

The second incompleteness theorem

Gödel's *second incompleteness* theorem may be paraphrased by saying that a consistent recursively axiomatizable theory that is sufficiently strong cannot prove its own consistency unless it is inconsistent. To prove it we must give a more explicit proof of the first incompleteness theorem.

Thruout this section we will be assuming T is a consistent recursively axiomatizable theory in a language that contains a symbol n for each $n \in \omega$. (More generally it would be enough to suppose these constants were nicely definable in T, as for example they are in set theory.)

We say that the partial recursive functions are *representable* in T iff for every partial recursive function $f : \omega \mapsto \omega$ there exists a formula $\theta(x, y)$ such that for every $n, m \in \omega$

$$f(n) = m \text{ iff } T \vdash \theta(n, m)$$

and furthermore

$$T \vdash \forall x \exists^{\leq 1} y\ \theta(x, y)$$

the quantifier $\exists^{\leq 1} y$ meaning there exists at most one y. Note that we do not demand that f is total.

We say that the recursively enumerable sets are representable in T iff for every recursively enumerable set $A \subseteq \omega$ there is a formula of $\theta(x)$ such that for every $n \in \omega$,

$$n \in A \text{ iff } T \vdash \theta(n)$$

We have already seen that a consistent recursively axiomatizable theory in which every recursively enumerable set is representable must be incomplete, since there exist recursively enumerable sets that are not recursive.

12.1 Show that if every partial recursive function is representable in T then every recursively enumerable set is representable in T.

Now since T is recursively axiomatizable the formulas of the language of T are a recursive set. We use $\theta \mapsto \ulcorner\theta\urcorner$ to mean a recursive map from the sentences of the language of T to ω. One could get tricky and do this in some unnatural way, but we assume here that is done simply, as follows. First number all the symbols of T and the logical symbols. We assume that the type and arity of each symbol is given by recursive functions and also our special symbols 'n' correspond to a recursive set. The map that takes n to the number that codes the symbol n should be recursive. Then each formula is a finite string of symbols and so corresponds to a sequence of numbers $\langle a_1, a_2, \ldots, a_n \rangle$. Map this to $2^{a_1} 3^{a_2} \ldots p_n^{a_n}$ where p_n is the n^{th} prime number. Such a map (from sentences to numbers) is called a Gödel numbering.

Fixed point lemma: Suppose T is a consistent recursively axiomatizable theory in which every partial recursive function is representable. Let $\psi(x)$ be any formula with one free variable. Then there exists a sentence θ such that

$$T \vdash \text{`}\theta \iff \psi(\ulcorner \theta \urcorner)\text{'}$$

proof:

The sentence θ says in effect "I have the property defined by ψ". Consider the recursive map $\rho(x) \mapsto \rho(\ulcorner \rho \urcorner)$. That is, given a formula $\rho(x)$ with one free variable map it to the formula gotten by substituting the Gödel number of ρ into the free variable of ρ. Since this function is recursive there exist a formula $\chi(x, y)$ such that for any τ and $\rho(x)$ we have that

$$\tau = \rho(\ulcorner \rho \urcorner) \text{ iff } T \vdash \chi(\ulcorner \tau \urcorner, \ulcorner \rho \urcorner)$$

and

$$T \vdash \forall x \exists^{\leq 1} y \, \chi(x, y)$$

Now define $\sigma(x) = \exists y(\chi(x, y) \wedge \psi(y))$ and let $\theta = \sigma(\ulcorner \sigma \urcorner)$. We claim that $T \vdash \psi(\ulcorner \theta \urcorner) \iff \theta$. To see this let $\mathfrak{A}$ be any model of T. If $\mathfrak{A} \models \theta$ then $\mathfrak{A} \models \exists y(\chi(\ulcorner \sigma \urcorner, y) \wedge \psi(y))$, by the definition of θ. But $T \vdash \chi(\ulcorner \sigma \urcorner, \ulcorner \theta \urcorner)$, hence $\mathfrak{A} \models \psi(\ulcorner \theta \urcorner)$. Alternatively suppose $\mathfrak{A} \models \psi(\ulcorner \theta \urcorner)$ then $\mathfrak{A} \models \chi(\ulcorner \sigma \urcorner, \ulcorner \theta \urcorner) \wedge \psi(\ulcorner \theta \urcorner)$ Hence $\mathfrak{A} \models \exists y \, \chi(\ulcorner \sigma \urcorner, y) \wedge \psi(\ulcorner \theta y \urcorner)$, and so $\mathfrak{A} \models \theta$.

$\square$

12.2 (Tarski, truth is not definable.) Suppose T is a consistent recursively axiomatizable theory in which every partial recursive function is representable and $\mathfrak{A}$ is a model of T. Let Truth$= \{\ulcorner \theta \urcorner : \mathfrak{A} \models \theta\}$. Show that Truth is not definable in $\mathfrak{A}$.

12.3 (The First Incompleteness Theorem.) If T is a consistent recursively axiomatizable theory in which every partial recursive function is representable, then the theorems of T are a recursively enumerable set. Hence there exists a formula in the language of T such that for any θ a sentence in the language of T

$$T \vdash \mathrm{Prf}(\ulcorner\theta\urcorner) \text{ iff } T \vdash \theta$$

Show there exists a sentence θ that asserts its own unprovability, that is

$$T \vdash \text{`}\theta \iff \neg\mathrm{Prf}(\ulcorner\theta\urcorner)\text{'}$$

Show that this θ is neither provable nor refutable in T.

The Second Incompleteness Theorem: From now on fix a consistent recursively axiomatizable theory T in which every partial recursive function is representable. The statement which stands for T is consistent can now be thought of as a sentence in the language of T:

$$con(T) = \neg\mathrm{Prf}(\ulcorner 0 = 1\urcorner)$$

We wish to show that T does not prove $con(T)$. However there are many different Prf formulas and for some of these it may be possible to prove $con(T)$!

12.4 Let $\mathrm{Prf}^*(x)$ be the formula $(\mathrm{Prf}(x) \wedge x \neq \ulcorner 0 = 1\urcorner)$. Show that $T \vdash con^*(T)$ where $con^*(T) = \neg\mathrm{Prf}^*(\ulcorner 0 = 1\urcorner)$, and for every sentence θ we have $T \vdash \theta$ iff $T \vdash \mathrm{Prf}^*(\ulcorner\theta\urcorner)$.

We say that the $\mathrm{Prf}(x)$ is a *reasonable proof predicate* for a theory T iff it satisfies:

1. $T \vdash \mathrm{Prf}(\ulcorner\theta\urcorner)$ iff $T \vdash \theta$.
2. $T \vdash$‘$(\mathrm{Prf}(\ulcorner\theta\urcorner) \wedge \mathrm{Prf}(\ulcorner(\theta \to \rho)\urcorner) \to \mathrm{Prf}(\ulcorner\rho\urcorner)$’.
3. $T \vdash$‘$\mathrm{Prf}(\ulcorner\theta\urcorner) \to \mathrm{Prf}(\ulcorner\mathrm{Prf}(\ulcorner\theta\urcorner)\urcorner)$’.

The second condition is kind of a Modus ponens for proofs while the last condition basically says that if you can prove θ, then you can prove that you can prove θ.

12.5 Show that ZFC has a reasonable proof predicate.

12.6 Suppose that T is a theory with a reasonable proof predicate $\mathrm{Prf}(x)$ and θ is a sentence which asserts its own unprovability:

$$T \vdash \theta \iff \neg\mathrm{Prf}(\ulcorner\theta\urcorner)$$

Show that $T \vdash con(T) \to \theta$. Conclude that $T \not\vdash con(T)$.

12.7 (Löb) Let θ be a sentence of ZFCsuch that

$$\text{ZFC} \vdash \text{`}\theta \iff \text{Prf}(\ulcorner\theta\urcorner)\text{'}$$

Show that ZFC$\vdash \theta$.

12.8 Suppose ZFCis consistent. Show that there exists a recursively axiomatizable consistent T extending ZFCsuch that $T \vdash \neg con(T)$.

12.9 Find a consistent theory $T \supseteq$ ZFC and a first order sentence (in any language) such that

$$T \vdash \theta \text{ has a finite model}$$

but θ does not have a finite model.

12.10 Suppose T is a recursively axiomatizable theory contained in $Th(\omega, +, \cdot)$. Show that there exists a model $\mathfrak{A}$ of T and a formula $\theta(x)$ such that the least $n \in A$ such that $\mathfrak{A} \models \theta(n)$ is infinite.

Further Reading

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13.6 Set Theory, T.Jech, Academic Press, 1978.

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13.8 Self-Reference and Modal Logic, C.Smoryński, Springer-Verlag, 1985.

13.9 Undecidable Theories, A.Tarski, A.Mostowski, and R.M.Robinson, North-Holland, 1953.

13.10 Journal of Symbolic Logic, Annals of Pure and Applied Logic, Fundamenta Mathematicae.

13.11 Electronic preprints in Logic: email listserv@math.ufl.edu with “HELP settheory” in the body of the message. Or you can anonymously ftp to

ftp.math.ufl.edu

and look in directory /pub/logic.