29. Suppose $f : I \rightarrow J$, $U$ is an ultrafilter on $I$, and

$$V = \{ X \subseteq J : f^{-1}(X) \in U \}$$

Show that $V$ is an ultrafilter on $J$. This is the definition of the Rudin-Keisler ordering on ultrafilters and is written $V \leq_{\text{RK}} U$. Show that for any structure $\mathfrak{A}$ there is an elementary embedding of $\mathfrak{A}^I/V$ into $\mathfrak{A}^J/U$.

30. An ultrafilter $U$ on $I$ is called $(\kappa, \omega)$-regular iff there exists $I_\alpha \in U$ for $\alpha < \kappa$ such that for any infinite $X \subseteq \kappa$

$$\bigcap\{ I_\alpha : \alpha \in X \} = \emptyset$$

Show that $U$ is $(\kappa, \omega)$-regular iff there exists a regular ultrafilter $V$ on $[\kappa]^{<\omega}$ such that $V \leq_{\text{RK}} U$. $V$ regular means that for every $\alpha < \kappa$

$$\{ F \in [\kappa]^{<\omega} : \alpha \in F \} \in V$$

31. Let $U$ be a $(\kappa, \omega)$-regular ultrafilter on $I$ ($\kappa$ an infinite cardinal) and let $\mathfrak{A}$ be any infinite $\mathcal{L}$-structure where $|\mathcal{L}| = \kappa$. Show that $\mathfrak{A}^I/U$ is weakly saturated (i.e. realizes every consistent type).

32. Show that if $\mathfrak{A} = (\omega_1, <)$ then $\mathfrak{A}^\omega/U$ is not $\omega_2$-saturated for any ultrafilter $U$ on $\omega$.

33. Let $T$ be the theory of $(P(X), \subseteq)$ where $X$ is an infinite set, $P(X)$ is the power set of $X$, and $\subseteq$ is the binary relation of inclusion restricted to $P(X)$. Show for any infinite cardinal $\kappa$, that any $\kappa^+$-saturated model of $T$ has cardinality at least $2^\kappa$.

34. Let $\kappa$ be an infinite singular cardinal. Show that there is no linear order of cardinality $\kappa$ which is $\kappa$-saturated.