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Martin Goldstern; Mark J. Johnson; Otmar Spinas

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TOWERS ON TREES

MARTIN GOLDSTERN, MARK J. JOHNSON, AND OTMAR SPINAS

(Communicated by Franklin D. Tall)

ABSTRACT. We show that (under MA) for any $\mathcal{C}$ many dense sets in Laver forcing $\mathbb{L}$ there exists a $\sigma$-centered $\mathcal{Q} \subseteq \mathbb{L}$ such that all the given dense sets are dense in $\mathcal{Q}$. In particular, MA implies that $\mathbb{L}$ satisfies MA and does not collapse the continuum and the additivity of the Laver ideal is the continuum.


1. INTRODUCTION

In [V] a model for ZFC has been constructed where Martin's Axiom holds for perfect set forcing ($\mathcal{S}$) and the continuum is arbitrarily large. It is clear that for this we cannot just iterate forcing with $\mathcal{S}$ since then we would have to use countable supports and hence would never get the continuum larger than $\omega_2$. Velickovic's method is roughly the following. For $\mathcal{P}$ a forcing notion and $\mathcal{C}$ a class (or property) of forcing notions consider the following statement $\mathcal{C}(\mathcal{P})$:

Whenever $\mathcal{D}$ is a family of at most $2^{\aleph_0}$ many dense sets in $\mathcal{P}$, $p \in \mathcal{P}$, then there is a $\mathcal{Q} \in \mathcal{C}$, $p \in \mathcal{Q} \subseteq \mathcal{P}$, such that for every $D \in \mathcal{D}$, $D \cap \mathcal{Q}$ is dense in $\mathcal{Q}$.

Velickovic constructs a finite support iteration where each iterand is alternately either a carefully chosen finite support product of ccc suborderings of $\mathcal{S}$ or an arbitrary ccc forcing. The delicate point is to show that each of those finite support products is ccc. In that way a model is obtained where simultaneously $\text{ccc}(\mathcal{S})$ (where ccc is the class of all partial orderings satisfying the countable chain condition) and MA hold. Then clearly MA($\mathcal{S}$) is also true.

Velickovic proves that MA is not enough to get MA($\mathcal{S}$), a fortiori MA does not imply ccc($\mathcal{S}$). In [V] it is even shown that PFA implies the failure of ccc($\mathcal{S}$).

Velickovic indicates that a construction similar to that he does for $\mathcal{S}$ works for all tree-like forcings. Here we show that at least for Mathias, Laver, and...
Miller forcing there is no need for such a construction. Writing $t$ for the length of the shortest maximal tower in $[\omega]^{\omega}/\text{finite}$, we show:

**Theorem.** (a) $t = \omega$ implies $\sigma$-centered($\text{Mathias}$).
(b) $t = \omega$ implies $\sigma$-centered($\text{Laver}$).
(c) $\text{MA}(\sigma$-centered) implies $\sigma$-centered($\text{Miller}$).

Our method of proof uses ideas from [JMS]. As corollaries of the theorem we obtain that if MA holds, then first, Mathias, Laver, and Miller forceings do not collapse the continuum—in the case of Laver forcing our proof is a slight improvement of the result "MA($\sigma$-centered) implies that Laver forcing does not collapse cardinals" from [JMS] (recall that MA($\sigma$-centered) easily implies $t = \omega$), for Miller forcing this provides a correction to the proof from [JMS]. Second, MA holds for these forceings, and third, the ideals associated with them have additivity continuum.

2. Capturing density

2.1. **Definition.** If $P$ is a forcing notion, $D \subseteq P$ dense, $Q \subseteq P$, then we say that $Q$ captures the density of $D$ if $D \cap Q$ is dense in $Q$.

If $\mathcal{D}$ is a family of dense sets, then we say that $Q$ captures the density of $\mathcal{D}$ if and only if $Q$ captures the density of each $D \in \mathcal{D}$.

2.2. **Definition.** If $\mathcal{F}$ is a class of forcing notions, $P$ a forcing notion, then we write $\mathcal{F}(P)$ for the statement

Whenever $\mathcal{D}$ is a family of at most $2^{\aleph_0}$ many dense sets in $P$, $p \in P$, then there is a $Q \in \mathcal{F}$, $p \in Q \subseteq P$, which captures the density of $\mathcal{D}$.

(This notion is due to [V].)

The motivation for this concept is given by the following two facts:

2.3. **Fact.** If MA($\mathcal{F}$) and $\mathcal{F}(P)$, then MA($P$).

(Here, MA($P$) means that for any collection of less than $2^{\aleph_0}$ many dense sets in $P$ there exists a filter intersecting all of them. MA($\mathcal{F}$) means that MA($P$) holds for all $P$ in $\mathcal{F}$. Thus, the usual MA is MA(ccc).)

2.4. **Fact.** If $\text{ccc}(P)$, then $P$ does not change the cofinality of any cardinal $\leq 2^{\aleph_0}$.

*Proof.* Assume $\text{ccc}(P)$; let $\lambda < \text{cf}(\kappa)$, $\kappa \leq \omega$; and assume that

\[ p_0 \vdash f : \lambda \rightarrow \kappa \text{ is cofinal}. \]

For $\alpha < \lambda$ let

\[ D_\alpha := \{ p : p \perp p_0 \text{ or } \exists \beta p \vdash f(\check{\alpha}) = \check{\beta} \}, \]

and for $\gamma < \kappa$ let

\[ E_\gamma := \{ p : p \perp p_0 \text{ or } \exists \alpha \exists \beta > \gamma p \vdash f(\check{\alpha}) = \check{\beta} \}. \]

Clearly all sets $D_\alpha$ and all sets $E_\gamma$ are dense.
Let \( Q \subseteq P \) be a ccc set capturing the density of all \( D_\alpha \) and all \( E_\gamma \), with \( p_0 \in Q \). Let \( G \subseteq Q \) be generic, \( p_0 \in G \). Since \( Q \) is ccc,

\[
V[G] \models \text{cf}(\kappa) > \lambda.
\]

Working in \( V[G] \), let

\[
R := \{ (\alpha, \beta) : \alpha < \lambda, \beta < \kappa, \exists p \in G, V \models p \Vdash_{F} \check{\alpha} = \check{\beta} \}.
\]

It is easy to see that (in \( V[G] \)):

1. \( R \) is a function (with domain \( \lambda \)) and
2. the range of \( R \) is cofinal in \( \kappa \).

So \( V[G] \models \text{cof}(\kappa) \leq \lambda \), a contradiction.

3. LAVER FORCING

3.1. Theorem. \( t = c \) implies \( \sigma \)-centered(Laver).

3.2. Notation. \( \mathbb{L} \) will be the set of conditions in Laver forcing. For \( p \in \mathbb{L} \) we let \( \text{stem}(p) \in ^{<\omega}\omega \) be the stem of \( p \) and \( p^{-} := \{ s \in p : \text{stem}(p) \subseteq s \} \), \( \text{succ}_p(s) := \{ i : s \upharpoonright i \in p \} \).

Note that we write forcing "upwards": \( p \geq q \) if and only if \( p \subseteq q \) if and only if \( p \) "extends" \( q \) if and only if \( p \) "has more information than" \( q \).

We write \( p \geq^{0} q \) if and only if \( p \geq q \) and \( \text{stem}(p) = \text{stem}(q) \).

3.3. Definition. Let \( \kappa \) be an ordinal. A sequence \( \mathcal{A} = (A_\alpha : \alpha < \kappa) \) is called a tower of height \( \kappa \) if

1. \( \forall \alpha : A_\alpha \subseteq \omega \) and
2. \( \forall \alpha < \beta : A_\beta \subseteq^* A_\alpha \), i.e., \( A_\beta - A_\alpha \) is finite.

\( \mathcal{A} \) is called maximal if there is no set \( A_\kappa \) such that \( (A_\alpha : \alpha < \kappa + 1) \) is a tower.

We let \( t \) be the minimal height of a maximal tower. It is well known that \( \text{MA} \) (or even \( \text{MA}(\sigma \text{-centered}) \)) implies \( t = c \).

3.4. Definition. Let \( \mathcal{A} = (A_s : s \in ^{<\omega}\omega) \) be a family of infinite sets. We let

\[
\mathbb{L}_\mathcal{A} := \{ p \in \mathbb{L} : (\forall s \in p^{-}) A_s \subseteq^* \text{succ}_p(s) \}.
\]

3.5. Definition. For \( \mathcal{A} \) and \( \mathcal{B} \) as in 3.4 we write \( \mathcal{A} \geq^* \mathcal{B} \) if \( \forall s \in ^{<\omega}\omega A_s \subseteq^* B_s \).

Clearly, if \( p, q \in \mathbb{L}_\mathcal{A} \) and \( \text{stem}(p) = \text{stem}(q) \), then also \( p \cap q \in \mathbb{L}_\mathcal{A} \). Hence \( \mathbb{L}_\mathcal{A} \) is \( \sigma \)-centered.

Proof of Theorem 3.1. Let \( \mathcal{D} \) be a collection of \( c \) many dense sets in \( \mathbb{L} \), and \( p_0 \) a condition, and enumerate \( \mathbb{L} \times \mathcal{D} \) as

\[
\mathbb{L} \times \mathcal{D} = \{ (p_\alpha, D_\alpha) : \alpha < c \}.
\]

We will define a sequence

\[
(\mathcal{A}^\alpha : \alpha < c) \quad \mathcal{A}^\alpha = (A^\alpha_s : s \in ^{<\omega}\omega)
\]

satisfying

\[
(\star) \quad \forall s : (A^\alpha_s : \alpha < c) \text{ is a tower}.
\]
The construction proceeds by induction on \( \alpha \). For \( \alpha = 0 \), let
\[
A^\alpha_s := \begin{cases} 
\omega & \text{if } s \notin (p_0)^-, \\
\text{succ}_{p_0}(s) & \text{if } s \in (p_0)^-.
\end{cases}
\]

If \( \alpha \) is a limit, we define \( A^\alpha_s \) such that (\*) is satisfied, using \( t = c \).

If \( \alpha = \beta + 1 \), we distinguish two cases:

**Case 1.** For no \( \bar{A} \geq^* \bar{A}^\beta \), \( p_\beta \in L_{\bar{A}} \). In this case we let \( A^\alpha_s = A^\beta_s \) for all \( s \).

**Case 2.** Otherwise, let \( s_\beta := \text{stem}(p_\beta) \) and
\[
p'_\beta := \{ s \in p_\beta : \forall i \in \text{dom}(s) (i < |s_\beta| \text{ or } s(i) \in A^\beta_{s(i)}) \}.
\]

We claim that \( p'_\beta \) is a condition, \( p'_\beta \geq p_\beta \), and \( \text{stem}(p'_\beta) = \text{stem}(p_\beta) \), and for all \( s \in p'_\beta \) we have \( \text{succ}_{p'_\beta}(s) = \text{succ}_{p_\beta}(s) \cap A^\beta_s \). For this it is enough to see that for each \( s \in p^{-}_{\bar{A}} \), \( \text{succ}_{p_\beta}(s) \cap A^\beta_s \) is infinite. If not, then there is \( s \in p^{-}_{\bar{A}} \) such that
\[
\text{succ}_{p_\beta}(s) \cap A^\beta_s \text{ is finite.}
\]

Since \( p_\beta \in L_{\bar{A}} \) for some \( \bar{A} \geq^* \bar{A}^\beta \) and hence \( A_s \subseteq^* \text{succ}_{p_\beta}(s) \), we conclude that \( A_s \cap A^\beta_s \) is finite—a contradiction, since \( A_s \subseteq^* A^\beta_s \). Now we can find a condition \( q_\beta \geq p'_\beta \), \( q_\beta \in D_\beta \) and define
\[
A^\alpha_s := \begin{cases} 
\text{succ}_{q_\beta}(s) & \text{if } s \in q^{-}_\beta, \\
A^\beta_s & \text{otherwise}.
\end{cases}
\]

Note that \( q_\beta \in L_{\bar{A}^0} \).

This concludes the construction of the sequence of \( \bar{A}^\alpha \)’s.

Now we let \( Q := \bigcup\alpha \in \mathcal{L}_{\bar{A}^0} \), and we claim that:

1. \( \forall \beta < \alpha : L_{\bar{A}^0} \subseteq L_{\bar{A}^\beta} \),
2. \( Q \) is \( \sigma \)-centered, and
3. \( Q \) captures the density of \( \mathcal{D} \).

The proof of these claims will finish the proof of the theorem, since clearly \( p_0 \in L_{\bar{A}^0} \subseteq Q \).

**Proof of (1).** Obvious.

**Proof of (2).** If \( p, q \in Q \) with \( \text{stem}(p) = \text{stem}(q) \), then there is \( \alpha \) such that \( p, q \in L_{\bar{A}^\alpha} \). So \( p \cap q \in L_{\bar{A}^0} \subseteq Q \).

**Proof of (3).** Let \( D \in \mathcal{D} \), \( p \in Q \). We claim that there is \( q \in Q \cap D \), \( q \geq p \).

Assume that \( (D, p) = (D_\beta, p_\beta) \). We only have to show that \( q_\beta \) is well defined. But this is obvious since by \( p_\beta \in Q \) at stage \( \beta + 1 \) we must have been in Case 2.

3.6. **Remark.** A similar construction, using only one tower instead of a system of towers, shows the analogous result for Mathias forcing.

3.7. **Definition.** Let \( \ell^0 \) denote the \( \sigma \)-ideal of all \( X \subseteq \omega^{\omega} \) such that \( \forall p \in L \exists q \in L (q \geq p \text{ and } [q] \cap X = \emptyset) \). Remember that \( \text{add}(\ell^0) \) is the minimal cardinality of a family of members of \( \ell^0 \) whose union does not belong to \( \ell^0 \).
3.8. Corollary. \( MA \) implies \( \text{add}(\ell^0) = \kappa. \)

\text{Proof.} Let \( \langle \ell^n : \alpha < \kappa \rangle \) be a family in \( \ell^0 \) and \( \kappa < \kappa. \) Let \( D_\alpha = \{ p \in \mathbb{L} : [p] \cap X_\alpha = \emptyset \}. \) Observe that \( D_\alpha \) is \( \geq^0 \)-dense in \( \mathbb{L}, \) i.e., \( \forall p \in \mathbb{L} \exists q \in D_\alpha (q \geq p \) and \( \text{stem}(p) = \text{stem}(q)) \). Now choose a ccc \( Q \subseteq \mathbb{L} \) which captures the density of \( \langle D_\alpha : \alpha < \kappa \rangle. \)

Define “amoeba forcing” for \( Q \) as follows: \( \mathcal{S}(Q) = \{ (p, n) : p \in Q, 1 \leq n < \omega \} \) ordered by

\[(p, n) \geq (q, m) \text{ if and only if } p \geq q \text{ and } n \geq m \text{ and } \forall i < m p(i) = q(i)\]

where \( p(\cdot) \) is the canonical enumeration of \( p^- \). In particular, this implies \( \text{stem}(p) = p(0) = q(0) = \text{stem}(q). \)

If we know that \( D_\alpha^* \) is dense in \( \mathcal{S}(Q) \) and \( \mathcal{S}(Q) \) is ccc, then applying MA to \( \mathcal{S}(Q) \) and \( \langle D_\alpha^* : \alpha < \kappa \rangle \) we could obtain a Laver tree whose branches are disjoint from every \( X_\alpha, \) and since the whole argument could be done above a given tree, we would be done.

But, in fact, without loss of generality we may assume that each \( D_\alpha^* \) is dense. For in the construction of \( Q \) in the proof of 3.1,using the observation that each \( D_\alpha \) is \( \geq^0 \)-dense, at stage \( \alpha = \beta + 1 \) we may choose \( q_\beta \in D_\beta \) such that \( q_\beta \geq^0 p_\beta \). But then each \( D_\alpha \) is \( \geq^0 \)-dense in \( Q \) and hence, as can be easily checked, \( D_\alpha^* \) is dense in \( \mathcal{S}(Q). \) Furthermore, two conditions in \( \mathcal{S}(Q), \) say \( (p, n), (q, m) \) are compatible provided \( \text{stem}(p) = \text{stem}(q), \) \( n = m, \) and \( \forall i < n p(i) = q(i) \). This is true since two conditions in \( Q \) with the same stem \( s \) have an extension in \( Q \) with stem \( s^+. \) This shows that \( \mathcal{S}(Q) \) is even \( \sigma \)-centered.

4. Miller forcing

4.1. Theorem. \( MA(\sigma \)-centered) implies \( \sigma \)-centered(Miller).

4.2. Notation. \( \mathcal{F} \) is Miller forcing (also called rational perfect sets). Conditions \( p \in \mathcal{F} \) are superperfect trees \( p \subseteq ^{<\omega} \omega, \) that is, trees which have infinite splitting along every branch. We will consider only the dense subset of superperfect trees \( p \) with the property

\[ \forall s \in p : |\text{succ}_p(s)| \in \{ 1, \infty \}. \]

For \( p \in \mathcal{F}, \) \( \text{split}(p) \) is the set of splitting nodes of \( p, \) with smallest element \( \text{stem}(p). \) This set is also partially ordered by \( \subseteq, \) and we write \( \text{Succ}_p(s) \) for the set of direct successors of \( s \) in \( \text{split}(p). \)

Again we recall that \( p \geq q \) means “\( p \) is stronger than \( q, \)”, i.e., \( p \subseteq q \).

In 4.3–4.10 we modify the argument of [JMS] to find a \( \leq^* \) which is transitive.

4.3. Definition. Call a sequence \( \langle P_s : s \in ^{<\omega} \omega \rangle \) good if and only if

1. each \( P_s \subseteq ^{<\omega} \omega \) is infinite,
2. \( t \in P_s \) implies \( s \subseteq t, \) and
3. for \( s \in ^{<\omega} \omega \) if \( t, t' \in P_s \) and \( t \neq t', \) then \( t(n) \neq t'(n). \)

4.4. Definition. Given any good sequence \( P = \langle P_s : s \in ^{<\omega} \omega \rangle \) we determine \( \langle p_s \in \mathcal{F} : s \in ^{<\omega} \omega \rangle \) as follows. For each \( s \) let \( S \) be the smallest subset of \( ^{<\omega} \omega \) such that \( s \in S \) and if \( t \in S \) then \( P_s \subseteq S. \) Then \( p_s \) is the unique condition in \( \mathcal{F} \) such that \( S = \text{split}(p_s). \) In other words, \( s = \text{stem}(p_s) \), and if \( t \in \text{split}(p_s) \) then \( \text{Succ}_{p_s}(t) = P_t. \)

If \( Q, P', \) etc., are good, then \( q_s, p'_s, \) etc., will be defined similarly.
4.5. **Definition.** Define \( \langle P_s : s \in \prec \omega \rangle \approx \langle Q_s : s \in \prec \omega \rangle \) if and only if \( P_s \subseteq q_s \) for each \( s \in \prec \omega \). An equivalent definition would be: For all \( s \), \( P_s \subseteq \text{split}(q_s) \).

4.6. **Definition.** Define \( \langle P_s : s \in \prec \omega \rangle \approx \langle Q_s : s \in \prec \omega \rangle \) if and only if

(a) \( \forall s P_s = * Q_s \) and

(b) \( \forall \omega s P_s = Q_s \).

Clearly this is an equivalence relation.

4.7. **Definition.** Let \( \bar{P} \) and \( \bar{Q} \) be good. We will write \( \bar{P} \geq * \bar{Q} \) if and only if

1. There exists \( P' \approx P \) such that \( P' \geq \bar{Q} \).

4.8. **Lemma.** If there exists \( Q' \) such that \( P \geq \bar{Q} \approx Q \); then \( P \geq * \bar{Q} \), i.e., there is \( P' \) such that \( P \approx P' \geq \bar{Q} \). Moreover, we can choose \( P' \) such that \( P' \geq P \).

Schematically, we can write this as follows:

\[
P \geq \approx Q' \iff \exists P' \ \text{ such that } P' \geq \approx Q
\]

**Proof.** Assume we have \( \bar{P} \geq \bar{Q} \approx \bar{Q} \). Recall that for all \( s \in \prec \omega \) we have \( P_s \subseteq \text{split}(q_s) \). We can define \( P' \) by

\[
P'_s := P_s \cap \text{split}(q_s).
\]

Fix \( s \in \prec \omega \). To understand why \( P_s = * P'_s \) we consider the function \( \rho_s \) defined on \( \text{split}(q'_s) - \text{split}(q_s) \) as follows:

For any \( t \in \text{split}(q'_s) \) we can find a finite sequence \( s = r_0 \subseteq r_1 \subseteq \cdots \subseteq r_n = t \), where for all \( k < n \) we have \( r_{k+1} \in Q_{r_k} \).

For \( t \in \text{split}(q'_s) - \text{split}(q_s) \) we let

\[
\rho_s(t) := \text{ the minimal } r_{k+1} \text{ with } r_{k+1} \notin Q_{r_k}
\]

Note that \( s \subseteq \rho_s(t) \subseteq t \), so \( \rho_s(t)(|s|) = t(|s|) \), hence (by 4.3(3)) the function \( \rho_s \downharpoonright P_s \) is one-to-one. Hence

\[
|\{ t \in P_s : t \notin \text{split}(q_s) \}| \leq \sum_{t} |Q'_t - Q_t| = \text{finite}.
\]

So for all \( s \) we have \( P_s = * P'_s \), in particular, we get that \( P' \) is good.

Let \( A := \{ t \in \prec \omega : \exists s \in \prec \omega : t \subseteq s \wedge Q_s \neq Q'_t \} \). \( A \) is finite (and downward closed), and for \( s \notin A \) we have \( q_s = q'_s \) and hence \( P'_s = P_s \). So \( P' \approx P \).

Finally, it is clear that \( P' \geq \bar{Q} \) and \( P' \geq \bar{P} \).

4.9. **Remark.** This shows that

\[
P \geq * \bar{Q} \iff \text{there is } P' \approx P, P' \geq \bar{Q}, P' \geq P.
\]

**Proof.** If \( P \approx P_1 \geq \bar{Q} \), then we can apply 4.8 to the relation \( P \geq P \approx P_1 \) and get \( P' \) such that

\[
P \geq \bar{P} \approx P_1 \geq \bar{Q}
\]

4.10. **Corollary.** (a) \( \leq * \) is transitive.

(b) If \( P_1 \geq * P_2 \geq * \cdots \geq * P_n \), then there exists \( P^* \) such that for \( i = 1, \ldots, n \) we have \( P^* \geq P_i \).
Proof. If \( P^* \gtrsim Q \gtrsim R \), then there are \( P' \) and \( Q' \) such that
\[
P \approx P' \gtrsim Q \approx Q' \gtrsim R.
\]
By 4.9 we may assume \( P \leq P' \). By 4.8 we can find \( P^* \) such that
\[
P
\]
\[
f \approx \quad P'
\]
\[
f \approx \quad Q
\]
\[
f \approx \quad Q'
\]
\[
f \approx \quad R
\]
This proves (a), and also (b) for the case \( n = 3 \). For general \( n \) the proof of (b) is similar.

4.11. Fact. For \( P \) good, \( r \in F \), the following are equivalent:

1. \( \forall s \in \text{split}(r) : \text{split}(p_s) \subseteq \text{split}(r) \).
2. \( \forall s \in \text{split}(r) : P_s \subseteq \text{split}(r) \).
3. \( \forall s \in \text{split}(r) : \text{split}(p_s) \subseteq r \).
4. \( \forall s \in \text{split}(r) : p_s \subseteq r \).

Proof. (1) \( \Rightarrow \) (2) As \( P_s \subseteq \text{split}(p_s) \) for all \( s \).
(2) \( \Rightarrow \) (3) By induction on the height of \( t \in \text{split}(p_s) \) we can prove \( t \in \text{split}(r) \).
(3) \( \Rightarrow \) (4) As every node in \( p_s \) is below some node in \( \text{split}(p_s) \) and hence also in \( r \).
(4) \( \Rightarrow \) (1) As splitting points of a stronger condition are also splitting points of the weaker condition.

4.12. Definition. For \( P \) good, we let
\[
F_P := \{ r \in F : \exists P' \approx P, \forall s \in \text{split}(r) : \text{split}(p'_s) \subseteq \text{split}(r) \}.
\]

4.13. Fact. (a) \( F_P \) is \( \sigma \)-centered.
(b) If \( P \approx Q \), then \( F_P = F_Q \).
(c) If \( P \gtrsim Q \), then \( F_P \supseteq F_Q \).

Proof. For (a), assume \( r^1, r^2 \in F_P \), witnessed by \( P^1, P^2 \) respectively, and \( \text{stem}(r^1) = \text{stem}(r^2) = s \). Since \( P^1 \approx P^2 \), we may find \( p^3_s \) such that \( P^3_s = P^i_s \) for \( i = 1, 2 \) and \( \forall t \in P^3_s \forall u(t \subseteq u \rightarrow P^1_u = P^2_u) \). Define \( P^3_t = P^i_t \) for \( t \neq s \). Now clearly we have \( p^3_s \in F_P \), and \( p^3_s \) extends \( r^1, r^2 \). (b) can easily be checked. For (c), consider \( r \in F_Q \). By hypothesis find \( P', Q' \) such that \( P \approx P' \geq Q \approx Q' \), and \( \forall s \in \text{split}(r) \) we have \( q'_s \subseteq r \). By 4.8 we can find \( P'' \approx P' \), \( Q' \leq P'' \). So \( \forall s \in \text{split}(r) \) we have \( p''_s \subseteq q'_s \subseteq r \), hence \( r \in F_{p''} = F_P \).

4.14. Lemma. For all \( p \in F \), all good \( P \), and all dense sets \( D \subseteq F \): If there is \( Q \gtrsim P \) such that \( p \in F_Q \), then there is \( Q \gtrsim P \), \( p \in F_Q \), such that for some \( q \geq p \) we have \( q \in D \cap F_Q \). Moreover, if \( D \) is \( \geq^0 \)-dense, then we may assume \( \text{stem}(p) = \text{stem}(q) \).

Proof. Let \( Q^0 \gtrsim P \) such that \( p \in F_{Q^0} \). Find \( Q^1 \approx Q^0 \) such that \( \forall s \in \text{split}(p) \), \( \text{split}(q^1_s) \subseteq \text{split}(p) \). (So also \( Q^1 \gtrsim P \).) Find \( q \in D \), \( q \subseteq q^1_{\text{stem}(p)} \). So \( \text{split}(q) \subseteq \text{split}(p) \).
Now define $\bar{Q}$ as follows:

1. If $s \in \text{split}(q)$, then $Q_s := \text{Succ}_q(s)$.
2. Otherwise, $Q_s := Q_1$.

So in any case we have $Q_s \subseteq \text{split}(q^1_s)$ and hence $\bar{Q} \geq Q^1 \geq^* P$, and clearly $q \in F_{\bar{Q}}$.

4.15. **Lemma.** Assume $\text{MA}_\kappa(\sigma\text{-centered})$. If $(P_\alpha : \alpha < \kappa)$ is a $\geq^*$-descending sequence, then there exists $P_\kappa$ such that for all $\alpha$, $P_\kappa \geq^* P_\alpha$.

**Proof.** Given a sequence $(P^\alpha : \alpha < \kappa)$, we define the following forcing notion: Elements are of the form $(F, T_s : s \in S)$, where $F$ is a finite subset of $\kappa$, $S$ a finite subset of $<\omega\omega$, and each $T_s$ a finite subset of $<\omega\omega$ with

1. $t \in T_s$ implies $s \subseteq t$ and
2. for $s \in \omega^n$, if $t, t' \in T_s$ and $t \neq t'$, then $t(n) \neq t'(n)$.

We let $(F, T_s : s \in S) \leq (F', T'_s : s \in S')$ if and only if

1. $F \subseteq F'$,
2. $S \subseteq S'$,
3. $\forall s \in S : T_s \subseteq T'_s$, and
4. $\forall s \in S' \forall \alpha \in F : T'_s - T_s \subseteq \text{split}(p^s_\alpha)$ where we let $T_s = \emptyset$ for $s \notin S$.

This forcing is $\sigma$-centered (since conditions with the same $(T_s : s \in S)$ are always compatible), and a generic filter $G$ describes a good $P$ via

$$P_t := \{ \eta \in <\omega\omega : \exists (F, T_s : s \in S) \in G, t \in S, \eta \in T_t \}.$$ 

To check that each $P_t$ is infinite, we use 4.10(b) and a density argument.

Similarly as in the proof for Laver forcing, 4.15 and 4.14 imply that $\text{MA}$ (or indeed $\text{MA}(\sigma\text{-centered})$) implies $\text{ccc}(F)$ and that the additivity of the Miller ideal is $\epsilon$.

**References**


