Math 873
Fall 2009

For a review of forcing see Kunen [10] Chapter VII.

Lemma 1 *(Fusion) (Perfect set forcing (Sacks) forcing). Suppose* $p_{n+1} \leq_n p_n$ *for* $n \in \omega$*. Then the fusion*

$$q = \bigcap_{n \in \omega} p_n$$

is a perfect tree and $p \leq_n p_n$ *for all* n*.*

Lemma 2 *(Sacks forcing) (uniformly finite antichains) Suppose* $n < \omega$ *and* $p \Vdash \tau \in M$ *then there exists* $F \in M$ *with* $|F| \leq 2^{n+1}$ *and* $q \leq_n p$ *such that* $q \Vdash \tau \in \check{F}$*.*

Definition 3 *(Sacks property) Suppose* G *is Sacks-generic over* M*. Then for every* $f \in M^\omega \cap M[G]$ *there exists* $F \in M$ *such that*

$$\forall n < \omega \quad f(n) \in F(n) \text{ and } |F(n)| \leq 2^{n+1}.$$

Corallary 4 *Sacks forcing does not collapse* ω_1*. If the ground model satisfies CH, then no cardinal is collapsed.*

Theorem 5 *(Sacks [25]) Suppose* $x \in 2^\omega$ *is Sacks over* M*, then for every* $y \in M[x] \cap 2^\omega$ *either* $y \in M$ *or* $M[y] = M[x]$*. Furthermore in the latter case,* y *is itself is Sacks generic over* M*.*

Let $\mathbb{P}$ be the partial order of superperfect subtrees $p \subseteq \omega^{<\omega}$. This ordering satisfies the Fusion Lemma and the uniformly countable antichain Lemma and hence does not collapse ω_1. Also called Miller forcing.

Theorem 6 *(Miller [17]) Forcing with superperfect trees* $\mathbb{P}$ *gives a minimal degree. Any unbounded* $g \in M[G] \cap \omega^\omega$ *is itself superperfect generic.*

Remark 7 ω*-Silver forcing* $(p : D \to \omega)$ *collapses the continuum to* ω*.* ω*-Superperfect forcing (splitting nodes have all splits in) adds a Cohen real.*

Theorem 8 *(Baumgartner-Laver[2]) If* G *is Sacks generic over* M*, then no* $X \in [\omega]^\omega \cap M[G]$ *splits all* $Y \in [\omega]^\omega \cap M$ *(i.e.,* $|Y \cap X| = |Y \setminus X| = \omega$*.*

Same is true for superperfect set forcing [17].

Theorem 9 *(Silver) Silver forcing satisfies Fusion, the Sacks property, and is minimal.*

For the proof see Grigorieff [5]. The result is attributed to Silver in Mathias [14].

Mathias forcing [12] can be thought of as Silver conditions $p : D \to 2$ with $D \subseteq \omega$ and $\overline{D}$ infinite with the additional property that $p^{-1}(1)$ is finite. It adds an $X \subseteq \omega$ which reaps all $Y \subseteq \omega$ in the ground model, i.e., $X \subseteq^* Y$ or $X \subseteq^* \overline{Y}$. The enumeration of X is a dominating real.

Basic facts about product forcing, see Solovay [28].

Theorem 10 *(Adamowicz [1]) For $\mathbb{P}$ Sacks forcing, if $G_1 \times G_2$ $\mathbb{P}^2$-generic over M, then for every $x \in 2^\omega \cap M[G_1, G_2]$, $M[x]$ is either M, $M[G_1]$, $M[G_2]$, or $M[G_1, G_2]$. This fails for Silver forcing.*

Side-by-side Sacks forcing (products with countable support) $\mathbb{P}^{(\kappa)}$ has the Sacks property. See Groszek and Slaman [7] for an application of this forcing to Turing degrees.

Theorem 11 *(Shelah) It is consistent to have a model of $ZFC + \clubsuit + \neg CH$.*

(Miller) Forcing with $(\omega^{<\omega})^{(\omega_3)}$ over a model of $V = L$ yields a model $\clubsuit + \neg CH$.

Proposition 12 *The following are equivalent for models $M \subseteq N$.*

1. *$(cnd)^M$ is cofinal in $(cnd)^N$*
2. *$(meager)^M$ is cofinal in $(cnd)^N$*
3. *$(meager)^M$ is cofinal in $(meager)^N$*

Theorem 13 *For models $M \subseteq N$ we have that (1) implies (2) implies (3).*

1. *$M \subseteq N$ has the Sacks property.*
2. *$(cnd)^M$ is cofinal in $(cnd)^N$.*
3. *$M \subseteq N$ is bounded, i.e., for every $f \in N \cap \omega^\omega$ there is $g \in M \cap \omega^\omega$ with $f(n) < g(n)$ all n,*

Shelah proved (1) implies (2). (2) implies (3) is the dual form of Thm 1.2 p.94 [18]. Modified Silver forcing (see Miller [18] p.106) satisfies (2) but not (1), random real forcing satisfies (3) but not (2).

Theorem 14 *(Laver [11]) Laver forcing adds a dominating real and iterating it makes the Borel conjecture true.*

Proposition 15 *If $M \subseteq N$ has the Laver property and*

$$M \models X \subseteq [0,1] \text{ does not have SMZ}$$

then

$$M \models X \text{ does not have SMZ}$$

Theorem 16 *(Miller [15]) No Q-points in Lavers model.*

Theorem 17 *(Grigorieff [5]) If $\mathcal{U}$ is a P-point, then forcing with $\mathbb{P}_{\mathcal{U}}$ is bounding.*

Remark. (Grigorieff) If $\mathcal{U}$ not a P-point, then forcing with $\mathbb{P}_{\mathcal{U}}$ collapses the continuum or at least changes its cofinality to ω.

Theorem 18 *(Shelah, see Wimmers[24], Shelah [26]) Suppose $\mathcal{U}$ is a P-point in M and G is $\mathbb{P}_{\mathcal{U}}^{\omega}$-generic over M. Then no ultrafilter in $M[G]$ extending $\mathcal{U}$ is a P-point.*

Theorem 19 *(Ketonen [9]) Suppose $\mathfrak{d} = \mathfrak{c}$, then every filter on ω generated by few than continuum many sets can be extended to a P-point. The converse is also true.*

Theorem 20 *(Mathias [13]) If $\mathfrak{d} = \omega_1$, then there exists a Q-point.*

Corallary 21 *If $\mathfrak{c} \leq \omega_2$, then there exists a P-point or there exists a Q-point.*

Q. Does ZFC prove that there exists a P-point or there exists a Q-point?

Theorem 22 *(Charles Grey, see [6, 8]) Laver forcing is minimal.*

Theorem 23 *(Namba [23]) Namba forcing changes the cofinality of ω_2 to ω without adding a new real (assuming CH).*

The Prikry collapse of ω_1 is the poset of Prikry trees, subtrees $p \subseteq \omega_1^{<\omega}$ with the property that for every $s \in p$ there exists $t \supseteq s$ with $t\hat{\ }\langle\alpha\rangle \in p$ for uncountably many $\alpha < \omega_1$.

Theorem 24 *(Prikry) If G is Prikry collapse of ω_1 - generic over M, then for any $f \in M[G]$ such that $M[G] \models f : \omega \to \omega_1$ is unbounded $G \in M[f]$ and f itself is a generic Prikry collapse.*

Theorem 25 *(Carlson, Kunen, Miller [3]). If the ground model M satisfies MA_{ω_1} and G is the Prikry collapse of ω_1 - generic over M, then for every $x \in 2^\omega \cap M[G]$ either $x \in M$ or $G \in M[x]$.*

The Carlson collapse of ω_1 is the poset of subtrees $p \subseteq \omega_1^{<\omega}$ with the property that there is a root $s \in p$ such that for every $s \subseteq t \in p$ we have $t\hat{\ }\langle\alpha\rangle \in p$ for uncountably many $\alpha < \omega_1$.

Theorem 26 *(Carlson) If the ground model M satisfies MA_{ω_1} and G is the Carlson collapse of ω_1 - generic over M, then for every $f \in \omega^\omega \cap M[G]$ there exists $g \in \omega^\omega \cap M$ such that $f(n) < g(n)$ for all $n < \omega$.*

Theorem 27 *(Miller [22]) If the ground model M satisfies MA_{ω_1} and G is the Carlson collapse of ω_1 - generic over M, then for every $x \in 2^\omega \cap M[G]$ either $x \in M$ or $G \in M[x]$.*

The Shelah-Woodin collapse of ω_2 is the poset of subtrees $p \subseteq \omega_2^{<\omega}$ with the property that for every $s \in p$ there is $t \supseteq s$ such that $t\hat{\ }\langle\alpha\rangle \in p$ for ω_2 many $\alpha < \omega_2$. This is used in the proof of:

Theorem 28 *(Shelah-Woodin [27]) If there is a transitive model of ZFC, then there is a countable transitive model W of ZFC+CH and $a \subseteq \omega$ in a generic extension of W such that $W[x]$ models $\neg CH$ and $\omega_1^W = \omega_1^{W[a]}$.*

$$\frac{\text{Laver}}{\omega} = \frac{\text{Carlson}}{\omega_1} = \frac{\text{Namba}}{\omega_2}$$

$$\frac{\text{Miller}}{\omega} = \frac{\text{Prikry}}{\omega_1} = \frac{\text{Shelah-Woodin}}{\omega_2}$$

Theorem 29 *(Hausdorff 1936) The Hausdorff gap. 2^ω can be partitioned into ω_1 pairwise disjoint $F_{\sigma\delta}$-sets.*

Theorem 30 *(Fremlin-Shelah [4]) 2^ω can be partitioned into ω_1 pairwise disjoint G_δ-sets iff 2^ω is the union of ω_1 meager sets.*

Theorem 31 *(Miller [16]) It is consistent that 2^ω can be partitioned into ω_1 disjoint G_δ sets but cannot be partitioned into ω_1 disjoint closed sets.*

Theorem 32 *(Galvin-Prikry) For any Borel set $B \subseteq [\omega]^\omega$ there exists $H \in [\omega]^\omega$ with $[H]^\omega \subseteq B$ or $[H]^\omega \cap B = \emptyset$.*

Theorem 33 *(Ellentuck) Completely Ramsey is the same as having the property of Baire in the Ellentuck topology. Ramsey null is the same as meager which is the same as nowhere dense.*

For references for Galvin-Prikry and Ellentuck, see Miller [19].

Theorem 34 *(Mathias [12]) Mathias forcing has the Laver property. It can be decomposed as countable $*$ ccc. Every infinite subset of a Mathias generic is Mathias generic. Mathias forcing is not minimal.*

Theorem 35 *(Zapletal [29], see also Miller [21]). If $A \subseteq \omega^\omega$ is analytic then either A is disjoint from the infinite branches of a Hechler tree or A contains the infinite branches of a Laver tree.*

Theorem 36 *(Miller [20]) In the Cohen real model the hierarchy of ω_1-Borel sets has length at least $\omega_1 + 1$ but no more than $\omega_1 + 2$.*

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