

Instructions: Do two E problems and two problems in the area C, M, or S in which you signed up.

Write your letter code on **all** of your answer sheets.

If you think that a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation in your solution. In such cases, do not interpret the problem in such a way that it becomes trivial.

E1. Suppose T_i for $i < n$ (with $n < \omega$) are L -theories such that every L -structure M satisfies exactly one of the T_i . Prove that each T_i is finitely axiomatizable. If $n = \omega$ must this still be true? Prove or give a counterexample.

E2. Let $(A, <)$ be a dense total order without endpoints, and assume that A is homogeneous in the sense that (a, b) is isomorphic to A whenever $a, b \in A$ with $a < b$ (examples: \mathbb{R} , \mathbb{Q}). Let $\alpha(A)$ be the least ordinal which is not isomorphic to any subset of A . Prove that $\alpha(A)$ is a regular uncountable cardinal. Then, give examples of such A, B with $|A| = |B| = \aleph_1$ and $\alpha(A) = \omega_1$ and $\alpha(B) = \omega_2$.

E3. Prove that the set of validities in the language with two unary operation symbols is undecidable. You may assume without proof that the set of validities in the language with one binary relation symbol is undecidable.

Recall that a binary relation on a set A is a set $R \subseteq A \times A$. A unary operation on a set A is a function $f : A \rightarrow A$.

Model Theory

M1. Let L be the language whose signature contains a single unary function symbol. Show that the empty theory T in L has a model companion.

Recall that a theory U in L is a model companion of T iff U is model-complete, every model of T has an extension which is a model of U , and every model of U has an extension which is a model of T . A theory U is model-complete iff every embedding between its models is an elementary embedding.

M2. Show T has the JEP iff whenever ϕ and ψ are universal sentences and $T \vdash \phi \vee \psi$, then $T \vdash \phi$ or $T \vdash \psi$.

Recall that T has the joint embedding property (JEP) iff whenever A and B are models of T , there is some C modeling T such that both A and B are embeddable into C .

M3. Let T be a complete theory in a countable language with infinite models. Show that T has a countable model A such that for every tuple \bar{a} from A , there is a formula $\psi(\bar{x})$ with $A \models \psi(\bar{a})$ such that either (1) ψ generates a complete type over T or (2) no principal complete type over T contains ψ .

Recall that a type Φ is principal iff there is a formula ψ consistent with T which generates Φ , i.e., $T \vdash \psi \rightarrow \rho$ for every $\rho \in \Phi$.

Sketchy Answers or Hints

E1. For $n < \omega$ use compactness. For $n = \omega$ any T_0 which is not finitely axiomatizable but has a countable axiomatization can be turned into a counterexample. This problem was on the qual in 1983.

E2 ans. Note that if ξ is isomorphic to a subset of A , then by homogeneity, ξ is isomorphic to a bounded subset. It follows that $\alpha(A)$ is a limit ordinal. Now, suppose that $\text{cf}(\alpha(A)) = \theta < \alpha(A)$. Choose $a_\xi \in A$ for $\xi < \theta$ such that $\xi < \eta \rightarrow a_\xi < a_\eta$. For each $\xi < \theta$, choose a well-ordered $E_\xi \subseteq (a_\xi, a_{\xi+1})$ such that $\sup\{\text{type}(E_\xi) : \xi < \theta\} = \alpha(A)$. But then $\text{type}(\bigcup_\xi E_\xi) = \alpha(A)$, a contradiction.

For the examples, note that $\alpha(\mathbb{R}) = \omega_1$. This is homogeneous, with the isomorphisms given by rational functions. Of course, $|\mathbb{R}|$ is 2^{\aleph_0} , not \aleph_1 , so let A be an elementary submodel of the ordered field of real numbers of size \aleph_1 . Then, let B be an ordered field of size \aleph_1 which elementarily equivalent to \mathbb{R} but contains an increasing ω_1 -sequence. Then $\alpha(B) > \omega_1$, so $\alpha(B) = \omega_2$.

E3. Given $R \subseteq A^2$ consider the structure (U, f_1, f_2) where U is the disjoint union of A and R and $f_1(\langle a, b \rangle) = a$, $f_2(\langle a, b \rangle) = b$, and both are the identity on A .

M1,2,3. These problems are exercises in Hodge's big model theory book.