MODEL THEORY QUAL PROBLEMS

January 2004

M1. Prove that the following are equivalent for a cardinal κ:

a. $\kappa \geq 2^{\aleph_0}$.

b. Whenever $\mathcal{F}$ is a family of elementarily equivalent countable structures for a countable language, there is a model $\mathfrak{B}$ with $|\mathfrak{B}| = \kappa$ such that each $\mathfrak{A} \in \mathcal{F}$ is elementarily embeddable into $\mathfrak{B}$.

M2. Let $\mathcal{L} = \{+, -, \cdot, 0, 1\}$. Let Σ in $\mathcal{L}$ be the theory of algebraically closed fields of characteristic 0. Let $\mathcal{L}' = \mathcal{L} \cup \{U\}$ where U is 1-place. Let Σ' in $\mathcal{L}'$ be the theory of pairs of models of Σ, so a model of Σ' is an algebraically closed field of characteristic 0 in which U is an algebraically closed proper subfield. Prove that Σ' is complete and model-complete.

M3. Assume that $\mathcal{L}$ contains only predicate symbols. Say that a structure $\mathfrak{A}$ is *partitionable* iff it is the disjoint union of two substructures each of which is isomorphic to $\mathfrak{A}$. Prove or disprove: If $\mathfrak{A}$ is partitionable and $\mathfrak{A} \equiv \mathfrak{B}$, then $\mathfrak{B}$ is partitionable.

Here, $\mathfrak{A}$ is the *disjoint union* of $\mathfrak{A}_1, \mathfrak{A}_2$ iff $\mathfrak{A}_1, \mathfrak{A}_2$ are submodels of $\mathfrak{A}$ and A is the disjoint union of A_1, A_2.

August 2003

M1. Let $(F; +, \cdot, <)$ be an ordered field. Prove that $\cdot$ is not first-order definable in $(F; +, <)$. Here, "first-order definable" allows the use of a fixed finite list of elements of F as parameters.

Hint. Prove that the theory of ordered abelian divisible groups is model complete.

M2. Let T be a complete $\mathcal{L}$-theory with infinite models. Assume that $\mathcal{L}$ contains the symbol $<$ and that T contains the axioms that $<$ is a total order. Assume that $|\mathcal{L}| = \aleph_1$. Prove that T is not $\aleph_2$–categorical.

M3. Let $\mathcal{M} = (M; <, \dots)$ be an expansion of a dense linear order without endpoints, and assume that any $\mathcal{N}$ elementarily equivalent to $\mathcal{M}$ is o-minimal. Prove that every definable (with parameters) subset of M^2 is a finite union of definable cells.

Hint. To simplify things, you may use the weak version of the Monotonicity Theorem: if $f : (a, b) \to M$ is definable, where $a, b \in M \cup \{-\infty, +\infty\}$, then there are $a_0 = a < a_1 < \cdots < a_k < a_{k+1} = b$ such that for every $i \in \{0, \dots, k\}$, the restriction of f to the interval (a_i, a_{i+1}) is continuous.

Terminology. A cell in M is either a point or an open interval with endpoints in $M \cup \{-\infty, +\infty\}$. A set $C \subseteq M^2$ is a cell if its projection I on the first coordinate is a cell and there are definable, continuous $f, g : I \to M$ such that either:

a. $C = \{(x, y) : x \in I\, ,\ y = f(x)\}$, or
b. $C = \{(x, y) : x \in I\, ,\ y > f(x)\}$, or
c. $C = \{(x, y) : x \in I\, ,\ y < f(x)\}$, or
d. $C = \{(x, y) : x \in I\, ,\ g(x) < y < f(x)\}$ and $g(x) < f(x)$ for all $x \in I$.

A model M is o-minimal iff every definable with parameters subset of M is a finite union of cells.

January 2003

Mystery Problem 1. Let $\mathcal{L}$ be a first-order language. We say that an $\mathcal{L}$-theory T has definable Skolem functions if for any $\mathcal{L}$-formula $\phi(y, x)$, where y is a finite tuple of variables and x is a single variable, there is an $\mathcal{L}$-formula $\psi(y, x)$ such that

$$T \models \forall y \exists x \psi(y, x),$$
$$T \models \forall y \forall x \forall z ((\psi(y, x) \wedge \psi(y, z)) \to x = z),$$
$$T \models \forall y (\exists x \phi(y, x) \to \exists x (\psi(y, x) \wedge \phi(y, x)).$$

Show (without using cell decomposition) that if T is an o-minimal theory extending the theory of divisible, ordered, abelian groups, then T has definable Skolem functions.

M1. Let $\mathcal{M} = (M, <, +, 0, \dots)$ be an o-minimal expansion of a divisible, ordered, abelian group. Show from scratch that $\mathcal{M}$ has definable Skolem functions; that is, for every $n \in \mathbb{N}$ and every definable set $A \subseteq M^{n+1}$, there is a definable function $f : \Pi_n(A) \longrightarrow M$ such that $(x, f(x)) \in A$

for all $x \in \Pi_n(A)$, where $\Pi_n : M^{n+1} \longrightarrow M^n$ denotes the projection on the first n coordinates.

Hint: if $a, b \in M$ are such that $a < b$, then one can canonically pick an element from the interval (a, b) by choosing $\frac{1}{2}(a + b)$.

M2. Let $\overline{\mathbb{C}} := (\mathbb{C}, +, -, 0, 1)$ be the field of complex numbers, and let $\mathbb{A} \subset \mathbb{C}$ be the set of all algebraic numbers. Given a formula without parameters $\phi(x)$, where $x = (x_1, \ldots, x_n)$ denotes the tuple of all free variables in ϕ, we define

$$\dim \phi(\mathbb{C}^n) := \max\{\dim(a) : \ a \in \phi(M^n),\ \mathcal{M} \succeq \overline{\mathbb{C}}\},$$

where $\dim(a)$ is the pregeometry dimension of the tuple a obtained from the algebraic closure operation. We call a point $a \in \phi(\mathbb{C})$ generic if $\dim(a) = \dim \phi(\mathbb{C})$. Prove that if $n > 1$, $p(x) \in \mathbb{Z}[x]$ and $\phi(x)$ is the formula $p(x) = 0$, then $\phi(\mathbb{A}^n)$ contains no generic point, but $\phi(\mathbb{C}^n)$ does.

Mystery Problem 2. Prove that the theory of the group $(\mathbb{Z}/2\mathbb{Z})^\omega$ is categorical in every uncountable cardinal.

Hint: Show first that the theory of $(\mathbb{Z}/2\mathbb{Z})^\omega$ in the language $\mathcal{L} = (+, 0)$ admits quantifier elimination.

M3. $\mathbb{Z}_6 = \mathbb{Z}/6\mathbb{Z}$ denotes the group of integers modulo 6. Prove that the theory of the group $(\mathbb{Z}_6)^\omega$ is decidable.

August 2002

M1. Let $\mathcal{L}$ be a first-order language, $\phi(x, y)$ an $\mathcal{L}$-formula and $\mathfrak{M}$ an ω_1-saturated $\mathcal{L}$-structure. Assume that there is a sequence $(a_i : \ i \in \mathbb{N})$ of elements of $\mathfrak{M}$ such that

$$\mathfrak{M} \models \phi(a_i, a_j) \quad \Longleftrightarrow \quad i < j \qquad \text{for all } i, j \in \mathbb{N}.$$

(a) Prove that there is a set $(b_i : \ i \in \mathbb{Q})$ of elements of $\mathfrak{M}$ such that

$$\mathfrak{M} \models \phi(b_i, b_j) \quad \Longleftrightarrow \quad i < j \qquad \text{for all } i, j \in \mathbb{Q}.$$

(b) Conclude that $\mathfrak{M}$ is not ω-stable, that is, there is a countable $B \subseteq M$ with uncountably many 1-types over B.

For the next problem, we need the following **definitions:** let $\mathcal{F}$ be the collection of all functions $f : \mathbb{R} \longrightarrow \mathbb{R}$. We define an equivalence relation $\sim$ on $\mathcal{F}$ by

$$f \sim g \qquad \Longleftrightarrow \qquad \text{there is } a \in \mathbb{R} \text{ such that } f(x) = g(x) \text{ for all } x > a.$$

Given $f \in \mathcal{F}$, we denote by $[f]$ the equivalence class of f under $\sim$ (called the **germ** of f at $= \infty$), and we put $\mathcal{HH} = \mathcal{F}/\sim$. We also let $<$ be a partial ordering on $\mathcal{HH}$ defined by

$$[f] < [g] \qquad \Longleftrightarrow \qquad \text{there is } a \in \mathbb{R} \text{ such that } f(x) < g(x) \text{ for all } x > a.$$

M2. Let $\mathcal{R}$ be an expansion of the real field $(\mathbb{R}, <, +, -, \cdot, 0, 1)$, and put $\mathcal{HH}(\mathcal{R}) = \{[f] : \; f : \mathbb{R} \longrightarrow \mathbb{R} \text{ is definable in } \mathcal{R}\} \subseteq \mathcal{HH}$. Prove that $\mathcal{R}$ is o-minimal iff $\mathcal{HH}(\mathcal{R})$ is totally ordered by $<$. (Hint: use the Monotonicity Theorem for $\Longrightarrow$ and characteristic functions for $\Longleftarrow$.)

M3. Let $\mathcal{L}$ be a first-order language and $\mathcal{M}$ an infinite, strongly minimal $\mathcal{L}$-structure. Below we let $\Pi_{n-1} : M^n \longrightarrow M^{n-1}$ be the projection on the first $n-1$ coordinates, and for a set $S \subseteq M^n$ and $x' \in M^{n-1}$, we put $S_{x'} = \{x_n \in M : \; (x', x_n) \in S\}$.

We define by induction on $n \in \mathbb{N}$ what it means for a definable set $B \subseteq M^n$ to be **σ-finite**, where $\sigma \in \{0,1\}^{\{1,\dots,n\}}$:

- if $n = 1$, then B is 0-finite iff B is finite and 1-finite iff B is cofinite;
- if $n > 1$ and $\sigma \in \{0,1\}^{\{1,\dots,n\}}$, then B is σ-finite iff the set $\Pi_{n-1}(B)$ is σ'-finite, where $\sigma' \in \{0,1\}^{\{1,\dots,n-1\}}$ is given by $\sigma'(i) = \sigma(i)$ for all $i = 1, \dots, n-1$, and $B_{x'}$ is
  - finite for all $x' \in \Pi_{n-1}(B)$ if $\sigma(n) = 0$,
  - cofinite for all $x' \in \Pi_{n-1}(B)$ if $\sigma(n) = 1$.

(a) Let $B = \phi(M^n)$, where $n \in \mathbb{N}$ and $\phi(x_1, \dots, x_n)$ is an $\mathcal{L}$-formula. Assume that $\sigma \in \{0,1\}^{\{1,\dots,n\}}$ and B is σ-finite, and put $\operatorname{gdim}(B) = \sum_{i=0}^{n} \sigma(i)$. Prove that

$$\operatorname{gdim}(B) = \max\{\dim(a) : \; a \in \phi((M^*)^n), \; \mathcal{M} \preceq \mathcal{M}^*\},$$

where $\dim(a)$ is the dimension of the tuple a in the sense of the pregeometry defined on $\mathcal{M}$ by the (model-theoretic) algebraic closure operation.

(b) Let $A = \phi(M^n)$, where $n \in \mathbb{N}$ and $\phi(x_1, \dots, x_n)$ is an $\mathcal{L}$-formula, and put $\operatorname{gdim}(A) = \max\left\{\operatorname{gdim}(B) : \; B \subseteq A \text{ and } B \text{ is } \sigma\text{-finite for some } \sigma\right\}$. Prove that

$$\operatorname{gdim}(A) = \max\{\dim(a) : \; a \in \phi((M^*)^n), \; \mathcal{M} \preceq \mathcal{M}^*\}.$$

January 2002

M1. Let F be a field of characteristic zero, and let L be the first-order language with a constant symbol 0, a one-place function symbol f_λ for each $\lambda \in F$ and a two-place function symbol $+$. Let also V be a nontrivial vector space over F, and consider

$$V = (V, +, 0, f_\lambda)_{\lambda \in F}$$

as an L-structure where $+$ is vector addition, 0 is the zero vector, and each $f_\lambda : V \to V$ is scalar multiplication by λ.

(1) Show that the theory of V admits quantifier elimination. (You may use any standard facts from Linear Algebra.)
(2) Let $S \subseteq V$. Show that the algebraic closure in the model theoretic sense of S in V is equal to the linear subspace of V generated by S.

The algebraic closure in the model theoretic sense of S in V is defined to be the smallest subset A of V such that $S \subseteq A$ and for every first order formula $\varphi(x)$ with parameters from A if there are only finitely many $v \in V$ such that $\varphi(v)$ holds in V, then all of these v are in A.

M2. Let L be a first-order language and T an L-theory, and assume that T is model-complete and universally axiomatizable. Let p be a complete 1-type (over the empty set) consistent with T, and let $\phi(x)$ be an L-formula without parameters with at most one free variable x. The formula $\phi(x)$ isolates p with respect to T if and only if $\phi(x)$ is in p and

$$T \vdash \phi(x) \to \psi(x)$$

for every formula $\psi(x)$ in p. For any L-structure A and any $a \in A$ we denote by $\langle a \rangle$ the substructure of A generated by a.

Show that $\phi(x)$ isolates p with respect to T if and only if for any $M \models T$, $N \models T$, $a \in M$ and $b \in N$ such that $M \models \phi[a]$ and $N \models \phi[b]$, there is an L-isomorphism $f : \langle a \rangle \longrightarrow \langle b \rangle$ such that $f(a) = b$.

M3. Let L be the language with one binary relation symbol $<$ and one unary operation symbol f. Let T be the L-theory stating that $<$ is a dense linear ordering without endpoints and f is an order preserving bijection such that $f(x) > x$ for all x.

(1) Prove that T admits quantifier elimination.
(2) Prove that every model of T is o-minimal.

(3) Give, with justification, two functions $f, g : \mathbb{R} \longrightarrow \mathbb{R}$ such that the structures $(\mathbb{R}, <, f)$ and $(\mathbb{R}, <, g)$ are models of T, but the structure

$$(\mathbb{R}, <, f, g)$$

is not o-minimal.

A structure is o-minimal iff any subset of it which is definable with parameters is a finite union of sets each of which is a point, or an open interval with end points in the structure, or a ray with end point in the structure.

September 1, 2000

M1. Suppose that $\mathcal{L}$ is a language which contains among its symbols a unary relation symbol $\underline{\omega}$ and constants $\ulcorner 0 \urcorner, \ulcorner 1 \urcorner, \ulcorner 2 \urcorner, \ldots, \ulcorner n \urcorner, \ldots$ $(n \in \omega)$. An $\omega-$*model* for $\mathcal{L}$ is a model in which $\underline{\omega}$ is interpreted by ω and each $\ulcorner n \urcorner$ by n.

Find $\aleph_2$ first-order sentences in some language $\mathcal{L}$ such that every $\aleph_1$ of them has an ω-model, but the whole collection doesn't.

Hint: Consider linear orders with countable initial segments.

M2. Let $\mathcal{B} = (B; \wedge, \vee, \ ', U)$, where $(B; \wedge, \vee, \ ')$ is an atomless boolean algebra and U is an ultrafilter on $\mathcal{B}$ (viewed as a unary predicate). Prove that the theory of $\mathcal{B}$ is decidable.

M3. Let $\mathcal{L}$ be a first-order language and T an $\mathcal{L}$-theory. Assume that T is model complete and universally axiomatizable. Fix a model $\mathbb{A} = (A; \ldots)$ of T and a function $f : A \rightarrow A$ which is definable in $\mathbb{A}$ without using parameters. Show that f is piecewise given by $\mathcal{L}$-terms; that is, there are finitely many $\mathcal{L}$-terms $t_1(x), \ldots, t_k(x)$ each with at most one free variable x such that

$f(a) \in \{t_1(a), t_2(a), \ldots, t_k(a)\}$ for every $a \in A$.

August 1997

M1. Let U be a distinguished unary predicate in the language L. An L-structure has type (κ, λ) iff the universe has cardinality κ and the interpretation of U in the structure has cardinality λ. Let $\kappa_0 = \omega$ and for every $n < \omega$ let $\kappa_{n+1} = 2^{\kappa_n}$. Let $\kappa = \sup_{n<\omega} \kappa_n$. Let $\mathfrak{c}$ be the cardinality of the continuum. Assume that at least one of $|L|$ and κ^ω is no more than κ^+. Prove that every L-structure of type $(\kappa, \mathfrak{c})$ has an elementary extension of type $(\kappa^+, \mathfrak{c})$.

M2. Let T be defined as follows:

(a) T has unary predicates P and Q and a three place predicate E, written as yE_xz,

(b) the universe of any model of T is the disjoint union of P and Q, each infinite,

(c) if yE_xz, then $P(x)$, $Q(y)$ and $Q(z)$,

(d) for any fixed x in P, E_x is an equivalence relation on Q with infinitely many equivalence classes, and

(e) if $n < \omega$ and $x_1, \ldots, x_n \in P$ with no repetition, and $y_1, \ldots y_n \in Q$, then for some $y \in Q$ we have that for all $1 \leq l \leq n$ the relation $yE_{x_l}y_l$ holds.

(f) If $n, m < \omega$ and $x_1, \ldots x_n \in P$, while $A_1, \ldots A_m$ are disjoint finite subsets of Q, there is $x \in P$ distinct from $x_1, \ldots x_n$ such that $A_1, \ldots A_m$ are subsets of different E_x equivalence classes.

Note: we obtain a logically equivalent theory if we demand that y in (e) is different than each $y_1, \ldots y_n$.

Show that T has elimination of quantifiers.

M3. Prove that a countable complete theory which has uncountably many types has continuum many pairwise nonisomorphic countable ω-homogeneous models.

January 16, 1997

M1. Without assuming the Continuum Hypothesis, do the following:

1. Describe two structures, $\mathfrak{A}$ and $\mathfrak{B}$, for a finite language, such that: $\mathfrak{A}$ and $\mathfrak{B}$ are elementarily equivalent, $|A| = |B| = \aleph_2$, and such that there are no ultrafilters $\mathcal{U}, \mathcal{V}$ on ω with $\mathfrak{A}^\omega/\mathcal{U}$ isomorphic to $\mathfrak{B}^\omega/\mathcal{V}$.

2. Describe two structures, $\mathfrak{A}$ and $\mathfrak{B}$, for a finite language, such that: $\mathfrak{A}$ and $\mathfrak{B}$ are not isomorphic, $|A| = |B| = \aleph_2$, and such that $\mathfrak{A}^\omega/\mathcal{U}$ is isomorphic to $\mathfrak{B}^\omega/\mathcal{V}$ whenever $\mathcal{U}, \mathcal{V}$ are any non-principal ultrafilters on ω.

M2. Let $\mathfrak{M}$ be an infinite saturated $\mathcal{L}$-structure. Assume $X \subseteq M$ is definable with parameters $\vec{a} \in M^{<\omega}$; that is, for some $\mathcal{L}$-formula $\theta(x, \vec{y})$:

$$X = \{m \in M : \mathfrak{M} \models \theta(m, \vec{a})\} \quad .$$

Assume also that every automorphism f of $\mathfrak{M}$ satisfies $f(X) = X$. Prove that X is definable without parameters; that is, for some $\mathcal{L}$-formula $\psi(x)$:

$$X = \{m \in M : \mathfrak{M} \models \psi(m)\} \quad .$$

M3. Let $\mathcal{L}$ contain the symbol $<$, and let $\mathfrak{A}$ be an $\mathcal{L}$-structure in which $<_{\mathfrak{A}}$ is a total order with no largest element. Prove that $\mathfrak{A}$ has an elementary extension, $\mathfrak{B}$ such that:

1. $\mathfrak{B}$ has a non-trivial automorphism.
2. $<_{\mathfrak{B}}$ has uncountable cofinality (that is, every countable subset of B is bounded).

August 1996

M1. Let $\mathfrak{A}$ be a structure for $\mathcal{L}$, and let U be a unary predicate symbol. Assume $|U_{\mathfrak{A}}| = \mathfrak{c}$ (where $\mathfrak{c} = 2^{\aleph_0}$). Prove that $\mathfrak{A}$ has an elementary extension, $\mathfrak{B}$, such that $|U_{\mathfrak{B}}| = \mathfrak{c}^+$. Note that we are assuming nothing about $|\mathfrak{A}|$ or the size of $\mathcal{L}$.

M2. Let $\mathcal{L} = \{<\}$. Describe a complete theory T in $\mathcal{L}$ such that

1. In every model $\mathfrak{A}$ for $\mathcal{L}$, $<_A$ totally orders A
2. There are $2^{\aleph_0}$ different 1-types consistent with T.

M3. Let T be a complete theory with infinite models. Assume that T has some model with an automorphism σ of order 2 (that is, σ^2 is the identity but σ isn't). Let $\mathfrak{A}$ be any model of T. Prove that $\mathfrak{A}$ has an elementary extension $\mathfrak{B}$ such that $\mathfrak{B}$ has an automorphism of order 2.

August 1995

M1. Let L and L' be first order languages such that $L' \subseteq L$. Let T be a theory in L. Suppose that for any two models M, N for L whose L'-reducts M' and N' are isomorphic, M is a model of T if and only if N is a model of T. Prove that T is equivalent to a theory in L'.

M2. Let T be a model complete theory in a countable language. Suppose R is a binary relation in the language of T, and let K be the class of all models M of T such that M is well ordered by R^M. We say that N is an *end extension* of M if N is a proper extension of M and $N \models R(a, b)$ for all $a \in M$ and $b \in N - M$.

Suppose that K is nonempty and each $M \in K$ has an end extension $N \in K$. Prove that for each uncountable cardinal κ there exists $M \in K$ such that R^M has order type κ.

M3. Let J be an uncountable set, let A be the set of all finite subsets of J, and let $M = \langle A, R\rangle$ where R is the subset relation on A. Let $N = \langle B, S\rangle$ be a countably indexed ultrapower of M such that the natural embedding $d : M \prec N$ is proper. Prove that:

a) For each $b \in B$ the set $E_b = \{a \in A : S(d(a), b)\}$ is at most countable.

b) For each countable subset $C \subset B$, there exists $b \in B$ such that $S(c, b)$ for all $c \in C$.

August 1993

M1. Prove that there is an uncountable model for PA which is ω-homogeneous but not ω_1-homogeneous.

M2. Given two models

$$A = (U, R_1, R_1, \ldots), \qquad B = (V, S_1, S_2, \ldots)$$

of models for the same language such that U and V are disjoint, define the union to be

$$A \cup B = (U \cup V, R_1 \cup S_1, R_2 \cup S_2, \ldots).$$

Suppose that $A_1 \equiv A_2$, $B_1 \equiv B_2$, A_1, B_1 have disjoint universes, and A_2, B_2 have disjoint universes. Prove that $A_1 \cup B_1 \equiv A_2 \cup B_2$.

August 1992

M1. Prove that the theory of torsion free abelian groups is complete.

Note: An abelian group G is divisible if for each integer $n > 0$,

$$G \models (\forall x)(\exists y)ny = x.$$

G is torsion free if for each $n > 0$,

$$G \models (\forall x)[nx = 0 \;\Rightarrow\; x = 0].$$

M2. If $\mathcal{U}$ is an ultrafilter, and $\mathfrak{A}$ is a structure, let $\Pi_{\mathcal{U}}\mathfrak{A}$ be the ultrapower of $\mathfrak{A}$ modulo $\mathcal{U}$. Let $\mathbb{N}$ be the standard model of arithmetic. Prove that there exist ultrafilters $\mathcal{U}$ and $\mathcal{V}$ (possibly on different index sets) such that the models $\Pi_{\mathcal{U}}(\Pi_{\mathcal{V}}\mathbb{N})$ and $\Pi_{\mathcal{V}}(\Pi_{\mathcal{U}}\mathbb{N})$ are not isomorphic.

M3. Let $\mathfrak{A}$ be an uncountable model of Peano arithmetic and let $a \in A$. Prove that $\mathfrak{A}$ has an elementary extension $\mathfrak{B}$ with a countable sequence of elements $b_n, n \in \omega$, such that

$$\mathfrak{B} \models a + n < b_n$$

for all $n \in \omega$, and there is no element $c \in B$ such that

$$\mathfrak{B} \models a + n < c < b_n$$

for all $n \in \omega$.

January 1987

M1. Find $T_i, \Gamma_i, L_i \;\; i < 2$ such that:

(1) T_i is complete theory in L_i, $i < 2$;
(2) Γ_i complete non-principal type of T_i, $i < 2$;
(3) $T_0 \cup T_1$ is a consistent theory in $L_0 \cup L_1$; and
(4) there exists a $L_0 \cup L_1$ formula $\theta(\bar{x})$ which is consistent with $T_0 \cup T_1$ and for every formula $\psi(\bar{x}) \in (\Gamma_1(\bar{x}) \cup \Gamma_2(\bar{x}))$

$$(T_0 \cup T_1) \vdash \theta(\bar{x}) \implies \psi(\bar{x})$$

M2. Assume T is a complete consistent theory such that no complete consistent expansion of T by finitely many constants has a complete principal type. Prove that every model of T has a proper elementary substructure.

M3. $L(T) = \{<\} \cup \{c_i \mid i < \omega\}$. T is a complete consistent theory which says that $<$ is a dense linear order without endpoints and the c_i's are distinct constants. What are the possible cardinalities of the class of countable isomorphism types of models of T?

M4. Let T be the theory with countably many unary relation symbols $\{P_n : n \in \omega\}$ and all axioms of the form:

$$\exists x(\bigwedge_{n\in A} P_n \wedge \bigwedge_{n\in B} \neg P_n)$$

where A and B are disjoint finite subsets of ω. Show that T is a complete theory.

August 1986

(1) Let T be the theory with binary relation $<$ and unary operation f and axioms stating that:
- (a) $<$ is a strict linear ordering;
- (b) $<$ is dense and has no greatest or least element;
- (c) f is an automorphism of the ordering $<$; and
- (d) $\forall x\, x < f(x)$.

Show that T is complete.

(2) Let $\mathfrak{A}$ be a model for a countable language with E and other relations such that E is an equivalence relation with each class countably infinite.
- (a) Prove that $\mathfrak{A}$ has a countable elementary substructure $\mathfrak{B}$ such that any element of $\mathfrak{A}$ which is E-equivalent to an element of $\mathfrak{B}$ is an element of $\mathfrak{B}$.
- (b) Prove that $\mathfrak{A}$ has an elementary extension in which each E-equivalence class has cardinality the continuum. (Warning: $\mathfrak{A}$ may be larger than the continuum.)

(3) Suppose that T is a countable theory with an infinite model, but no countable saturated model. Show that T has uncountably many pairwise nonisomorphic models in every infinite power.