MODEL THEORY QUAL SOLUTIONS

January 2004 Answers

M1. Prove that the following are equivalent for a cardinal κ:

a. $\kappa \geq 2^{\aleph_0}$.

b. Whenever $\mathcal{F}$ is a family of elementarily equivalent countable structures for a countable language, there is a model $\mathfrak{B}$ with $|\mathfrak{B}| = \kappa$ such that each $\mathfrak{A} \in \mathcal{F}$ is elementarily embeddable into $\mathfrak{B}$.

Solution: For (a) $\rightarrow$ (b): Let Σ be the theory of the structures in $\mathcal{F}$, let $\mathfrak{B}_0$ be an $\aleph_1$–saturated model of Σ with $|\mathfrak{B}_0| = 2^{\aleph_0}$, and let $\mathfrak{B}$ be an elementary extension of $\mathfrak{B}_0$ of size κ.

For (b) $\rightarrow$ (a): Let $\mathfrak{A}_0 = (\omega; +, \cdot, 0, S)$. Let $\mathcal{F}$ be the family of all countable models which are elementarily equivalent to $\mathfrak{A}_0$. If each $\mathfrak{A} \in \mathcal{F}$ is elementarily embeddable into $\mathfrak{B}$, then $\mathfrak{B}$ realizes all types over $\{S^n(0) : n \in \omega\}$, so that $|\mathfrak{B}| \geq 2^{\aleph_0}$.

M2. Let $\mathcal{L} = \{+, -, \cdot, 0, 1\}$. Let Σ in $\mathcal{L}$ be the theory of algebraically closed fields of characteristic 0. Let $\mathcal{L}' = \mathcal{L} \cup \{U\}$ where U is 1-place. Let Σ' in $\mathcal{L}'$ be the theory of pairs of models of Σ, so a model of Σ' is an algebraically closed field of characteristic 0 in which U is an algebraically closed proper subfield. Prove that Σ' is complete and model-complete.

Solution: Completeness follows from model-completeness because there is a model $\mathfrak{M} \models \Sigma$ which embeds into every model of Σ — namely, let $U_{\mathfrak{M}}$ be the algebraic numbers and let M have transcendence degree 1.

To prove model-completeness, it is sufficient to assume $\mathfrak{A} \subset \mathfrak{B}$ with $\mathfrak{A}, \mathfrak{B} \models \Sigma$, and prove that every existential sentence true in $\mathfrak{B}_A$ is true in $\mathfrak{A}_A$. This is a little easier if you assume (WLOG) that the pair $(\mathfrak{A}, \mathfrak{B})$ is $\aleph_1$–saturated. Note that there are two cases; $U_{\mathfrak{B}}$ may or may not equal $U_{\mathfrak{A}}$. Σ does not have quantifier-elimination; for example, the formula $\exists x(xy = z \wedge U(x))$ is not equivalent to a quantifier-free formula of $\mathcal{L}$.

M3. Assume that $\mathcal{L}$ contains only predicate symbols. Say that a structure $\mathfrak{A}$ is *partitionable* iff it is the disjoint union of two substructures each of which is isomorphic to $\mathfrak{A}$. Prove or disprove: If $\mathfrak{A}$ is partitionable and $\mathfrak{A} \equiv \mathfrak{B}$, then $\mathfrak{B}$ is partitionable.

Here, $\mathfrak{A}$ is the *disjoint union* of $\mathfrak{A}_1, \mathfrak{A}_2$ iff $\mathfrak{A}_1, \mathfrak{A}_2$ are submodels of $\mathfrak{A}$ and A is the disjoint union of A_1, A_2.

Solution: This is false. In the language with countable many unary predicates P_n let T be the (complete) theory which says that P_0 is everything, and each P_{n+1} is an infinite coinfinite subset of P_n. Let $\mathfrak{A}$ be the countable model of T where the intersection of the P_n is empty and $\mathfrak{B}$ be any model of T where the intersection of the P_n has size one.

August 2003 Answers

M1. Let $(F; +, \cdot, <)$ be an ordered field. Prove that $\cdot$ is not first-order definable in $(F; +, <)$. Here, "first-order definable" allows the use of a fixed finite list of elements of F as parameters.

Hint. Prove that the theory of ordered abelian divisible groups is model complete.

Solution: Assume that $\cdot$ is first-order definable, using $p_1, \ldots, p_n$ as parameters. WLOG, $0 < p_1 < \cdots < p_n$. Taking an elementary extension of $(F; +, \cdot, <)$, we may assume, WLOG, that F is non-archimedean. Fix an infinitely large $a \in F$ with $a > p_n$, and let G be the set of all rational combinations of $a, p_1, \ldots, p_n$. Then $(G; +, <)$ is an ordered divisible abelian group, so $(G; +, <) \prec (F; +, <)$. Under the assumption that product is definable, the restriction of $\cdot$ to G would make $(G; +, \cdot, <)$ into an ordered field, which is impossible, since $a^2 \notin G$.

Proof of Hint: This is an exercise in Chang and Keisler.

If we expand the language to be $+$, $<$, $>$, $\leq$, $\geq$, 0, $-$ with obvious definitions, then the theory of ordered abelian divisible groups eliminates quantifiers. To see this, let $\exists x\ \psi(x)$ be a formula such that $\psi(x)$ is conjunction of atomic formulas of the form

$n_i x \leq y_i$, $n_i x \geq y_i$, $n_i x < y_i$, or $n_i x > y_i$

where each n_i is a positive integer and y_i is a term not involving x. We can reduce to this case by replacing $u \neq v$ by $u < v$ or $v < u$ and $u = v$ by $u \leq v$ and $v \leq u$, and $\neg u \leq v$ by $v < u$ and so forth.

Now since $nx < y$ iff $mnx < my$, and so forth, by taking the product of all the n_i we may assume that they are all the same. Since the groups are divisible we can replace n by 1. Finally to eliminate x replace $x < y_i$ and $x > y_j$ by $y_j < y_i$ and so forth for $\leq$, etc.

M2. Let T be a complete $\mathcal{L}$-theory with infinite models. Assume that $\mathcal{L}$ contains the symbol $<$ and that T contains the axioms that $<$ is a total order. Assume that $|\mathcal{L}| = \aleph_1$. Prove that T is not $\aleph_2$–categorical.

Solution: First assume that there is no greatest element. Using a compactness argument, it is easy to prove that for any model M of size $\aleph_2$ there exist an elementary extension N of size $\aleph_2$ with some element of N strictly greater than all elements of M. By using elementary chains of length either ω or ω_1 we can get models of size $\aleph_2$ with cofinality ω and ω_1. Clearly they can't be isomorphic. If there is a last element, then either there is a greatest element with no immediate predescessor and only finitely many successors or every model ends in an ω^*. In either case we add new elements just below this tail segment and above all the old elements.

Another way to do this is to use the fact that there must be models of size $\aleph_2$ which realize only $\aleph_1$ types over every subset of size $\aleph_1$. However, for a theory with a total order, one can always construct models which fail to have this property.

M3. Let $\mathcal{M} = (M; <, \dots)$ be an expansion of a dense linear order without endpoints, and assume that any $\mathcal{N}$ elementarily equivalent to $\mathcal{M}$ is o-minimal. Prove that every definable (with parameters) subset of M^2 is a finite union of definable cells.

Hint. To simplify things, you may use the weak version of the Monotonicity Theorem: if $f : (a, b) \to M$ is definable, where $a, b \in M \cup \{-\infty, +\infty\}$, then there are $a_0 = a < a_1 < \cdots < a_k < a_{k+1} = b$ such that for every $i \in \{0, \dots, k\}$, the restriction of f to the interval (a_i, a_{i+1}) is continuous.

Terminology. A cell in M is either a point or an open interval with endpoints in $M \cup \{-\infty, +\infty\}$. A set $C \subseteq M^2$ is a cell if its projection I on the first coordinate is a cell and there are definable, continuous $f, g : I \to M$ such that either:

a. $C = \{(x, y) : x \in I\,,\ y = f(x)\}$, or
b. $C = \{(x, y) : x \in I\,,\ y > f(x)\}$, or
c. $C = \{(x, y) : x \in I\,,\ y < f(x)\}$, or
d. $C = \{(x, y) : x \in I\,,\ g(x) < y < f(x)\}$ and $g(x) < f(x)$ for all $x \in I$.

A model M is o-minimal iff every definable with parameters subset of M is a finite union of cells.

Solution: Let $A \subseteq M^2$ be definable, and consider the definable set

$$B := \{(x, y) : \ y \text{ is in the boundary of the fiber } A_x\} \quad .$$

By o-minimality, each fiber B_x is finite, so by the assumption and the compactness theorem, there is $k \in \mathbb{N}$ such that $|B_x| \leq k$ for all $x \in M$. Hence by o-minimality, M can be partitioned into finitely many cells such that if I is one of these cells, there is an $l \in \{0, \dots, k\}$ such that $|B_x| = l$ for all $x \in I$. So there are definable functions $f_1, \dots, f_l : I \to M$ such that $f_1(x) < \cdots < f_l(x)$ for all $x \in I$ and $f_i(x) \in B_x$ for all x and i. By the Monotonicity Theorem, after shrinking I if necessary we may assume that each f_i is continuous. Hence $A \cap (I \times M)$ is a finite union of cells, and since I was arbitrary, the claim follows.

January 2003 Answers

Mystery Problem 1. Let $\mathcal{L}$ be a first-order language. We say that an $\mathcal{L}$-theory T has definable Skolem functions if for any $\mathcal{L}$-formula $\phi(y, x)$, where y is a finite tuple of variables and x is a single variable, there is an $\mathcal{L}$-formula $\psi(y, x)$ such that

$$T \models \forall y \exists x \psi(y, x),$$
$$T \models \forall y \forall x \forall z ((\psi(y, x) \wedge \psi(y, z)) \to x = z),$$
$$T \models \forall y (\exists x \phi(y, x) \to \exists x (\psi(y, x) \wedge \phi(y, x))).$$

Show (without using cell decomposition) that if T is an o-minimal theory extending the theory of divisible, ordered, abelian groups, then T has definable Skolem functions.

Mystery Solution: ?????

M1. Let $\mathcal{M} = (M, <, +, 0, \dots)$ be an o-minimal expansion of a divisible, ordered, abelian group. Show from scratch that $\mathcal{M}$ has definable Skolem functions; that is, for every $n \in \mathbb{N}$ and every definable set $A \subseteq M^{n+1}$, there is a definable function $f : \Pi_n(A) \longrightarrow M$ such that $(x, f(x)) \in A$ for all $x \in \Pi_n(A)$, where $\Pi_n : M^{n+1} \longrightarrow M^n$ denotes the projection on the first n coordinates.

Hint: if $a, b \in M$ are such that $a < b$, then one can canonically pick an element from the interval (a, b) by choosing $\frac{1}{2}(a + b)$.

Solution: Let $n \in \mathbb{N}$ and $A \subseteq M^{n+1}$ definable. For $x \in M^n$ put $A_x := \{y \in M : (x, y) \in A\}$, a subset of M. By o-minimality, for each $x \in M^n$ the topological boundary B_x of A_x is definable and finite. Put

$a(x) := \min B_x$ for each $x \in \Pi_n(A)$, then $a : \Pi_n(A) \longrightarrow M$ is a definable function. Consider the disjoint definable sets

$$\begin{aligned} A_1 &:= \{x \in \Pi_n(A) : \ a(x) \in A_x\}, \\ A_2 &:= \{x \in \Pi_n(A) : \ \forall y \in A_x (y > a(x))\}, \\ A_3 &:= \Pi_n(A) \setminus (A_1 \cup A_2). \end{aligned}$$

Note that $(-\infty, a(x)) \subseteq A_x$ for all $x \in A_3$. We further partition A_2:

$$\begin{aligned} A_{21} &:= \{x \in A_2 : \ (a(x), +\infty) \subseteq A_x\}, \\ A_{22} &:= A_2 \setminus A_{21}. \end{aligned}$$

Define $b : A_{22} \longrightarrow M$ by $b(x) := \min B_x \setminus \{a(x)\}$. We can now define a Skolem function $f : \Pi_n(A) \longrightarrow M$ for A:

$$f(x) := \begin{cases} a(x) & \text{if } x \in A_1, \\ a(x) + 1 & \text{if } x \in A_{21}, \\ \frac{a(x)+b(x)}{2} & \text{if } x \in A_{22}, \\ a(x) - 1 & \text{if } x \in A_3. \end{cases}$$

M2. Let $\overline{\mathbb{C}} := (\mathbb{C}, +, -, 0, 1)$ be the field of complex numbers, and let $\mathbb{A} \subset \mathbb{C}$ be the set of all algebraic numbers. Given a formula without parameters $\phi(x)$, where $x = (x_1, \ldots, x_n)$ denotes the tuple of all free variables in ϕ, we define

$$\dim \phi(\mathbb{C}^n) := \max\{\dim(a) : \ a \in \phi(M^n), \ \mathcal{M} \succeq \overline{\mathbb{C}}\},$$

where $\dim(a)$ is the pregeometry dimension of the tuple a obtained from the algebraic closure operation. We call a point $a \in \phi(\mathbb{C})$ generic if $\dim(a) = \dim \phi(\mathbb{C})$. Prove that if $n > 1$, $p(x) \in \mathbb{Z}[x]$ and $\phi(x)$ is the formula $p(x) = 0$, then $\phi(\mathbb{A}^n)$ contains no generic point, but $\phi(\mathbb{C}^n)$ does.

Solution: Any tuple $a \in \mathbb{A}^n$ has dimension 0, because $\emptyset$ is the largest algebraically independent subset of $\{a_1, \ldots, a_n\}$. On the other hand, there is $a \in \mathbb{C}^n$ such that $\dim(a) = n$: this is proved by induction on n, noting that for any $a_1, \ldots, a_{n-1} \in \mathbb{C}$, the algebraic closure of $\{a_1, \ldots, a_{n-1}\}$ in $\mathbb{C}$ is countable, so that there is $a_n \in \mathbb{C}$ that is not algebraic over $\{a_1, \ldots, a_{n-1}\}$.

Now let $n > 1$ and $p(x) \in \mathbb{Z}[x]$, and let $\mathcal{M} \succeq \overline{\mathbb{C}}$ with underlying set M. Then for any $a \in M^n$ such that $p(a) = 0$, the latter equation implies that $\dim(a) \leq n-1$. Hence $\dim \phi(\mathbb{C}^n) \leq n-1$. Now pick $a' = (a_1, \ldots, a_{n-1}) \in \mathbb{C}^{n-1}$ such that $\dim(a') = n-1$. Since $\mathbb{C}$ is algebraically closed, there is $a_n \in \mathbb{C}$ such that $p(a_1, \ldots, a_n) = 0$. Hence $n-1 \leq \dim(a) \leq \dim \phi(\mathbb{C}^n) \leq n-1$. It follows that a is a generic point of $\phi(\mathbb{C}^n)$, and since $n-1 > 0$, no point of $\phi(\mathbb{A}^n)$ is generic.

Mystery Problem 2. Prove that the theory of the group $(\mathbb{Z}/2\mathbb{Z})^\omega$ is categorical in every uncountable cardinal.

Hint: Show first that the theory of $(\mathbb{Z}/2\mathbb{Z})^\omega$ in the language $\mathcal{L} = (+, 0)$ admits quantifier elimination.

Less Mysterious Mystery Solution: Let $\phi(x_1, \dots, x_n, y)$ be a quantifier-free $\mathcal{L}$-formula in which exactly the variables $x_1, \dots, x_n$ and y occur. We need to show that $\psi(x) := \exists y \phi(x_1, \dots, x_n, y)$ is equivalent in the theory T of the $\mathcal{L}$-structure $\mathcal{M} := (\mathbb{Z}/2\mathbb{Z})^\omega$ to a quantifier-free formula. Since $\mathcal{M}$ has characteristic 2, $\phi(x_1, \dots, x_n, y)$ is equivalent in T to either $x_1 + \cdots + x_n + y = 0$ or $x_1 + \cdots + x_n + y \neq 0$. But for any $a_1, \dots, a_n \in M$, we have

$$a_1 + \cdots + a_n + (a_1 + \cdots + a_n) = 0,$$

while

$$a_1 + \cdots + a_n + (1 + a_1 + \cdots + a_n) = 1 \neq 0,$$

where 1 is the element $(1/2\mathbb{Z}, 1/2\mathbb{Z}, \dots)$ (in fact, any element different from 0 will do). So in both cases, $\psi(x_1, \dots, x_n)$ is equivalent to the $\mathcal{L}$-formula $0 = 0$. This proves quantifier elimination.

It follows that $\mathcal{M}$ is strongly minimal: let $\phi(x, y)$ be an $\mathcal{L}$-formula with a single variable x and a tuple of variables $y = (y_1, \dots, y_n)$, and let $\mathcal{M}' \models T$ with underlying set M'. By quantifier elimination, $\phi(x, y)$ is equivalent in T to either $x + y_1 + \cdots + y_n = 0$ or $x + y_1 + \cdots + y_n \neq 0$. In the first case, for every $a \in M^n$ the set $\phi(M, a)$ has exactly one element (because $\mathcal{M}'$ is a group), while in the second case the set $M \setminus \phi(M, a)$ has exactly one element.

Since T is strongly minimal and $\mathcal{L}$ is finite, it follows that T is categorical in every uncountable cardinal by the usual dimension theory for strongly minimal structures (which closely mimics the categoricity argument for algebraically closed fields). More precisely, let $\mathcal{M}_1, \mathcal{M}_2 \models T$ such that $|\mathcal{M}_1| = |\mathcal{M}_2| = \kappa > \omega$. Since T is strongly minimal, the (model-theoretic) algebraic closure relation gives rise to a pregeometry. Let B_1 and B_2 be maximal algebraically independent subsets of M_1 and M_2, respectively. Since $\mathcal{L}$ is finite and $|\mathcal{M}_1| = |\mathcal{M}_2| > \omega$, we must have $|B_1| = |B_2| = \kappa$, so there is a bijection $f : B_1 \longrightarrow B_2$. Since maximal algebraically independent subsets are indiscernible, the map f is elementary. Since $M_1 = \mathrm{acl}(B_1)$ and $M_2 = \mathrm{acl}(B_2)$, the map f extends to an isomorphism $\tilde{f} : \mathcal{M}_1 \longrightarrow \mathcal{M}_2$.

M3. $\mathbb{Z}_6 = \mathbb{Z}/6\mathbb{Z}$ denotes the group of integers modulo 6. Prove that the theory of the group $(\mathbb{Z}_6)^\omega$ is decidable.

Solution: Let T be the theory of abelian groups of exponent 6 (that is, $\forall x(x+x+x+x+x+x=0)$) such that there are infinitely many elements of order 2 and infinitely many elements of order 3. Then every model of T is a direct sum of an infinite abelian group of exponent 2 and an infinite abelian group of exponent 3. So, T is $\aleph_0$ – categorical, and hence complete.

August 2002 Answers

M1. Let $\mathcal{L}$ be a first-order language, $\phi(x, y)$ an $\mathcal{L}$-formula and $\mathfrak{M}$ an ω_1-saturated $\mathcal{L}$-structure. Assume that there is a sequence $(a_i : i \in \mathbb{N})$ of elements of $\mathfrak{M}$ such that

$$\mathfrak{M} \models \phi(a_i, a_j) \iff i < j \qquad \text{for all } i, j \in \mathbb{N}.$$

(a) Prove that there is a set $(b_i : i \in \mathbb{Q})$ of elements of $\mathfrak{M}$ such that

$$\mathfrak{M} \models \phi(b_i, b_j) \iff i < j \qquad \text{for all } i, j \in \mathbb{Q}.$$

(b) Conclude that $\mathfrak{M}$ is not ω-stable, that is, there is a countable $B \subseteq M$ with uncountably many 1-types over B.

Solution: By compactness, let $\mathfrak{B}$ be an elementary extension of $\mathfrak{M}$ containing elements b_i as in part (a). List $\mathbb{Q}$ as $\{i_n : n \in \omega\}$. Since $\mathfrak{M}$ is ω_1-saturated, we can inductively find elements $d_{i_n} \in M$ such that $(b_{i_0}, b_{i_1}, \ldots, b_{i_n})$ and $(d_{i_0}, d_{i_1}, \ldots, d_{i_n})$ realize the same type.

For part (b), if $B = \{d_i : i \in \mathbb{Q}\}$, then there is at least one 1-type over B corresponding to each proper Dedekind cut in $\mathbb{Q}$.

For the next problem, we need the following definitions: let $\mathcal{F}$ be the collection of all functions $f : \mathbb{R} \longrightarrow \mathbb{R}$. We define an equivalence relation $\sim$ on $\mathcal{F}$ by

$$f \sim g \iff \text{there is } a \in \mathbb{R} \text{ such that } f(x) = g(x) \text{ for all } x > a.$$

Given $f \in \mathcal{F}$, we denote by $[f]$ the equivalence class of f under $\sim$ (called the **germ** of f at $= \infty$), and we put $\mathcal{HH} = \mathcal{F}/\sim$. We also let $<$ be a partial ordering on $\mathcal{HH}$ defined by

$$[f] < [g] \iff \text{there is } a \in \mathbb{R} \text{ such that } f(x) < g(x) \text{ for all } x > a.$$

No solutions given for M2 or M3.

January 2002 Answers

M1. Let F be a field of characteristic zero, and let L be the first-order language with a constant symbol 0, a one-place function symbol f_λ for each $\lambda \in F$ and a two-place function symbol $+$. Let also V be a nontrivial vector space over F, and consider

$$V = (V, +, 0, f_\lambda)_{\lambda \in F}$$

as an L-structure where $+$ is vector addition, 0 is the zero vector, and each $f_\lambda : V \to V$ is scalar multiplication by λ.

(1) Show that the theory of V admits quantifier elimination. (You may use any standard facts from Linear Algebra.)

(2) Let $S \subseteq V$. Show that the algebraic closure in the model theoretic sense of S in V is equal to the linear subspace of V generated by S.

The algebraic closure in the model theoretic sense of S in V is defined to be the smallest subset A of V such that $S \subseteq A$ and for every first order formula $\varphi(x)$ with parameters from A if there are only finitely many $v \in V$ such that $\varphi(v)$ holds in V, then all of these v are in A.

Solution:

(1) Let $\exists x \ \phi(x, y_1, ..., y_n)$ be a formula such that ϕ is a conjunction of atomic and negation of atomic formulas. By using elementary linear algebra we may assume each of these conjunctions is of the form

$x = \alpha_1 y_1 + \cdots + \alpha_n y_n$ or $x \neq \alpha_1 y_1 + \cdots + \alpha_n y_n$

If the first case ever occurs, then just substitute $\alpha_1 y_1 + \cdots + \alpha_n y_n$ for x in all the others and hence eliminate x. If all the conjunctions are $\neq$ then the formula is equivalent to True.

(2) Suppose $\theta(x, a_1, \ldots, a_n)$ has only finitely many solutions. Then by part (1) it is clear that θ is logically equivalent to saying that x is one of a finite set of linear combinations of the a_i.

M2. Let L be a first-order language and T an L-theory, and assume that T is model-complete and universally axiomatizable. Let p be a complete 1-type (over the empty set) consistent with T, and let $\phi(x)$ be an L-formula without parameters with at most one free variable x. The formula $\phi(x)$ isolates p with respect to T if and only if $\phi(x)$ is in p and

$$T \vdash \phi(x) \to \psi(x)$$

for every formula $\psi(x)$ in p. For any L-structure A and any $a \in A$ we denote by $\langle a \rangle$ the substructure of A generated by a.

Show that $\phi(x)$ isolates p with respect to T if and only if for any $M \models T$, $N \models T$, $a \in M$ and $b \in N$ such that $M \models \phi[a]$ and $N \models \phi[b]$, there is an L-isomorphism $f : \langle a \rangle \longrightarrow \langle b \rangle$ such that $f(a) = b$.

Solution: Suppose $\phi(x)$ isolates p. Given a, b define $f : \langle a \rangle \longrightarrow \langle b \rangle$ by $f(\tau(a)) = \tau(b)$ where $\tau(x)$ is any term with one free variable. Then since p is complete we have that $\tau(a) = \tau'(a)$ iff $\tau(b) = \tau'(b)$ and so f is well-defined and similarly it is an isomorphism.

Suppose on the other hand that $\phi(x)$ does not isolate p, then there exists $M \models T$, $N \models T$, $a \in M$ and $b \in N$ such that $M \models \phi[a]$ and $N \models \phi[b]$, where p is the type of a in M but not the type of b in N. Since $\langle a \rangle$ and $\langle b \rangle$ are elementary substructures there can be no isomorphism f taking a to b.

M3. Let L be the language with one binary relation symbol $<$ and one unary operation symbol f. Let T be the L-theory stating that $<$ is a dense linear ordering without endpoints and f is an order preserving bijection such that $f(x) > x$ for all x.

(1) Prove that T admits quantifier elimination.
(2) Prove that every model of T is o-minimal.
(3) Give, with justification, two functions $f, g : \mathbb{R} \longrightarrow \mathbb{R}$ such that the structures $(\mathbb{R}, <, f)$ and $(\mathbb{R}, <, g)$ are models of T, but the structure
$$(\mathbb{R}, <, f, g)$$
is not o-minimal.

A structure is o-minimal iff any subset of it which is definable with parameters is a finite union of sets each of which is a point, or an open interval with end points in the structure, or a ray with end point in the structure.

Solution:

(1) Let $\exists x\ \phi(x, y_1, ..., y_n)$ be a formula such that ϕ is a conjunction of atomic and negation of atomic formulas. Temporarily add the symbol f^{-1} to the language. By using the properties of a linear order and f (ie. we can replace $x < f(x)$ by True) these conjunctions can be taken to be of the form

$x = f^n(y_i)$, $x < f^n(y_i)$ or $x > f^n(y_i)$ where n is an integer (possibly negative or zero).

If the "=" case occurs, then we may substitute and eliminate x. If one of the other cases doesn't occur then the formula is equivalent to True. If both of the other cases occur then just replace each pair $x < f^n(y_i)$, $x > f^m(y_j)$ by $f^n(y_i) > f^m(y_j)$. To get rid of negative exponents just apply f repeatedly to both sides of the equation or inequality, e.g. replace $f^{-3}(y_1) = f^2(y_2)$ by $y_1 = f^5(y_2)$, etc.

(2) Each atomic formula defines either a point or ray or empty set or the whole model. Hence by qe every definable set is a finite boolean combination of these.

(3) Let $f(x) = x+2$ and $g(x) = f(x)+\sin(x)$. Then the set of x where $f(x) = g(x)$ is the set of multiples of π.

August 2000 Answers

M1. Suppose that $\mathcal{L}$ is a language which contains among its symbols a unary relation symbol $\underline{\omega}$ and constants $\ulcorner 0 \urcorner, \ulcorner 1 \urcorner, \ulcorner 2 \urcorner, \ldots, \ulcorner n \urcorner, \ldots \quad (n \in \omega)$. An $\omega-$*model* for $\mathcal{L}$ is a model in which $\underline{\omega}$ is interpreted by ω and each $\ulcorner n \urcorner$ by n.

Find $\aleph_2$ first-order sentences in some language $\mathcal{L}$ such that every $\aleph_1$ of them has an ω-model, but the whole collection doesn't.

Hint: Consider linear orders with countable initial segments.

Solution: Let $\mathcal{L}$ have a symbol for $<$ and a constant $\ulcorner \alpha \urcorner$ for each $\alpha \leq \omega_2$, plus a binary function F, plus a constant c. Consider the structure

$$(\omega_2 + 1; <, F, \alpha)_{\alpha \leq \omega_2}.$$

In this structure, F is chosen so that when $\omega \leq \alpha < \omega_1$, the map $x \mapsto F(\alpha, x)$ defines a 1-1 map from α into ω and when $\omega_1 \leq \alpha < \omega_2$, the map $x \mapsto F(\alpha, x)$ defines a 1-1 map from α into ω_1. Let Σ be the complete diagram of this structure, plus the sentence $c < \ulcorner \omega_2 \urcorner$, plus the sentence $c \neq \ulcorner \alpha \urcorner$ for each $\alpha \leq \omega_2$.

Then, in an ω-model for Σ, every $\alpha < \omega_1$ must be standard. But then ω_1 must be standard as well, since the theory says that every element below ω_1 can be injected into ω. Repeating this argument with the ordinals between ω_1 and ω_2, we see that ω_2 must be standard as well. Thus, Σ cannot have an ω-model, but there will be an ω-model for Σ with any of the sentences $c \neq \ulcorner \alpha \urcorner$ deleted.

M2. Let $\mathcal{B} = (B; \wedge, \vee, \,', U)$, where $(B; \wedge, \vee, \,')$ is an atomless boolean algebra and U is an ultrafilter on $\mathcal{B}$ (viewed as a unary predicate). Prove that the theory of $\mathcal{B}$ is decidable.

Solution: The theory of atomless boolean algebras with an ultrafilter is complete; in fact, $\aleph_0$-categorical.

M3. Let $\mathcal{L}$ be a first-order language and T an $\mathcal{L}$-theory. Assume that T is model complete and universally axiomatizable. Fix a model $\mathbb{A} = (A; \ldots)$ of T and a function $f : A \to A$ which is definable in $\mathbb{A}$ without using parameters. Show that f is piecewise given by $\mathcal{L}$-terms; that is, there are finitely many $\mathcal{L}$-terms $t_1(x), \ldots, t_k(x)$ each with at most one free variable x such that

$f(a) \in \{t_1(a), t_2(a), \ldots, t_k(a)\}$ for every $a \in A$.

Solution: By the compactness theorem, it is enough to show that for any $a \in A$ there is an $\mathcal{L}$-term $t(x)$ with at most one free variable x such that $f(a) = t(a)$. Fix $a \in A$, and let

$$B := \{t(a) : \ t(x) \text{ is an } \mathcal{L}\text{-term with at most one free variable } x\}.$$

Then $B \subseteq A$, and if F is an n-ary function symbol of $\mathcal{L}$ and $c \in B^n$, then $f(c) \in B$. Hence B is the underlying set of the substructure $\mathcal{B}$ of $\mathbb{A}$ obtained by interpreting each symbol S of $\mathcal{L}$ as $S^{\mathcal{B}} := S^{\mathbb{A}}|_B$. Since T is universally axiomatizable we get $\mathcal{B} \models T$, and since T is model complete it follows that $\mathcal{B}$ is an elementary substructure of $\mathbb{A}$. Thus, $f(a)$ being the unique image of a under f in $\mathbb{A}$ implies that $f(a) \in B$, that is, $f(a) = t(a)$ for some $\mathcal{L}$-term $t(x)$ with at most one free variable x, as desired.

August 1997 Answers

M1. Let U be a distinguished unary predicate in the language L. An L-structure has type (κ, λ) iff the universe has cardinality κ and the interpretation of U in the structure has cardinality λ. Let $\kappa_0 = \omega$ and for every $n < \omega$ let $\kappa_{n+1} = 2^{\kappa_n}$. Let $\kappa = \sup_{n<\omega} \kappa_n$. Let $\mathfrak{c}$ be the cardinality of the continuum. Assume that at least one of $|L|$ and κ^ω is no more than κ^+. Prove that every L-structure of type $(\kappa, \mathfrak{c})$ has an elementary extension of type $(\kappa^+, \mathfrak{c})$.

Answer M1. Take an ω-ultrapower of the model. In the case the language is small take an elementary substructure.

M2. Let T be defined as follows:

(a) T has unary predicates P and Q and a three place predicate E, written as yE_xz,

(b) the universe of any model of T is the disjoint union of P and Q, each infinite,

(c) if yE_xz, then $P(x)$, $Q(y)$ and $Q(z)$,

(d) for any fixed x in P, E_x is an equivalence relation on Q with infinitely many equivalence classes, and

(e) if $n < \omega$ and $x_1, \ldots, x_n \in P$ with no repetition, and $y_1, \ldots y_n \in Q$, then for some $y \in Q$ we have that for all $1 \leq l \leq n$ the relation $yE_{x_l}y_l$ holds.

(f) If $n, m < \omega$ and $x_1, \dots x_n \in P$, while $A_1, \dots A_m$ are disjoint finite subsets of Q, there is $x \in P$ distinct from $x_1, \dots x_n$ such that $A_1, \dots A_m$ are subsets of different E_x equivalence classes.

Note: we obtain a logically equivalent theory if we demand that y in (e) is different than each $y_1, \dots y_n$.

Show that T has elimination of quantifiers.

Answer M2. Solution 1: Do the elimination of quantifiers directly, defining an arrangement of $x_1, \dots x_n$ as a formula which specifies which x_i's are P and which Q, which x_i, x_j are equal to each other, and for each x_k in P and x_i, x_j in Q, tells if $x_i E_{x_k} x_j$ or not. Use axioms (d) and (e) in the crucial step of the argument.

Solution 2: Use the notion of model completeness. First observe that T is a consistent theory. Notice that $T_\forall$ is the theory of equivalence relations, which has the amalgamation property. It suffices to show that T is the model completion of $T_\forall$. This can be done by considering all models of $T_\forall$ which are existentially closed for $T_\forall$. Else, we can show that T is model complete by Robinson's test, or we can use a syntactical characterization (see e.g Theorem 3.5.17 in Chang-Keisler's book).

M3. Prove that a countable complete theory which has uncountably many types has continuum many pairwise nonisomorphic countable ω-homogeneous models.

Answer M3. If a countable theory has uncountably many types, then it has continuum many n-types for some n. Every countable model in a countable language extends to a countable ω-homogeneous model, so every n-type is realized in some countable ω-homogeneous model. Since each countable model can realize only countably many types, there must be continuum many non-isomorphic countable ω-homogeneous models.

January 1997 Answers

M1. Without assuming the Continuum Hypothesis, do the following:

1. Describe two structures, $\mathfrak{A}$ and $\mathfrak{B}$, for a finite language, such that: $\mathfrak{A}$ and $\mathfrak{B}$ are elementarily equivalent, $|A| = |B| = \aleph_2$, and such that there are no ultrafilters $\mathcal{U}, \mathcal{V}$ on ω with $\mathfrak{A}^\omega/\mathcal{U}$ isomorphic to $\mathfrak{B}^\omega/\mathcal{V}$.

2. Describe two structures, $\mathfrak{A}$ and $\mathfrak{B}$, for a finite language, such that: $\mathfrak{A}$ and $\mathfrak{B}$ are not isomorphic, $|A| = |B| = \aleph_2$, and such that $\mathfrak{A}^\omega/\mathcal{U}$ is isomorphic to $\mathfrak{B}^\omega/\mathcal{V}$ whenever $\mathcal{U}, \mathcal{V}$ are any non-principal ultrafilters on ω.

Answer M1. For 1: Let $\mathfrak{A}$ and $\mathfrak{B}$ both code three total orders. In $\mathfrak{A}$, the orders have cofinalities $\omega, \omega_1, \omega_2$. In $\mathfrak{B}$, they all have cofinality ω.

For 2: Let $\mathfrak{A}$ and $\mathfrak{B}$ consist of just a set (unary relation). In $\mathfrak{A}$, the set has size $\aleph_0$. In $\mathfrak{B}$, the set has size $\aleph_1$.

M2. Let $\mathfrak{M}$ be an infinite saturated $\mathcal{L}$-structure. Assume $X \subseteq M$ is definable with parameters $\vec{a} \in M^{<\omega}$; that is, for some $\mathcal{L}$-formula $\theta(x, \vec{y})$:

$$X = \{m \in M : \mathfrak{M} \models \theta(m, \vec{a})\} \quad .$$

Assume also that every automorphism f of $\mathfrak{M}$ satisfies $f(X) = X$. Prove that X is definable without parameters; that is, for some $\mathcal{L}$-formula $\psi(x)$:

$$X = \{m \in M : \mathfrak{M} \models \psi(m)\} \quad .$$

Answer M2. Let Γ be the complete $\mathcal{L}$-type of $\vec{a}$ in $\mathfrak{M}$. Consider the set of formulas in $\mathcal{L}_{\vec{a}}$:

$$\Sigma(\vec{y}) = \Gamma(\vec{y}) \cup \Gamma(\vec{a}) \cup \{\exists x[\theta(x, \vec{y}) \leftrightarrow \neg\theta(x, \vec{a})]\}.$$

If $\Sigma(\vec{y})$ is consistent, then by saturation, it is realized in $\mathfrak{M}$ by some $\vec{d}$, and by saturation again, there is an automorphism f which satisfies $f(\vec{a}) = \vec{d}$. By the definition of Σ, $f(X) \neq X$. Therefore, Σ is inconsistent. Thus, there is a $\varphi(\vec{y}) \in \Gamma(\vec{y})$ such that

$$\mathfrak{M} \models \forall x\vec{y}\vec{z}[[\varphi(\vec{y}) \wedge \varphi(\vec{z})] \rightarrow [\theta(x, \vec{y}) \leftrightarrow \theta(x, \vec{z})]] \quad .$$

so

$$X = \{m \in M : \mathfrak{M} \models \exists\vec{y}[\varphi(\vec{y}) \wedge \theta(m, \vec{y})]\} \quad .$$

M3. Let $\mathcal{L}$ contain the symbol $<$, and let $\mathfrak{A}$ be an $\mathcal{L}$-structure in which $<_{\mathfrak{A}}$ is a total order with no largest element. Prove that $\mathfrak{A}$ has an elementary extension, $\mathfrak{B}$ such that:

1. $\mathfrak{B}$ has a non-trivial automorphism.
2. $<_{\mathfrak{B}}$ has uncountable cofinality (that is, every countable subset of B is bounded).

Answer M3. By the usual Eherenfeuct-Mostowski argument, get an elementary extension with a non-trivial automorphism. Then, add a name for the automorphism to the language and take elementary extensions ω_1 times to construct $\mathfrak{B}$.

August 1996 Answers

M1. Let $\mathfrak{A}$ be a structure for $\mathcal{L}$, and let U be a unary predicate symbol. Assume $|U_{\mathfrak{A}}| = \mathfrak{c}$ (where $\mathfrak{c} = 2^{\aleph_0}$). Prove that $\mathfrak{A}$ has an elementary extension, $\mathfrak{B}$, such that $|U_{\mathfrak{B}}| = \mathfrak{c}^+$. Note that we are assuming nothing about $|\mathfrak{A}|$ or the size of $\mathcal{L}$.

Answer M1. Form an elementary chain of $\mathfrak{c}^+$ models, starting with $\mathfrak{A}$. At limits, take unions. At successors, take an ultrapower using an ultrafilter on ω.

M2. Let $\mathcal{L} = \{<\}$. Describe a complete theory T in $\mathcal{L}$ such that

1. In every model $\mathfrak{A}$ for $\mathcal{L}$, $<_A$ totally orders A
2. There are $2^{\aleph_0}$ different 1-types consistent with T.

Answer M2. List the rationals, $\mathbb{Q}$, as $\{q_n : n \in \omega\}$. Form A by replacing each q_n by a sequence of n points, and let T be the theory or $\mathbb{Q}$.

M3. Let T be a complete theory with infinite models. Assume that T has some model with an automorphism σ of order 2 (that is, σ^2 is the identity but σ isn't). Let $\mathfrak{A}$ be any model of T. Prove that $\mathfrak{A}$ has an elementary extension $\mathfrak{B}$ such that $\mathfrak{B}$ has an automorphism of order 2.

Answer M3. Let $\mathfrak{C}$ be the model with the automorphsim. Then, embed $\mathbb{A}$ into a $|\mathbb{A}|$ - saturated elementary extension of $(\mathfrak{C}, \sigma)$.

August 1995 Answers

M1. Let L and L' be first order languages such that $L' \subseteq L$. Let T be a theory in L. Suppose that for any two models M, N for L whose L'-reducts M' and N' are isomorphic, M is a model of T if and only if N is a model of T. Prove that T is equivalent to a theory in L'.

Answer M1. Let T' be the set of all consequences of T in L'. Let M be a model of T'. Let S' be the set of all sentences in L' true in M. Then $S' \bigcup T$ is finitely satisfiable. By the compactness theorem $S' \bigcup T$ has a model N. The L'-reducts of M and N are elementarily equivalent. Then M and N have elementary extensions M_1 and N_1 whose L'-reducts are isomorphic. N_1 is a model of T, so by hypothesis M_1 and hence M is a model of T. Therefore T' is a theory in L' which is equivalent to T.

M2. Let T be a model complete theory in a countable language. Suppose R is a binary relation in the language of T, and let K be the class of all models M of T such that M is well ordered by R^M. We say that N is an *end extension* of M if N is a proper extension of M and $N \models R(a, b)$ for all $a \in M$ and $b \in N - M$.

Suppose that K is nonempty and each $M \in K$ has an end extension $N \in K$. Prove that for each uncountable cardinal κ there exists $M \in K$ such that R^M has order type κ.

Answer M2. Since T is model complete, whenever $M \subseteq N$ and $M, N \in K$, N is an elementary extension of M. By the elementary chain theorem, the union of a chain of end elementary extensions of M is again an elementary extension and thus a model of T. Since each extension is an end extension, this union is also well ordered by R and hence belongs to K. Given a cardinal κ, by transfinite recursion we may form a sequence of models $M_\alpha \in K, \alpha \leq \kappa$ such that $M_\alpha = \bigcup_{\beta<\alpha} M_\beta$ for limit ordinals α, and M_α is an end extension of M_β. By the downward Lowenheim-Skolem theorem we may also take M_α to be of cardinality $\omega \cup |\alpha|$. Then $M_\kappa \in K$ has order type κ.

M3. Let J be an uncountable set, let A be the set of all finite subsets of J, and let $M = \langle A, R \rangle$ where R is the subset relation on A. Let $N = \langle B, S \rangle$ be a countably indexed ultrapower of M such that the natural embedding $d : M \prec N$ is proper. Prove that:

a) For each $b \in B$ the set $E_b = \{a \in A : S(d(a), b)\}$ is at most countable.

b) For each countable subset $C \subset B$, there exists $b \in B$ such that $S(c, b)$ for all $c \in C$.

Answer M3. a) $N = \Pi_U M$ for some nonprincipal ultrafilter U over ω. Each $b \in B$ has the form $b = \langle b_n : n \in \omega \rangle_U$, and each b_n belongs to A and hence is finite. If $a \in E_b$, then $\{n \in \omega : a \subseteq b_n\} \in U$. Therefore E_b is contained in the countable set $\bigcup_n P(b_n)$. b) For each $n \in \omega$ the model M satisfies the sentence ϕ_n which says that for each $C \subset A$ of size $\leq n$ there exists $a \in A$ such that $S(c, a)$ for all $c \in C$. By Łos' theorem, $N \models \phi_n$ for each n. Then b) follows because N is ω_1-saturated.

M1. Prove that there is an uncountable model for PA which is ω-homogeneous but not ω_1-homogeneous.

Answer M1. Start with an ω_1 saturated model and build an ω-chain of ω-homogenous models each with a new element on the end. Then cofinal a ω sequence has same type as some bounded sequence from first model.

M2. Given two models

$$A = (U, R_1, R_1, \ldots), \qquad B = (V, S_1, S_2, \ldots)$$

of models for the same language such that U and V are disjoint, define the union to be

$$A \cup B = (U \cup V, R_1 \cup S_1, R_2 \cup S_2, \ldots).$$

Suppose that $A_1 \equiv A_2$, $B_1 \equiv B_2$, A_1, B_1 have disjoint universes, and A_2, B_2 have disjoint universes. Prove that $A_1 \cup B_1 \equiv A_2 \cup B_2$.

Answer M2. Add extra unary relations for the universes of A_i and B_i, and form the model pairs (A_i, B_i). Then take special or saturated models $(A_i', B_i') \equiv (A_i, B_i), i = 1, 2$ of the same sufficiently large cardinality and prove that (A_1', B_1') is isomorphic to (A_2', B_2'). Finish up by taking reducts to the original language.

Alternatively, you can use Ehrenfeucht games.