Analytic subspace of the reals without an analytic basis.

A Hamel basis is a basis for the reals $\mathbb{R}$ considered as a vector space over the field of rationals $\mathbb{Q}$.

**Theorem 1.** (Erdos and Sierpinski [1]) There is no analytic Hamel base.

**Proof.** Suppose on the contrary that $B$ is such a basis. $\mathbb{R}$ is the countable union of sets of the form $q_1B + \ldots + q_nB$, where $q_1, \ldots, q_n \in \mathbb{Q}$. These sets are all analytic, hence have the property of Baire and thus, by the Baire category theorem, for some $q_1, \ldots, q_n \in \mathbb{Q}$, the set $A = q_1B + \ldots + q_nB$ is non meager. There is an interval $I \subseteq \mathbb{R}$ such that $A$ is comeager in $I$. Pick any distinct $x_1, \ldots, x_n \in B$. Let $q \in \mathbb{Q}$ and $J \subseteq I$ be a subinterval such that $q(x_1 + \ldots + x_n) + A$ is comeager in $J$. Note that $q$ can be chosen so that $q \neq q_i$, for each $i$. This will ensure that there will be none of the $x_i$ will be canceled out by elements of $A$. Then

$$W = (q(x_1 + \ldots + x_n) + A) \cap A \neq \emptyset$$

because both terms of the intersection above are comeager in $J$. Any element of $W$ will be at the same time a linear combination of at least $n + 1$ elements of $B$ and also $n$ elements of $B$. This contradicts the linear independence of $B$. \(\blacksquare\)

**Theorem 2.** Every proper analytic subspace of $\mathbb{R}$ is measure zero and meager.

**Proof.** For the first claim, suppose that $A$ is an analytic subspace of $\mathbb{R}$ has positive measure. Then by Steinhaus’ theorem, $A - A$ (the set of differences of elements of $A$) contains a nontrivial interval. Hence $A$ must be all of $\mathbb{R}$.

For the second claim, suppose that $A$ is an analytic subspace which is non-meager. $A$ has the property of Baire and hence there is an open interval $I$ in which $A$ is comeager. Fix any $\alpha \in \mathbb{R}$. Let $q \in \mathbb{Q}$ be small enough that $q\alpha$ is less than the length of $I$. Then $J = (q\alpha + I) \cap I$ is non empty and $q\alpha + A$ and $A$ are both comeager in $J$. In other words, there exist $x, y \in A$ such that $q\alpha + x = y$. Hence $\alpha = \frac{1}{q}(y - x) \in A$. We see that $A = \mathbb{R}$. \(\blacksquare\)
The following answers a question raised by Ashutosh Kumar.

**Theorem 3.** There exists a proper (and hence meager) analytic subspace of $\mathbb{R}$ with no analytic basis.

We begin by describing the subspace in question.

Let $\epsilon_n$ be a decreasing sequence of positive rational numbers such that for every $k$ and each $N \geq k$,

$$\sum_{n>N} k\epsilon_n \leq \frac{1}{4}\epsilon_N$$

This condition requires the $\epsilon_n$ to be a very rapidly decreasing sequence.

Now let $P$ be the set defined by

$$P = \left\{ \sum_{n \in \omega} x_n \epsilon_n : x_n \in \{-1, 0, 1\} \right\}$$

$P$ is essentially a very sparse Cantor set. We now take the subspace $A$ to be $\text{span}(P)$. Our first objective is to show that $A$ is a proper subspace of $\mathbb{R}$. To this end, we make the following observations: $A$ is the union of all sets of the form $q_1P + \ldots + q_nP$, where the $q_i$ are rational numbers. By taking common denominators, we can write such sets as $\frac{1}{m}(p_1P + \ldots + p_nP)$, for some $p_1, \ldots, p_n \in \omega$. If we let $k = p_1 + \ldots + p_n$, then

$$(p_1P + \ldots + p_nP) \subset \underbrace{P + P + \ldots + P}_{k \text{ times}}$$

We give this latter set the name $Q_k$. Observe that $Q_k$ can be described by

$$Q_k = \left\{ \sum_{n \in \omega} x_n \epsilon_n : x_n \in \{-k, -k + 1, \ldots, k - 1, k\} \right\}$$

$p_1P + \ldots + p_nP \subset Q_k$ and hence $\frac{1}{m}(p_1P + \ldots + p_nP) \subset \frac{1}{m}Q_k$. Note that of course each $\frac{1}{m}Q_k$ is also a subset of $A$.

Before proceeding, note that throughout we will use the notation $\hat{\sigma}$ for the rational number $\sum_{n<|\sigma|} \sigma(n)\epsilon_n$, where $\sigma$ is a finite sequence of integers.

We now show that $A$ is a proper subspace via the following two lemmas.
Lemma 4. Suppose that $\sigma, \tau \in \{-k, \ldots, k\}^{<\omega}$ such that $|\sigma| = |\tau| > k$. If $\sigma <_{lex} \tau$, then every point in $Q^\tau_k$ is less than every point in $Q^\tau_k$.

Proof. It suffices to prove this lemma for the case in which there exists $\gamma$ such that $\sigma = \gamma \cdot i$ and $\tau = \gamma \cdot (i + 1)$, for some $i \in \{-k, \ldots, k - 1\}$. Let $M = |\gamma|$. The greatest element of $Q^\gamma_k$ is

$$\alpha = \hat{\sigma} + \sum_{n>M} k\epsilon_n = \hat{\gamma} + i\epsilon_M + \sum_{n>M} k\epsilon_n$$

and the least element of $Q^\gamma_k$ is

$$\beta = \hat{\tau} - \sum_{n>M} k\epsilon_n = \hat{\gamma} + (i\epsilon_M - \sum_{n>M} k\epsilon_n)$$

Therefore,

$$\beta - \alpha = \epsilon_M - \sum_{n>M} 2k\epsilon_n \geq \frac{1}{2} \epsilon_M > 0$$

Lemma 5. Each $Q^\sigma_k$ is nowhere dense, for $|\sigma| > k$.

Proof. Fix any interval $I$ such that $I \cap Q^\sigma_k \neq \emptyset$. Choose $\tau \supseteq \sigma$ such that $\tau \cdot i, \tau \cdot (i + 1) \in I$, for some $i \in \{-k, \ldots, k - 1\}$. Then every element of $Q^\tau_k$ is less than every element of $Q^\tau_k$ by the previous lemma. Therefore, between all $Q^\sigma_k$ are closed sets, we may take an interval $J$ between $Q^\tau_k$ and $Q^\tau_k$. $J$ is disjoint from $Q^\tau_k$, because $Q^\tau_k$ is the disjoint union of $Q^\gamma_k$ for $|\gamma| = |\tau| + 1$ and by the previous lemma, no such $Q^\gamma_k$ intersects $J$.

This shows that each $Q^\sigma_k$ is nowhere dense. Hence $Q_k$ is as well, being a finite union of such $Q^\sigma_k$. It follows that each $\frac{1}{m}Q_k$ is nowhere dense and hence $A = \bigcup_{m,k \in \omega} \frac{1}{m}Q_k$ is meager. $A$ is therefore proper.

Now we get to our main claim.

Lemma 6. $A$ has no analytic basis as a vector space over $\mathbb{Q}$.
We begin with some remarks about the set $P$. As in the above lemma, for $\sigma \in \{-1, 0, 1\}^\omega$, we define

$$N_\sigma = \{ \sum_{n \in \omega} x_n \epsilon_n : x_n \in \{-1, 0, 1\} & \sigma(n) = x_n \text{ for } n < |\sigma| \}$$

Note that although the $N_\sigma$ are closed sets in $\mathbb{R}$ (and hence in $P$), they are also relatively open in $P$. In fact, they form a base for the relative topology on $P$.

**Proof of Lemma 6.** Suppose towards a contradiction that $B$ is an analytic basis for $A$. We may assume, without loss of generality, that $1 \in B$. Otherwise, suppose that $x_1, \ldots, x_n \in B$ and $q_1, \ldots, q_n \in \mathbb{Q}$ are such that $1 = q_1 x_1 + \ldots + q_n x_n$. Then $[(q_1 - 1)x_1 + q_2 x_2 + \ldots q_n x_n] + x_1 = 1$. Hence $[(q_1 - 1)x_1 + q_2 x_2 + \ldots q_n x_n] + B$ is an analytic basis for $A$ which contains 1.

Since $B$ is a basis, the generating set $P$ of $A$ must be covered by a union of set of the form $q_1(B \cap I_1) + \ldots + q_n(B \cap I_n)$ where $q_1, \ldots, q_n \in \mathbb{Q}$ and $I_1, \ldots, I_n$ are pairwise disjoint intervals with rational endpoints. To avoid confusion later on, we assume here that all $q_j$ are nonzero and that each $B \cap I_j$ is nonempty.

$P$ is an uncountable closed set and hence a Baire space when regarded as a topological subspace of $\mathbb{R}$. Because the union described above is countable, the Baire category theorem yields that there are $q_1, \ldots, q_n, I_1, \ldots, I_n$ as above such that $W = q_1(B \cap I_1) + \ldots + q_n(B \cap I_n)$ is non-meager in $P$. $W$ is analytic, hence has the Baire property. We therefore obtain $\sigma \in \{-1, 0, 1\}^\omega$ such that $W$ is comeager in $N_\sigma$. (Because the $N_\sigma$ are a base for the relative topology on $P$.)

We now define a homeomorphism $\pi$ of $N_\sigma$ as follows: If $z \in N_\sigma$, then $z = \hat{\sigma} + \sum_{n \geq |\sigma|} x_n \epsilon_n$, for some sequence $\langle x_n : n \in \omega \rangle \in \{-1, 0, 1\}^\omega$. We define $\pi(z) = \hat{\sigma} - \sum_{n \geq |\sigma|} x_n \epsilon_n$. It is clear that $\pi$ is an autohomeomorphism of $N_\sigma$.

It follows that $\pi^{-1}(W)$ is also comeager in $N_\sigma$ and hence $W \cap \pi^{-1}(W) \neq \emptyset$. Let $z \in W \cap \pi^{-1}(W)$. Then $z, \pi(z) \in W$. Note that we may assume that $z$
(and hence \(\pi(z)\)) are irrational. This follows from the fact that the rationals are meager in \(N_\sigma\).

We may now take \(x_j, y_j \in B \cap I_j\) such that

\[
z = q_1x_1 + \ldots + q_nx_n
\]

\[
\pi(z) = q_1y_1 + \ldots + q_ny_n
\]

Thus

\[
z + \pi(z) = q_1(x_1 + y_1) + \ldots + q_n(x_n + y_n)
\]

By the definition of \(\pi\), \(z + \pi(z) = \hat{\sigma} \in \mathbb{Q}\). Note that since the \(I_j\) are disjoint, for each \(j\) and \(i \neq j\) \(x_j \neq x_i, y_i\). Further, because \(z \notin \mathbb{Q}\), \(z \neq \pi(z)\), we have that for at least one \(j\), \(x_j \neq y_j\). We have therefore expressed a rational number (namely \(\hat{\sigma}\)) as a sum of \(n + 1\) distinct elements of the basis \(B\). On the other hand, \(1 \in B\) and any rational can be expressed as a rational scalar multiple of 1, i.e. a linear combination of length 1. By independence, such linear combinations are unique and so the above leads to a contradiction. \(\blacksquare\)

We conclude with some further notes.

**Theorem 7.** For all \(\alpha > 2\) there exists a \(\mathbb{Q}\)-subspace \(A\) of \(\mathbb{R}\) which is \(\Sigma^0_\alpha\), but not \(\Pi^0_\alpha\).

**Proof.** Let \(C \subseteq \mathbb{R}\) be a perfect, linearly independent set. Choose \(B \subseteq C\) which is \(\Sigma^0_\alpha\), but not \(\Pi^0_\alpha\). Take \(A\) to be the linear span of \(B\).

First of all, \(A\) is not \(\Pi^0_\alpha\). To see this, observe that, by the independence of \(C\), \(A \cap C = B\). If \(A\) were \(\Pi^0_\alpha\), then \(B\) would be as well.

Secondly, \(A\) is \(\Sigma^0_\alpha\). Observe that \(A\) is the union of sets of the form

\[
q_1(B \cap I_1) + \ldots + q_n(B \cap I_n)
\]

Where the \(q_i\) are nonzero rational numbers and the \(I_i\) are disjoint intervals with rational endpoints. We can define a homeomorphism

\[
\prod_{i=1}^n (C \cap I_i) \rightarrow q_1(C \cap I_1) + \ldots + (C \cap I_n)
\]

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by \(\langle x_1, \ldots, x_n \rangle \mapsto q_1 x_1 + \ldots + q_n x_n\). Under this map, \(\prod_{i=1}^n (B \cap I_i)\) maps onto \(q_1 (B \cap I_1) + \ldots + q_n (B \cap I_n)\). Hence this latter set of is of the same Borel class as \(B\), namely \(\Sigma_\alpha^0\). Since the union above is countable, \(A\) is also \(\Sigma_\alpha^0\).

Theorem 7 is also a consequence of Theorem 2.5 of Farah and Solecki [2] but has a shorter proof.

**Theorem 8.** For all \(\alpha > 3\), there exists a \(\mathbb{Q}\)-subspace \(W\) of \(\mathbb{R}\) which is \(\Pi_\alpha^0\) and not \(\Sigma_\alpha^0\).

**Proof.** Let \(C \subset \mathbb{R}\) be a perfect, independent set over \(\mathbb{Q}\). Let \(A_0 \supset A_1 \supset \ldots\) be subsets of \(C\) which are \(\Sigma_{<\alpha}^0\) and such that \(A = \bigcap_{n \in \omega} A_n\) is \(\Pi_\alpha^0 \setminus \Sigma_\alpha^0\). Let \(W_n = \text{span}_\mathbb{Q}(A_n)\) and \(W = \bigcap_{n \in \omega} W_n\). Then each \(W_n\) is \(\Sigma_{<\alpha}^0\) as in the proof of Theorem 7. Thus \(W\) is \(\Pi_\alpha^0\), but not \(\Sigma_\alpha^0\). If \(W\) were \(\Sigma_\alpha^0\), then \(A = W \cap C\) would be as well.

**References**
