Carlson Collapse is minimal under MA

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Namba forcing [5] may be regarded as a generalization of Laver forcing [2] to \( \omega_2 \). The analogous forcing for \( \omega_1 \) we call the Carlson collapse. We first encountered it when writing our joint paper: Carlson, Kunen, and Miller [1]. In that paper we used the Prikry collapse of \( \omega_1 \), which is analogous to superperfect tree forcing (e.g., Miller [3]) but with subtrees of \( \omega_1^{<\omega} \). We proved in [1] the analogue of Theorem 3 for the Prikry collapse, namely that assuming Martin’s axiom the generic extension is minimal. Our paper [1] did not include Theorem 4, although Carlson had already proved it with an easier proof than is given here. Lemma 2 was obtained while giving a topics course [4] on forcing.

One of my colleagues many years ago liked to joke about the referee report that said; “This paper fills a much needed gap in the literature.”

Definition 1

(1) A subtree \( p \subseteq \omega_1^{<\omega} \) is Carlson iff there exists \( s \in p \) called the root of \( p \) such that for all \( t \in p \) either \( s \subseteq t \) or \( t \subseteq s \) and for every \( t \in p \) with \( s \subseteq t \) there are uncountably many \( \alpha < \omega_1 \) with \( s \triangleleft \langle \alpha \rangle \in p \).

(2) Let \( \mathbb{P} \) be the partial order of Carlson trees under inclusion.

(3) We write \( p \leq_0 q \) iff \( p \leq q \) and \( \text{root}(p) = \text{root}(q) \).

(4) For \( s \in p \) define \( p_s = \{ t \in p : s \subseteq t \text{ or } t \subseteq s \} \).

(5) \( B(p) = \{ t \in p : \text{root}(p) \subseteq t \} \) (nodes beyond the root).

Lemma 2 Suppose MA\(_{\omega_1}\) and we are given \( (p_\alpha \in \mathbb{P} : \alpha < \omega_1) \) and \( \tau \) a \( \mathbb{P} \)-name such that for each \( \alpha < \omega_1 \)

\[
p_\alpha \models \tau \in 2^\omega \setminus V.
\]

Then there exists \( (q_\alpha \leq_0 p_\alpha : \alpha < \omega_1) \) and \( (C_\alpha : \alpha < \omega_1) \) pairwise disjoint closed subsets of \( 2^\omega \) such that for every \( \alpha < \omega_1 \)

\[
q_\alpha \models \tau \in C_\alpha.
\]

Theorem 3 If \( M \models \text{MA}_{\omega_1} \) and \( G \) is \( \mathbb{P} \)-generic over \( M \), then for every \( x \in 2^\omega \cap M[G] \) either \( x \in M \) or \( G \in M[x] \).
Proof
Given any $p$ such that $p \models \tau \in 2^\omega \setminus M$ construct $q \leq_0 p$ and closed sets $(C_s : s \in B(q))$ such that

1. $q_s \models \tau \in C_s$ for each $s \in B(q)$,
2. $C_s \subseteq C_t$ if $t \subseteq s$, and
3. $C_{s^{-(\alpha)}} \cap C_{s^{-(\beta)}} = \emptyset$ if $\alpha \neq \beta$.

This is an easy fusion argument combined with Lemma 2. We claim that

$q \models G \in M[\tau^G]$.

This is because $G$ is determined by the generic collapse map $g \in \omega_1^{\omega}$ defined by $g = \bigcap G$. Then

$G = \{p \in \mathbb{P} : g \in [p]\}$ and $g = \bigcup \{s : \tau^G \in C_s\}$.

QED

Theorem 4 (Carlson 1979) Suppose $M \models \text{MA}_{\omega_1}$ and $G$ is $\mathbb{P}$-generic over $M$. Then for every $f \in \omega^{\omega} \cap M[G]$ there exists $g \in M \cap \omega^{\omega}$ such that $\forall n \ f(n) < g(n)$.

Proof
Without loss we may suppose that

$p \models \phi \in \omega^{\omega} \setminus M$.

Let $E \subseteq 2^\omega$ be the eventually zero reals. Let $\Phi : 2^\omega \setminus E \to \omega^{\omega}$ be the natural homeomorphism and let $\tau$ be a name for $\Phi^{-1}(f)$. Letting $p_\alpha = p$ for all $\alpha < \omega_1$ we obtain $(q_\alpha \leq_0 p : \alpha < \omega_1)$ and closed pairwise disjoint $(C_\alpha \subseteq 2^\omega : \alpha < \omega_1)$ such that $q_\alpha \models \tau \in C_\alpha$ for all $\alpha < \omega_1$. Since the $C_\alpha$ are pairwise disjoint there must be $\alpha$ with $C_\alpha \cap E = \emptyset$. This implies that $\Phi(C_\alpha) = K_\alpha \subseteq \omega^{\omega}$ is a compact set and so we may find $g \in \omega^{\omega}$ such that $g$ dominates every element of $K_\alpha$. But then

$q_\alpha \models \forall n \ \phi_\alpha(n) < \dot{g}(n)$.

QED
Proof of Lemma 2

Claim 5 Given any sentence $\theta$ and condition $p$ there exists $q \leq_0 p$ such that

$q \Vdash \theta$ or $q \Vdash \neg \theta$.

Proof
This is the Laver Lemma. It is also true for Namba forcing and many others. QED

Claim 6 Suppose $p_0 \Vdash \tau \in 2^\omega \setminus M$. Then there exists $q_1, q_2 \leq_0 p_0$ and pairwise disjoint clopen sets $C_1, C_2$ such that $q_1 \Vdash \tau \in C_1$ and $q_2 \Vdash \tau \in C_2$.

Proof
For $p \leq p_0$ define $p$ is good iff there are $q_1, q_2 \leq_0 p$ and pairwise disjoint clopen sets $C_1, C_2$ such that $q_i \Vdash \tau \in C_i$ for $i = 1, 2$. Note that if $p$ is bad and $s = \text{root}(p)$ then for all but countably many $\alpha < \omega_1$ with $s^\frown (\alpha) \in p$ the condition $p_{s^\frown (\alpha)}$ is bad. This because there are only countably many pairs of disjoint clopen sets $C_1, C_2$.

By this observation if the Claim fails then we may construct $q \leq_0 p_0$ such that for every $t \in B(q) = \{ t \in q : \text{root}(q) \subseteq t \}$ the condition $q_i$ is bad. Using Claim 5 it follows that for every $s \in B(q)$ there exists a unique $x_s \in 2^\omega$ such that for every $n < \omega$ there exists $p \leq_0 q_s$ such that $p \Vdash \tau \upharpoonright n = x_s \upharpoonright n$. If there were two $x_s$ with this property, we could easily get a contradiction to the badness of $q_s$.

For every $s \in B(q)$ it must be that $x_s = x_{s^\frown (\alpha)}$ for all but countably many $s^\frown (\alpha) \in q$. To see this suppose not and let $Q(s) = \{ \alpha < \omega_1 : s^\frown (\alpha) \in q \}$. Then we would be able to find $t \in 2^{<\omega}$ with the property that uncountably many $\alpha \in Q(s)$ had $t \subseteq x_{s^\frown (\alpha)}$ but $t \neq t' = x_s \upharpoonright |t|$. But this means we can find $p \leq_0 q_s$ such that $p \Vdash t \subseteq \tau$. We can also find $p' \leq_0 q_s$ such that $p' \Vdash t^p \subseteq \tau$ by the definition of $x_s$. This contradicts the badness of $q_s$.

By the above arguments we can find $x \in 2^\omega$ and $q \leq_0 p_0$ such that $x_s = x$ for every $s \in B(q)$. This contradicts the assumption that $p_0 \Vdash \tau \neq \check{x}$.

QED

Claim 7 Suppose $p_i \Vdash \tau \in 2^\omega \setminus M$ for $i = 1, 2$. Then there exists $q_1 \leq_0 p_1$, $q_2 \leq_0 p_2$ and pairwise disjoint clopen sets $C_1, C_2$ such that $q_i \Vdash \tau \in C_i$ for $i = 1, 2$. 

3
Proof
Apply Claim 6 to \( p \) and obtain \( q_{1,i} \leq_0 p \) and disjoint clopen \( C_1, C_2 \) such that \( q_{1,i} \models \tau \in C_i \) for \( i = 1, 2 \). We may as well assume \( C_1, C_2 \) are complementary. Apply the Laver Lemma to \( p_2 \) and get \( q_2 \leq_0 p_2 \) such that either \( q_2 \models \tau \in C_1 \) or \( q_2 \models \tau \in C_2 \). If \( q_2 \models \tau \in C_2 \), take \( q_1 = q_{1,1} \), otherwise take \( q_1 = q_{1,2} \).

QED

Claim 8  Suppose \( n < \omega \) and \( p_i \models \tau \in 2^\omega \setminus M \) for \( i < n \). Then there exists \( (q_i \leq_0 p_i : i < n) \) and pairwise disjoint clopen sets \( (C_i : i < n) \) such that \( q_i \models \tau \in C_i \) for \( i < n \).

Proof
Iteratively apply Claim 7 to all pairs \( i < j < n \).

QED

Definition 9  For \( T \) a finite subtree of \( B(q) \) define

1. \( p \leq_T q \) iff \( p \leq_0 q \) and \( T \subseteq p \).
2. For each \( t \in T \) define
   \[
   q_{t,T} = \{ s \in q : s \subseteq t \text{ or } (t \subseteq s \text{ and } s \upharpoonright (|t| + 1) \notin T) \}
   \]

Note that \( \{ q_{t,T} : t \in T \} \) is a finite maximal antichain beneath \( q \). This is analogous to Laver’s \( q_\leq_0 p \) except there are uncountably many \( T \).

Claim 10  Suppose \( p \models \tau \in 2^\omega \setminus M \) and \( p' \models \tau \in 2^\omega \setminus M \) and \( T \subseteq B(p) \) and \( T' \subseteq B(p') \) are finite subtrees. Then there are \( q \leq_T p \) and \( q' \leq_{T'} p' \) and pairwise disjoint clopen sets \( C \) and \( C' \) such that \( q \models \tau \in C \) and \( q' \models \tau \in C' \).

Proof
Let \( p_i \) for \( i < m \) list all \( p_{T,t} \) for \( t \in T \) and let \( p_i \) for \( m \leq i < n \) list all \( p_{T',t'} \) for \( t' \in T' \). Apply Claim 6 to obtain \( q_i \) and \( C_i \). Let \( q = \bigcup\{ q_i : i < m \} \) and \( C = \bigcup\{ C_i : i < m \} \). Similarly put \( q' = \bigcup\{ q_i : m \leq i < n \} \) and \( C' = \bigcup\{ C_i : m \leq i < n \} \).

QED

Claim 11  Suppose \( p \models \tau \in 2^\omega \), then there exists \( q \leq_0 p \) such and \( \pi : q \to 2^{<\omega} \) such that for every \( s \in q \) \( |\pi(s)| = |s| \) and \( q_s \models \pi(s) \subseteq \tau \).
Proof
Use the Laver lemma (Claim 5) and fusion to get the result by considering the sequence of sentences “\( \tau(n) = 0 \)".

QED

**Definition 12** For \( q \) as in Claim 11 define the poset \( \mathbb{Q}(q) \) as follows:

\((T, C) \in \mathbb{Q}(q)\) iff \( T \) is a finite subtree of \( B(q) \), \( C \) is clopen subset of \( 2^\omega \), and there exists \( p \leq_T q \) such that \( p \Vdash \tau \in C \).

Define \((T_1, C_1) \leq (T_2, C_2)\) iff \( T_1 \supseteq T_2 \) and \( C_1 \subseteq C_2 \).

From now on for the poset \( \mathbb{Q}(q) \) the condition \( q \) will always have the property of Claim 11 and hence for any \( p \leq q \) and clopen set \( C \) we have that \( p \Vdash \tau \in C \) iff the range of the induced continuous map \( \pi : [p] \to 2^\omega \) is a subset of \( C \).

**Claim 13** \( \mathbb{Q}(q) \) has the ccc.

Proof
Since there are only countably many clopen sets it is enough to see that any pair with the same clopen set, \((T_1, C)\) and \((T_2, C)\) is compatible. We claim that \((T_1 \cup T_2, C) \in \mathbb{Q}(q)\). Note that for \( p \leq q \) that \( p \Vdash \tau \in C \) iff \(|\pi(s)| \cap C \neq \emptyset\) for every \( s \in p \). It follows that if \( p_1 \leq_{T_1} q \) and \( p_2 \leq_{T_2} q \) and each force \( \tau \in C \), then \((p_1 \cup p_2) \leq_{T_1 \cup T_2} q \) and \( p_1 \cup p_2 \Vdash \tau \in C \).

QED

**Definition 14** For \( G \) and \( \mathbb{Q}(q) \)-filter define

\[ q^G = \bigcup \{ T : \exists C \ (T, C) \in G \} \]

and

\[ C^G = \bigcap \{ C : \exists T \ (T, C) \in G \} \].

**Claim 15** We can find \( \mathcal{D} \) a family of dense subsets of \( \mathbb{Q}(q) \) with \( |\mathcal{D}| = \omega_1 \) such that for every \( G \) a \( \mathbb{Q}(q) \)-filter which meets each element of \( \mathcal{D} \) we have that \( q^G \leq_0 p \) and \( q^G \Vdash \tau \in C^G \).
Proof
Note that the trivial condition $\{\text{root}(q)\}, 2^\omega$ is always in $G$. For $s \in B(q)$ and $\alpha < \omega_1$ define

$$D_{s,\alpha} = \{(T, C) \in Q(q) \mid (T, C) \models s \notin q^G \text{ or } \exists \beta > \alpha \ s \hat{\langle} \beta \rangle \in T\}$$

To see that it is dense note that any condition can be extended to a condition $(T, C)$ such that either $(T, C) \models s \notin q^G$ or $(T, C) \models s \in q^G$. In the first case $(T, C) \in D_{s,\alpha}$ and we are done. In the second case we must be able to find $(T', C') \leq (T, C)$ with $s \in T'$. By the definition of $Q(q)$ there exists $p \leq_T q$ such that $p \models \tau \in C'$. Choose any $\beta > \alpha$ with $s \hat{\langle} \beta \rangle \in p$. Then $p$ witnesses that $(T' \cup \{s \hat{\langle} \beta \rangle\}, C')$ is in $Q(q)$ and $D_{s,\alpha}$.

Meeting all the $D_{s,\alpha}$ guarantees that $q^G \in \mathbb{P}$ and $q^G \leq_0 q$.

To see that $q^G \models \tau \in C^G$ it is enough to show that for every $(T, C) \in G$ that $q^G \models \tau \in C$. Choose $n$ large enough so that there exists $\Gamma \subseteq 2^n$ such that we may write $C = \bigcup\{[s] : s \in \Gamma\}$. We claim that $(T, C)$ forces that for every $s \in q^G \cap \omega^n_1$ that $\pi(s) \in \Gamma$. If this were not the case, then for some $(T', C') \leq (T, C)$ and $s \in T' \cap \omega^n_1$ we would have that $\pi(s) \notin \Gamma$. But this means that $q_s \models \tau \in [\pi(s)]$ and therefor $q_s \models \tau \notin C$ contradicting the definition of $Q(q)$ that $C' \subseteq C$ and there exists $p \leq_T q$ (so $p_s \leq_0 q_s$) and $p \models \tau \in C'$.

QED

Finally we prove Lemma 2. We assume that the $q^\alpha \leq_0 p^\alpha$ have the property as in Claim 11. We let

$$Q = \sum\{Q(q^\alpha) : \alpha < \omega_1\}$$

be the direct sum which has the ccc by MA$_{\omega_1}$. For any $\alpha < \beta < \omega_1$ let

$$D_{\alpha,\beta} = \{p \in Q : p^\alpha = (T^\alpha, C^\alpha), \ p^\beta = (T^\beta, C^\beta), \text{ and } C^\alpha \cap C^\beta = \emptyset\}$$

By Claim 10 this set is dense. By Claim 15 we may find a family $\mathcal{D}$ of dense subsets of $Q$ with $|\mathcal{D}| = \omega_1$ such that if $G$ is a $Q$-filter meeting each element of $\mathcal{D}$ then each $q^G_\alpha \leq_0 q^\alpha$ has the property that $q^G_\alpha \models \tau \in C^G_\alpha$. If $G$ meets all the $D_{\alpha,\beta}$ then the $C^G_\alpha$ will be pairwise disjoint. Applying MA$_{\omega_1}$ gives us the sequences $(q^\alpha \leq_0 q^\alpha \leq_0 p^\alpha : \alpha < \omega_1)$ and $(C^G_\alpha : \alpha < \omega_1)$ to prove Lemma 2.

QED
References

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