Souslin’s Hypothesis and Convergence in Category

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Abstract: A sequence of functions \( f_n : X \to \mathbb{R} \) from a Baire space \( X \) to the reals \( \mathbb{R} \) is said to converge in category iff every subsequence has a subsequence which converges on all but a meager set. We show that if there exists a Souslin Tree, then there exists a nonatomic Baire space \( X \) such that every sequence which converges in category converges everywhere on a comeager set. This answers a question of Wagner and Wilczynski who proved the converse.

Suppose that \( S \subseteq \mathcal{P}(X) \) is a \( \sigma \)-field of subsets of \( X \) and \( I \subseteq S \) is a \( \sigma \)-ideal. If \( I \) has the countable chain condition (ccc), i.e., every family of disjoint sets in \( S \setminus I \) is countable, then \( S/I \) is a complete boolean algebra. A boolean algebra is atomic iff there is an atom beneath every nonzero element.

A function \( f : X \to \mathbb{R} \) is \( S \)-measurable iff \( f^{-1}(U) \in S \) for every open set \( U \). A sequence of \( S \)-measurable functions \( f_n : X \to \mathbb{R} \) converges \( I \)-a.e. to a function \( f \) iff there exists \( A \in I \) such that \( f_n(x) \to f(x) \) for all \( x \in (X \setminus A) \). If \( (X, S, \mu) \) is a finite measure space, then a sequence of measurable functions \( f_n : X \to \mathbb{R} \) converges in measure to a function \( f \) iff for any \( \epsilon > 0 \) there exists \( N \) such that for any \( n > N \):

\[
\mu\left\{ x \in X : |f_n(x) - f(x)| > \epsilon \right\} < \epsilon.
\]

In this case if \( I \) is the ideal of measure zero sets, then \( f_n \) converges to \( f \) in measure iff every subsequence \( \{f_n : n \in A\} \) (where \( A \subseteq \mathbb{N} \) has a subsequence \( B \subseteq A \) such that \( \{f_n : n \in B\} \) converges \( I \)-a.e. This allows us to define convergence in measure without mentioning the measure, only the ideal \( I \).

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So in the abstract setting define the following: $f_n$ converges to $f$ with respect to $I$ iff every subsequence $\{f_n : n \in A\}$ has a subsequence $B \subseteq A$ such that $\{f_n : n \in B\}$ converges I-a.e. (where $A$ and $B$ range over infinite sets of natural numbers.) For more background on this subject in case $I$ is the ideal of meager sets, see Poreda, Wagner-Bojakoska, and Wilczyński [PWW] and Ciesielski, Larson, and Ostaszewski [CLO].

Marczewski [M] showed that if $(X, S, \mu)$ is an atomic measure and $I$ the $\mu$-null sets, then ‘I-a.e. convergence’ is the same as ‘convergence with respect to $I$’.

Gribanov [G] proved the converse, if $(X, S, \mu)$ is a finite measure space and $I$ the $\mu$-null sets, then if ‘I-a.e. convergence’ is the same as ‘convergence with respect to $I$’ then $\mu$ is an atomic measure.

Souslin’s Hypothesis (SH) is the statement that there are no Souslin lines. It is known to be independent (see Solovay and Tennenbaum [ST]). It was the inspiration for Martin’s Axiom.

**Theorem 1** (Wagner and Wilczyński [WW]) Assume SH. Then for any $\sigma$-field $S$ and ccc $\sigma$-ideal $I \subseteq S$ the following are equivalent:

- ‘I-a.e. convergence’ is the same as ‘convergence with respect to $I$’ for $S$-measurable sequences of real-valued functions, and
- the complete boolean algebra $S/I$ is atomic.

At the real analysis meeting in Łódź Poland in July 94, Wilczyński asked whether or not SH is needed for the Theorem above. We show here that the conclusion of Theorem 1 implies Souslin’s Hypothesis.

**Theorem 2** Suppose SH is false (so there exists a Souslin tree). Then there exists a regular topological space $X$ such that

1. $X$ has no isolated points,
2. $X$ is ccc (every family of disjoint open sets is countable),
3. every nonempty open subset of $X$ is nonmeager, and
4. if $I$ is the $\sigma$-ideal of meager subsets of $X$, then ‘I-a.e. convergence’ is the same as ‘convergence with respect to $I$’ for any sequence of Baire measurable real-valued functions.
Hence if $S$ is the $\sigma$-ideal of sets with the property of Baire and $I$ the $\sigma$-ideal of meager sets, then $S/I$ is ccc and nonatomic, but the two types of convergence are the same.

**Proof:** Define $(T,<)$ to be an $\omega_1$-tree iff it is a partial order and for each $s \in T$ the set $\{t \in T : t < s\}$ is well-ordered by $<$ with some countable order type, $\alpha < \omega_1$. We let

$$T_\alpha = \{s \in T : \{t \in T : t < s\} \text{ has order type } \alpha\}.$$  

Also

$$T_{<\alpha} = \bigcup\{T_\beta : \beta < \alpha\}.$$  

- Define $C \subseteq T$ is a chain iff for every $s,t \in C$ either $s \leq t$ or $t \leq s$.
- Define $A \subseteq T$ is an antichain iff for any $s,t \in A$ if $s \leq t$, then $s = t$, i.e. distinct elements are $\leq$-incomparable.
- Define $T$ is a Souslin tree iff $T$ is an $\omega_1$ tree in which every chain and antichain is countable. (Note that since $T_\alpha$ is an antichain it must be countable.)
- SH is equivalent to saying there is no Souslin tree. Every Souslin tree contains a normal Souslin tree, i.e., a Souslin tree $T$ such that for every $\alpha < \beta < \omega_1$ and $s \in T_\alpha$ there exists a $t \in T_\beta$ with $s < t$. (Just throw out nodes of $T$ which do not have extensions arbitrarily high in the tree.) For more on Souslin trees see Todorčević [T].

Now we are ready to define our space $X$. Let the elements of $X$ be maximal chains of $T$. For each $s \in T$ let

$$C_s = \{b \in X : s \in b\}$$

and let

$$\{C_s : s \in T\}$$

be an open basis for the topology on $X$. Note that $C_s \cap C_t$ is either empty or equal to either $C_s$ or $C_t$ depending on whether $s$ and $t$ are incomparable, or $t \leq s$ or $s \leq t$, respectively. Each $C_s$ is clopen since its complement is the union of $C_t$ for $t$ which are incomparable to $s$. $X$ has no isolated points, since given any $s \in T$ there must be incomparable extensions of $s$ (because $T$ is normal) and therefore at least two maximal chains containing $s$, so $C_s$ is not a singleton. Clearly $X$ has the countable chain condition.
**Lemma 3** Open subsets of $X$ are nonmeager. In fact, the intersection of countably many open dense sets contains an open dense set.

**Proof:** The proof is quite standard and can be found in the reference books: Kunen [K] or Jech [J]. For the convenience of the reader we include it.

Suppose $(U_n : n \in \omega)$ is a sequence of an open dense subsets of $X$. Let $A_n \subseteq T$ be an antichain which is maximal with respect to the property that $C_s \subseteq U_n$ for each $s \in A_n$. Since $U_n$ is open dense in $X$, $A_n$ will be a maximal antichain in $T$.

Let $V_n = \bigcup \{C_s : s \in A_n\}$. Then $V_n \subseteq U_n$ and $V_n$ is open dense. (It is dense, because given any $C_t$ there exists $s \in A_n$ and $r \in T$ with $t \leq r$ and $s \leq r$, hence $C_r \subseteq V_n \cap C_t$.)

Choose $\alpha < \omega_1$ so that for each $n \in \omega$ the (necessarily countable) antichain $A_n \subseteq T_{<\alpha}$. Let $U = \bigcup \{C_s : s \in T_\alpha\}$.

Note that since $T$ is normal $U$ is an open dense set. Also

$$U \subseteq \bigcap_{n<\omega} V_n \subseteq \bigcap_{n<\omega} U_n.$$  

($U \subseteq V_n$ because for any $b \in U$ if $b \in C_s$ for some $s \in T_\alpha$ there must be $t \in A_n$ comparable to it, since $A_n$ is a maximal antichain, and since $A_n \subseteq T_{<\alpha}$, it must be that $t < s$ and so $b \in C_t \subseteq V_n$.

**Lemma 4** Suppose $f : X \to \mathbb{R}$ is a real valued Baire function. Then there exists $\alpha < \omega_1$ such that for each $s \in T_\alpha$ the function $f$ is constant on $C_s$.

**Proof:** Let $\mathcal{B}$ be a countable open basis for $\mathbb{R}$. For each $B \in \mathcal{B}$ the set $f^{-1}(B)$ has the property of Baire (open modulo meager). So there exists an open $U_B$ such that $U_B \Delta f^{-1}(B)$ is meager.

By the proof of Lemma 3 we may assume that

$$U_B = \bigcup \{C_s : s \in A_B\}$$

for some countable set $A_B \subseteq T$. By the proof of Lemma 3 there exists an $\alpha < \omega_1$ such that
• each $A_B \subseteq T_{<\alpha}$ and

• if $U$ is the open dense set $\bigcup \{C_s : s \in T_\alpha\}$, then $U$ is disjoint from $U_B \Delta f^{-1}(B)$ for each $B \in \mathcal{B}$.

But now, $f$ is constant on each $C_s$ for $s \in T_\alpha$. Otherwise, suppose that $f(b) \neq f(c)$ for some $b, c \in C_s$ for some $s \in T_\alpha$. Then suppose that $f(b) \in B$ and $f(c) \notin B$ for some $B \in \mathcal{B}$. Because $b \in (f^{-1}(B) \cap U)$ and $U$ is disjoint from $U_B \Delta f^{-1}(B)$, it must be that $b \in U_B$. Hence there exists $t \in T_{<\alpha}$ such that $C_t \subseteq U_B$ and $b \in C_t$. Since $t < s$ it must be that $c \in C_t$ and so $c \in f^{-1}(B)$, which contradicts $f(c) \notin B$.

Steprans [S] shows that every continuous function on a Souslin tree takes on only countably many values.

**Lemma 5** Suppose $\{f_n : X \to \mathbb{R} : n \in \omega\}$ is a countable set of real valued Baire functions. Then there exists $\alpha < \omega_1$ such that for each $s \in T_\alpha$ and $n < \omega$ the function $f_n$ is constant on $C_s$.

**Proof:** Apply Lemma 4 countably many times and take the supremum of the $\alpha_n$.

Finally, we prove the theorem. The idea of the proof is to use the argument of the atomic case, where the ‘atoms’ are supplied by Lemma 5. Since ‘$I$-a.e. convergence’ always implies ‘convergence with respect to $I$’, it is enough to see the converse. So let $f_n : X \to \mathbb{R}$ be Baire functions which converge to $f : X \to \mathbb{R}$ with respect to $I$, i.e. every subsequence has a subsequence which converges on a comeager set to $f$. By Lemma 5 there exists $\alpha < \omega_1$ such that for each $s \in T_\alpha$ and $n < \omega$ the function $f_n$ is constant on $C_s$. Since every subsequence has a convergent subsequence, it must be that for each fixed $s \in T_\alpha$ the constant values of $f_n$ on $C_s$ converge to a constant value. It follows that the sequence $f_n(x)$ converges to $f(x)$ on the dense open set $\{C_s : s \in T_\alpha\}$.

References


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