A Dedekind Finite Borel Set

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Abstract

In this paper we prove three theorems about the theory of Borel sets in models of ZF without any form of the axiom of choice. We prove that if $B \subseteq 2^\omega$ is a $G_{\delta\sigma}$-set then either B is countable or B contains a perfect subset. Second, we prove that if 2^ω is the countable union of countable sets, then there exists an $F_{\sigma\delta}$ set $C \subseteq 2^\omega$ such that C is uncountable but contains no perfect subset. Finally, we construct a model of ZF in which we have an infinite Dedekind finite $D \subseteq 2^\omega$ which is $F_{\sigma\delta}$.

Contents

1 Introduction

In this paper we assume the theory ZF but we do not assume any form of the axiom of choice, in particular, we do not assume the countable axiom of choice (which says that choice functions exist for countable families of nonempty sets). For example, we do not assume that the countable union of countable sets is countable. Early researchers in set theory while critical

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of the use of the axiom of choice to prove that every set can be well-ordered seem to have been unaware of their own use of the countable axiom of choice (see Moore [16], Maddy [15] p.487-9).

A classical result is that (assuming the countable axiom of choice) every uncountable Borel set contains a perfect set. In fact, it is not hard to see, that assuming the countable axiom of choice every Borel subset of 2^ω is the projection of a closed subset of $2^\omega \times \omega^\omega$, i.e., an analytic set, and that every uncountable analytic set contains a perfect set.

Definition 1.1 *1. For $s \in 2^{<\omega}$ define the basic clopen set:*

$$[s] = \{x \in 2^\omega \;:\; s \subseteq x\}.$$

2. *A set $U \subseteq 2^\omega$ is open iff it is the union of basic clopen sets.*
3. *A set $A \subseteq 2^\omega$ is G_δ iff it is the intersection of a countable family of open sets.*
4. *A set $B \subseteq 2^\omega$ is $G_{\delta\sigma}$ iff it is the union of a countable family of G_δ-sets.*
5. *Similarly define F be the closed sets, i.e., complements of open sets, F_σ the countable unions of closed sets, and $F_{\sigma\delta}$ the countable intersections of F_σ's.*
6. *A subset $P \subseteq 2^\omega$ is perfect iff it is homeomorphic to 2^ω.*

Theorem 1.2 *If $A \subseteq 2^\omega$ is a $G_{\delta\sigma}$ set, then A is countable or contains a perfect set.*

Theorem 1.3 *Suppose that 2^ω is the countable union of countable sets. Then there exists an $F_{\sigma\delta}$ set $B \subseteq 2^\omega$ which is uncountable but contains no perfect subset.*

In the Feferman-Levy model 2^ω is the countable union of countable sets (see Cohen [1] p.143, Jech [9] p.142). Note that this implies that every set $B \subseteq 2^\omega$ is the countable union of countable sets. Since a countable subset of 2^ω is F_σ, it follows that every subset of 2^ω is $F_{\sigma\sigma}$, i.e., a countable union of countable unions of closed sets. By taking complements every subset of 2^ω is $G_{\delta\delta}$. So the set B in Theorem 1.3 is $F_{\sigma\delta}$, $F_{\sigma\sigma}$, and $G_{\delta\delta}$.

The fact that in the Feferman-Levy model, there is an uncountable Borel set without a perfect subset was observed by Juris Steprans (see Moore [16] p.103 footnote 27).

In ZF without using any choice at all there exists a $G_{\delta\sigma}$-set which is not $F_{\sigma\delta}$, see Theorem 2.1 of Miller [13].

A set D is Dedekind finite iff every one-one map of D into itself is onto. Equivalently, there is no one-one map of ω into D. Assuming the countable axiom of choice every Dedekind finite set is finite. The book Herrlich [7] pp.43-50 summarizes many of the basic results about Dedekind finite sets.

By infinite set we simply mean that the set is not finite, i.e., cannot be put into one-to-one correspondence with some finite ordinal $n \in \omega$.

Theorem 1.4 *Suppose that M is a countable transitive model of ZF and*

$$M \models D \subseteq 2^\omega \text{ is an infinite Dedekind finite set.}$$

Then there exists a symmetric submodel $\mathcal{N}$ of a generic extension of M such that

$$\mathcal{N} \models D \text{ is a Dedekind finite } F_{\sigma\delta}\text{-set.}$$

For our forcing terminology and basic facts about forcing over models of ZF see Miller [13] section 3.

Remark 1.5 *If 2^ω is the countable union of countable sets, then there are no infinite Dedekind finite $D \subseteq 2^\omega$. This is because the countable union of finite subsets of a linearly orderable set is countable.*

Besides the notion of Dedekind finite there are many other "definitions of finiteness", i.e., properties which are equivalent to finite assuming the axiom of choice (see Truss [18], Lévy [11], Howard and Yorke [8], De la Cruz [4]). Most of them are inconsistent with being an infinite subset of 2^ω. One exception is Δ_5 (see Truss [18]):

A set D is Δ_5 iff there does not exist an onto map $f : D \to D \cup \{*\}$ where $*$ is not an element of D.

It is possible to have an infinite Δ_5 subset of 2^ω. Let us say $D \subseteq 2^\omega$ has the density-Dedekind property iff it is a dense subset of 2^ω and for any $E \subseteq D$ there exists an open set $U \subseteq 2^\omega$ such that $d \in E$ iff $d \in U \cap D$ for all but finitely many $d \in D$. Density-Dedekind implies Δ_5. In the basic Cohen model of ZF in which choice fails (see Jech [9] p.66-68) there is a generic

Dedekind finite set $A \subseteq 2^\omega$. It is not hard to show that in fact A has the density-Dedekind property and hence is Δ_5. The notion of density-Dedekind seems to us to be analogous to that of Luzin set in set theory with choice.

We don't know if it is possible to have an infinite Borel Δ_5-set. Almost-disjoint sets forcing destroys the density-Dedekind property.

A set is amorphous iff every subset of it is finite or cofinite. This is analogous in model theory with the Baldwin and Lachlan notion of strongly minimal set (see Truss [19], Creed, Truss [2], Mendick, Truss [12], and Walczak-Typke [20]). An infinite $D \subseteq 2^\omega$ cannot be amorphous. We don't know if there could be an uncountable Borel set $D \subseteq 2^\omega$ such that every subset is countable or co-countable (i.e., quasi-amorphous, see Creed, Truss [3]).

Monro [17] constructed Dedekind finite sets which are large in the sense that they can be mapped onto a cardinal κ. The ones he constructed were subsets of 2^κ. It is possible to have a Dedekind finite Borel set which maps onto ω_1 (or any other larger ω_α if desired). By Theorem 1.4 it is enough to find a Dedekind finite set $D \subseteq 2^\omega$ which maps onto ω_α. Such a D can be constructed by using a slight variant of the second Cohen model, see Jech [9] pp. 68-71.

In computability theory, the notion of Dedekind finite is analogous to that of Dekker's notion of an isol. There are over 180 of papers on the theory of isols, although currently the subject seems to have fallen out of fashion. Two which connect the theory of isols and Dedekind finite cardinals are Ellentuck [6] and McCarty [14]. Perhaps there are analogies between Borel Dedekind finite sets and co-simple isols, i.e., complements of simple sets. See for example, Downey and Slaman [5] which contains work on co-simple isols.

2 Proof of Theorem 1.2

Definition 2.1 *Recall the following:*

1. *A nonempty* $T \subseteq 2^{<\omega}$ *is a tree iff* $\forall s, t \in 2^{<\omega}$ *if* $s \subseteq t \in T$*, then* $s \in T$.

2. *For* T *a tree*
$$[T] = \{x \in 2^\omega \;:\; \forall n < \omega \;\; x \restriction n \in T\}$$

3. *For* T *a tree and* $s \in T$
$$T(s) = \{t \in T \;:\; t \subseteq s \text{ or } s \subseteq t\}.$$

4. *A tree T is perfect iff $\forall s \in T \;\; \exists t \in T \;\; s \subseteq t$ and both $t\hat{}\langle 0\rangle \in T$ and $t\hat{}\langle 1\rangle \in T$.*

The proof of the following proposition is left to the reader.

Proposition 2.2 *A set $C \subseteq 2^\omega$ is closed iff there exists a tree $T \subseteq 2^{<\omega}$ such that $C = [T]$. A set $P \subseteq 2^\omega$ is perfect iff there is a perfect tree $T \subseteq 2^{<\omega}$ such that $P = [T]$. In both cases we may demand that the tree T have no terminal nodes, i.e., for any $s \in T$ either $s\hat{}\langle 0\rangle \in T$ or $s\hat{}\langle 1\rangle \in T$.*

Lemma 2.3 *Let $\mathcal{B}$ be the family of nonempty countable closed subsets of 2^ω. Then there is a function $\mathcal{F} : \mathcal{B} \to (2^\omega)^\omega$ such that if $\mathcal{F}(C) = f$, then $f : \omega \to C$ is an onto map.*

Proof

This argument is ancient set theory, the Cantor-Bendixson derivative. (Recall we must not use the axiom of choice.)

Let C be a nonempty countable closed set. Define

$$T = \{s \in 2^{<\omega} \;:\; [s] \cap C \neq \emptyset\}.$$

Hence $[T] = C$.

Inductively define a sequence of trees $T_\alpha \subseteq 2^{<\omega}$ for α an ordinal as follows:

1. $T_0 = T$
2. $T_\lambda = \bigcap_{\alpha<\lambda} T_\alpha$ if λ is a limit ordinal
3. $T_{\alpha+1} = T_\alpha \setminus \{t \in T_\alpha \;:\; |[T_\alpha(t)]| \leq 1\}$.

Note that $\alpha \leq \beta$ implies $T_\beta \subseteq T_\alpha$.

If $T_{\alpha+1} = T_\alpha$, then $T_\alpha = T_\beta$ for all $\beta > \alpha$. By the replacement axiom there must be an ordinal α such that $T_{\alpha+1} = T_\alpha$. Since $T_\alpha \subseteq T$ we have that $[T_\alpha] \subseteq C$ and since C is countable, it must be that T_α is empty, since otherwise it is easy to check that it is a perfect tree.

For each $x \in C$ there exists a unique ordinal $\alpha_x < \alpha$ such that

$$x \in [T_{\alpha_x}] \setminus [T_{\alpha_x+1}].$$

Let n be the least such that $x \upharpoonright n \notin T_{\alpha_x+1}$ and put $s_x = x \upharpoonright n$. We claim that the map $q : C \to 2^{<\omega}$ defined by $q(x) = s_x$ is one-to-one. To see this suppose that $s_x = s_y$. If $\alpha_x < \alpha_y$, then we get a contradiction since $s_x \notin T_{\alpha_x+1}$ and $T_{\alpha_y} \subseteq T_{\alpha_x+1}$. So $\alpha_x = \alpha_y$ and from the definition of T_{α_x+1} we see that $x = y$.

To get our onto map $f : \omega \to C$, let x_0 be the lexicographically least element of C and let $\{t_n : n < \omega\}$ be a fixed enumeration of $2^{<\omega}$. Given any n if $t_n = s_x$ for some $x \in C$ let $f(n) = x$ and otherwise let $f(n) = x_0$.

No choice is being used in our definition of f, so we may define $\mathcal{F}(C) = f$.
QED

Corollary 2.4 *The countable union of closed subsets of 2^ω each of which is countable is countable.*

Lemma 2.5 *Let $\mathcal{H}$ be the family of nonempty countable G_δ subsets of 2^ω. Then there is a function $\mathcal{G} : \mathcal{H} \to (2^\omega)^\omega$ such that if $\mathcal{G}(H) = g$, then the map $g : \omega \to H$ is onto.*

Proof
This argument is also ancient set theory (although perhaps not as well known), the Hausdorff difference hierarchy. Hausdorff proved that disjoint G_δ sets can be separated by a set which is in the difference hierarchy of closed sets (see Kechris [10] p.176).

Let $H, K \subseteq 2^\omega$ be disjoint G_δ-sets. Define closed sets $C_\alpha \subseteq 2^\omega$ for α an ordinal as follows:

$C_0 = \mathrm{cl}(H)$ (we use $\mathrm{cl}(X)$ to denote the closure of X)
$C_1 = \mathrm{cl}(C_0 \cap K)$
$C_2 = \mathrm{cl}(C_1 \cap H)$
$\vdots$
$C_\omega = \bigcap_{n<\omega} C_n$,
$C_{\omega+1} = \mathrm{cl}(C_\omega \cap K)$
$C_{\omega+2} = \mathrm{cl}(C_\omega \cap H)$
and so forth, in general, for λ a limit ordinal and $n < \omega$:

$$C_\lambda = \bigcap_{\alpha<\lambda} C_\alpha$$

$$C_{\lambda+n+1} = \begin{cases} \mathrm{cl}(C_{\lambda+n} \cap H) & \text{if } n \text{ is odd} \\ \mathrm{cl}(C_{\lambda+n} \cap K) & \text{if } n \text{ is even} \end{cases}$$

It is clear that if $\alpha \leq \beta$ then $C_\beta \subseteq C_\alpha$. Also if[2] $C_\alpha = C_{\alpha+2}$ then for all $\beta > \alpha$, $C_\beta = C_\alpha$. Hence there must be an ordinal α_0 such that $C_{\alpha_0} = C_\beta$ for all $\beta > \alpha_0$.

We claim that C_{α_0} is empty, otherwise, H and K are both dense in it and it would follow that $H \cap K \neq \emptyset$. To see this, let

$$T_{\alpha_0} = \{s \in 2^{<\omega} \; : \; [s] \cap C_{\alpha_0} \neq \emptyset\}.$$

Write $H = \bigcap_{n<\omega} U_n$ and $K = \bigcap_{n<\omega} V_n$ where U_n and V_n are open sets. Since H and K are dense in C_{α_0}, it must be that for every $s \in T_{\alpha_0}$ and $n < \omega$, there exists $t \in T_{\alpha_0}$ with $s \subseteq t$ and $[t] \subseteq U_n \cap V_n$. But now it is easy to construct $x \in [T_{\alpha_0}] \cap H \cap K$ without using the axiom of choice.

Since C_{α_0} is empty we have that the difference sets:

$$D = \bigcup \{(C_\alpha \setminus C_{\alpha+1}) \; : \; \alpha \text{ is even } \}$$

and

$$E = (2^\omega \setminus C_0) \cup \bigcup \{(C_\alpha \setminus C_{\alpha+1}) \; : \; \alpha \text{ is odd } \}.$$

are complements of each other.[3] We claim that $H \subseteq D$ and $K \subseteq E$. To see why, suppose that $x \in H$. Since $C_0 = \text{cl}(H)$ it must be that there is some ordinal α such that $x \in C_\alpha \setminus C_{\alpha+1}$. This α cannot be odd, since $C_{\alpha+1} = \text{cl}(C_\alpha \cap H)$. Similarly $K \subseteq E$.

Now suppose that H is a countable G_δ-set. Then $K = 2^\omega \setminus H$ is also a G_δ-set. From which it follows that

$$H \;\; = \;\; \bigcup \{(C_\alpha \setminus C_{\alpha+1}) \; : \; \alpha \text{ is even } \}.$$

Define

$$T_\alpha = \{s \in 2^{<\omega} \; : \; [s] \cap C_\alpha \neq \emptyset\}.$$

So for $\alpha < \alpha_0$ we have that each T_α is a nonempty tree without terminal nodes such that $C_\alpha = [T_\alpha]$. For $s \in 2^{<\omega}$ with length greater than 0, let $s^* \subseteq s$ with $|s^*| = |s| - 1$. Let

$$Q_\alpha = \{s \in T_\alpha \setminus T_{\alpha+1} \; : \; s^* \in T_{\alpha+1}\}$$

i.e, the minimal nodes of $T_\alpha \setminus T_{\alpha+1}$.

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[2]With a little more work it is enough that $C_\alpha = C_{\alpha+1}$.

[3]The ordinal α_0 must be countable and the unions could be taken over $\alpha \leq \alpha_0 + 2$ but we don't need this for our proof.

For each even α since $C_\alpha \setminus C_{\alpha+1} \subseteq H$ and H is countable we have that $[T_\alpha(s)]$ is a countable set for each $s \in Q_\alpha$. Note that the Q_α are pairwise disjoint. Let $Q \subseteq 2^{<\omega}$ be the set of all s such that $s \in Q_\alpha$ and α is even. For each $s \in Q$ define $f_s : \omega \to 2^{<\omega}$ by $\mathcal{F}([T_\alpha(s)]) = f_s$ where $s \in Q_\alpha$. It follows that the map $h : Q \times \omega \to H$ defined by $h(s,n) = f_s(n)$ is onto H and may easily be readjusted to an onto map $g : \omega \to H$. Put $\mathcal{G}(H) = g$.
QED

Corollary 2.6 *The countable union of countable G_δ subsets of 2^ω is countable.*

Proof
Suppose that $\bigcup_{n<\omega} H_n$ is given where each H_n is a countable G_δ-set. Let $\mathcal{G}(H_n) = g_n$. Then define an onto map

$$g : \omega \times \omega \to \bigcup_{n<\omega} H_n \quad \text{by} \quad g(n,m) = g_n(m).$$

QED

Proof of Theorem 1.2.

It follows immediately from Corollary 2.6 that we need only show that an uncountable G_δ-set $H \subseteq 2^\omega$ must contain a perfect set.

Define

$$H' = \{x \in H \ : \ \forall n < \omega \quad ([x \restriction n] \cap H) \text{ is uncountable } \}.$$

Note that H' is nonempty, since otherwise

$$H = \bigcup \{[s] \cap H \ : \ [s] \cap H \text{ is countable}, \quad s \in 2^{<\omega}\}$$

and since any set of the form $[x \restriction n] \cap H$ is G_δ and the countable union of G_δ-sets is countable, we would get a contradiction.

Define

$$T = \{s \in 2^{<\omega} \ : \ [s] \cap H' \neq \emptyset\}.$$

We claim that T is a perfect tree. To see this suppose that $s \in T$. Then s will have incompatible extensions in T unless $H' \cap [s] = \{x\}$. This would mean that for every extension t of s which is incomparable to x that $H \cap [t]$ is countable. But since $s \subseteq x$ we know that $H \cap [s]$ is uncountable. But this contradicts the fact that the countable union of countable G_δ-sets is countable.

Now suppose $H = \bigcap_{n<\omega} U_n$ where each U_n is open. We construct

$$(s_\sigma \in T : \sigma \in 2^{<\omega})$$

by induction on the length of σ. Given s_σ with $|\sigma| = n$ let $t \in T(s_\sigma)$ be the first in some fixed ordering of $2^{<\omega}$ with $[t] \subseteq U_n$. Then using that T is perfect, find $s_{\sigma\hat{}\langle i\rangle} \in T$ for $i = 0, 1$ incomparable extensions of t. Then

$$T' = \{t \ : \ \exists\sigma \ \ t \subseteq s_\sigma\}$$

is a perfect subtree of T such that $[T'] \subseteq H$.
QED

3 Proof of Theorem 1.3

Definition 3.1 *Let $\langle,\rangle : \omega \times \omega \to \omega$ be a fixed bijection, i.e., a pairing function. For each $n \in \omega$ define the map $\pi_n : 2^\omega \to 2^\omega$ by:*

$$\pi_n(x) = y \text{ iff } \forall m \in \omega \ \ y(m) = x(\langle n, m\rangle).$$

Lemma 3.2 *Suppose that 2^ω is the countable union of countable sets. Then there exists $(F_n \ : \ n \in \omega)$ such that*

1. $2^\omega = \bigcup_{n<\omega} F_n$ *and each* F_n *is countable,*
2. F_n *is a proper subset of* F_{n+1} *for each* n*, and*
3. F_n *is closed under* π_m *for all* $n, m < \omega$.

Proof
Define a map $H : \omega^{<\omega} \times 2^\omega \to 2^\omega$ inductively by

$$H(\langle\rangle, x) = x \text{ and } H(s\hat{}\langle m\rangle, x) = \pi_m(H(s, x)).$$

Given that $2^\omega = \bigcup_{n<\omega} L_n$ where each L_n is countable, let

$$F_n = H(\omega^{<\omega} \times \bigcup_{m \leq n} L_m).$$

Then the F_n are countable, increasing, cover 2^ω, and closed under the projection maps π_m. To get them to be properly increasing just pass to a subsequence.
QED

Define

$$B_n = \{x \in 2^\omega : \pi_n(x) \in F_{n+1} \setminus F_n \text{ or } [\pi_n(x) \in F_n \text{ and } \pi_n(x) = \pi_{n+1}(x)]\}.$$

Note that each B_n is an F_σ-set. Let $B = \bigcap_{n<\omega} B_n$.

The set B is uncountable because there is a map h from B onto 2^ω. Define h by $h(x) = \pi_n(x)$ iff $\pi_n(x) = \pi_m(x)$ for all $m > n$. Such an n must exist because for any x there exists n such that $x \in F_n$ and hence $\pi_m(x) \in F_n$ for all m. It is easy to check that h maps B onto 2^ω.

But B cannot contain a perfect set. Suppose for contradiction that $T \subseteq 2^{<\omega}$ is a perfect tree and $[T] \subseteq B$. For each $x \in [T]$ define $h(x) = n$ to be the least n so that $\pi_n(x) = \pi_m(x)$ for all $m > n$. For any n the set of all $x \in [T]$ with $h(x) \leq n$ is closed. By Corollary 2.6 it must be that for some n there exists a perfect subtree $T' \subseteq T$ such that $x \in [T']$ implies $h(x) < n$. But the map

$$k : [T'] \to \prod_{m<n} F_m \text{ defined by } k(x) = (\pi_m(x) : m < n)$$

would map a perfect set one-one into a countable set.
QED

Remark 3.3 *We don't really need Corollary 2.6 in the above proof, since it is easy to show that a perfect set cannot be the countable union of countable closed sets. For example, each would have to be nowhere dense.*

Remark 3.4 *In the Feferman-Levy model the set B has the stronger property that there is no one-one map (continuous or not) taking 2^ω into B. Also Lemma 3.2 is trivially true in that model, take $F_n = M[G_n] \cap 2^\omega$.*

Remark 3.5 *If 2^ω is the countable union of countable sets, then there is no universal $F_{\sigma\delta}$-set.*[4] *In fact, there cannot be an onto map $f : 2^\omega \to F_{\sigma\delta}$.*

To see this, it is enough to see that there is a map h from $F_{\sigma\delta}$ onto $\mathcal{P}(2^\omega)$. The existence of such an f would contradict Cantor's Theorem that there is no map from any set onto its power set. Define h by

$$h(Q) = \{\pi_n(x) : n < \omega \text{ and } x \in Q\}.$$

Given a nonempty $X \subseteq 2^\omega$ let $X = \bigcup_{n<\omega} X_n$ where each X_n is countable and $\bigcap_{n<\omega} X_n$ is nonempty. The set $Q = \prod_{n<\omega} X_n$ is $F_{\sigma\delta}$ and $h(Q) = X$.

4 Proof of Theorem 1.4

Definition 4.1 *A poset $\mathbb{P}$ is σ-centered iff there exists $(\Sigma_n : n < \omega)$ such that $\mathbb{P} = \bigcup_{n<\omega} \Sigma_n$ and each Σ_n is centered, i.e., for any finite $F \subseteq \Sigma_n$ there exists $p \in \mathbb{P}$ such that $p \leq q$ for every $q \in F$.*

We begin with a preservation lemma:

Lemma 4.2 *Suppose that M is a countable transitive model of ZF and*

$$M \models \mathbb{P} \textit{ is } \sigma\textit{-centered and } D \textit{ is Dedekind finite.}$$

Then for any G $\mathbb{P}$-generic over M

$$M[G] \models D \textit{ is Dedekind finite.}$$

Proof
Working in M let $(\Sigma_n : n < \omega)$ witness the σ-centeredness of $\mathbb{P}$. Suppose for contradiction that

$$p \Vdash \overset{\circ}{f}: \check{\omega} \to \check{D} \text{ is one-one.}$$

Define

$$D_{n,m} = \{x \in D \ : \ \exists q \in \Sigma_n \ \ q \leq p \text{ and } q \Vdash \overset{\circ}{f}(m) = \check{x}\}.$$

[4] This answers a question of Alessandro Andretta who had realized that in ZF there is a universal F_σ-set but there could fail to be a universal $F_{\sigma\delta\sigma}$ since $F_{\sigma\sigma} \subseteq F_{\sigma\delta\sigma}$.

Since Σ_n is centered, $|D_{n,m}| \leq 1$. Since D is Dedekind finite, the set

$$E = \bigcup_{n,m<\omega} D_{n,m}$$

is finite. But

$$p \Vdash \text{the range of } \overset{\circ}{f} \text{ is a subset of } \check{E}$$

which is a contradiction.
QED

Remark 4.3 *To preserve the Dedekind finiteness of $D \subseteq 2^\omega$ it would be enough to assume that $\mathbb{P} = \bigcup_{n<\omega} \Sigma_n$ where each Σ_n had the n-c.c., i.e., no antichain of size greater than n.*

Next we give a description of the well-known almost-disjoint sets forcing of Solovay.

Definition 4.4 *For $A \subseteq 2^\omega$ define*

$$\mathbb{P}(A) = \{\langle Q, F\rangle \ : \ Q \subseteq 2^{<\omega},\ F \subseteq A,\ \textit{and both } Q \textit{ and } F \textit{ are finite } \}.$$

For $p, q \in \mathbb{P}(A)$ define $p \leq q$ iff $Q_q \subseteq Q_p$, $F_q \subseteq F_p$, and $s \not\subseteq x$ for all $s \in Q_p \setminus Q_q$ and $x \in F_q$.

We use $\mathbf{1} = (\emptyset, \emptyset)$ to denote the trivial element of $\mathbb{P}(A)$.

Note that $\langle Q, F_1\rangle \leq \langle Q, F_2\rangle$ whenever $F_2 \subseteq F_1$. Hence given $Q \subseteq 2^{<\omega}$ finite, if we define

$$\Sigma_Q = \{p \in \mathbb{P}(A) \ : \ Q_p = Q\}$$

then Σ_Q is centered and

$$\mathbb{P}(A) = \bigcup\{\Sigma_Q \ : \ Q \subseteq 2^{<\omega} \text{ is finite } \}$$

shows that $\mathbb{P}(A)$ is σ-centered.

If G is $\mathbb{P}(A)$-generic over M, then we can define

$$R = R^G = \bigcup\{Q_p \ : \ p \in G\}.$$

Easy density arguments show that for every $x \in 2^\omega \cap M$

- if $x \in A$, then $\{n : x \upharpoonright n \in R\}$ is finite, and
- if $x \notin A$, then $\{n : x \upharpoonright n \in R\}$ is infinite.

Next we consider automorphisms of the poset $\mathbb{P}(A)$.

Definition 4.5 *A map $\hat{\pi} : 2^{<\omega} \to 2^{<\omega}$ is a tree automorphism iff $\hat{\pi}$ is a bijection such that for all $s, t \in 2^{<\omega}$*

$$s \subseteq t \text{ iff } \hat{\pi}(s) \subseteq \hat{\pi}(t).$$

A tree automorphism $\hat{\pi}$ induces a map from 2^ω to itself by letting $\hat{\pi}(x) = y$ where y is determined by $\hat{\pi}(x \upharpoonright n) = y \upharpoonright n$ for every $n < \omega$.

Lemma 4.6 *Suppose $\hat{\pi}$ is a tree automorphism such that $\hat{\pi}(x) \in A$ for every $x \in A$. Then $\pi : \mathbb{P}(A) \to \mathbb{P}(A)$ defined by*

$$\pi(Q, F) = (\{\hat{\pi}(s) : s \in Q\}, \{\hat{\pi}(x) : x \in F\})$$

is an automorphism of $\mathbb{P}(A)$.

Proof
We need to show that

$$p \leq q \text{ iff } \pi(p) \leq \pi(q).$$

It is easy to check that

$$Q_q \subseteq Q_p \text{ iff } \hat{\pi}(Q_q) \subseteq \hat{\pi}(Q_p)$$

and

$$F_q \subseteq F_p \text{ iff } \hat{\pi}(F_q) \subseteq \hat{\pi}(F_p).$$

For the third clause in definition 4.4 note that for $s \in 2^{<\omega}$ and $x \in 2^\omega$ that

$$s \subseteq x \text{ iff } \hat{\pi}(s) \subseteq \hat{\pi}(x).$$

QED

Definition 4.7 *For any $n < \omega$ define*

$$E_n = \{x \in 2^\omega \ : \ \forall k < n \ \ x(k) = 0 \ \textit{and} \ \exists l > n \, \forall k > l \ \ x(k) = 0\}$$

As usual for $x, y \in 2^\omega$ define $x + y$ to be their pointwise sum mod 2, i.e.,

$$\forall n \quad (x+y)(n) \equiv x(n) + y(n) \ \textit{mod } 2$$

and for $A, B \subseteq 2^\omega$ define

$$A + B = \{x + y \ : \ x \in A \ \textit{and} \ y \in B\}.$$

Lemma 4.8 *For $D \subseteq 2^\omega$ Dedekind finite*

$$D = \bigcap_{n<\omega} (D + E_n).$$

Proof
Since the constant zero function is in every E_n it is clear that

$$D \subseteq \bigcap_{n<\omega} (D + E_n).$$

Now suppose for contradiction that $x \in \bigcap_{n<\omega}(D + E_n)$ but $x \notin D$. Consider the equivalence class of x under "equal mod finite": $x + E_0$. Since this class can be well-ordered in type ω we know that the set:

$$F = D \cap (x + E_0)$$

is finite. Take $n < \omega$ large enough so that for all $u, v \in F \cup \{x\}$ if $u \restriction n = v \restriction n$ then $u = v$. But $x \in D + E_n$ which means that there exists $d \in D$ with $d \restriction n = x \restriction n$. But $d \in F$ which is a contradiction.
QED

Definition 4.9 *We define the poset $\mathbb{P}$ to be the direct sum of the posets: $\mathbb{P}(D + E_n)$, i.e.,*

$$\mathbb{P} = \Sigma_{n<\omega} \mathbb{P}(D + E_n).$$

This means $p \in \mathbb{P}$ iff $p = (p_n : n < \omega)$ where each $p_n \in \mathbb{P}(D+E_n)$ and $p_k = \mathbf{1}$ for all but finitely many k. It is ordered coordinatewise:

$$p \leq q \ \textit{iff} \ p_n \leq q_n \ \textit{for all } n.$$

As before, given any G a $\Sigma_{n<\omega}\mathbb{P}(D + E_n)$-filter and $n < \omega$ we define

$$R_n = R_n^G = \{s \in 2^{<\omega} \ : \ \exists p \in G \ \textit{with} \ s \in Q_{p_n}\}.$$

It is clear that for G a $\mathbb{P}$-generic filter over M that for every n and $x \in D + E_n$ there are at most finitely many $k < \omega$ with $x \upharpoonright k \in R_n$.

Lemma 4.10 *The poset $\mathbb{P} = \Sigma_{n<\omega}\mathbb{P}(D + E_n)$ is σ-centered.*

Proof
For any finite sequence $\vec{Q} = (Q_i : i < n)$ of finite subsets of $2^{<\omega}$ define

$$\Sigma_{\vec{Q}} = \{p \in \mathbb{P} \ : \ \forall i < n \ \ Q_{p_i} = Q_i \ \text{ and } \ \forall i \geq n \ \ p_i = \mathbf{1}\}.$$

Then each $\Sigma_{\vec{Q}}$ is centered and $\mathbb{P}$ is the countable union of them.
QED

Definition 4.11 *For $R \subseteq 2^{<\omega}$ define*

$$H(R) = \{x \in 2^\omega \ : \ \exists^\infty k \ \ x \upharpoonright k \in R\}$$

Here $\exists^\infty k$ stands for "there exists infinitely many k".

Lemma 4.12 *For $R \subseteq 2^{<\omega}$ the set $H(R)$ is a G_δ-set. Suppose $\mathcal{R}$ is a countable family of subsets of $2^{<\omega}$, then $\bigcap\{H(R) \ : \ R \in \mathcal{R}\}$ is a G_δ-set.*

Proof

$$H(R) = \bigcap_{n<\omega} \bigcup \{[s] \ : \ s \in R \text{ and } |s| > n\}.$$

Letting $\mathcal{R} = \{R_n \ : \ n < \omega\}$ we have that

$$\bigcap\{H(R) \ : \ R \in \mathcal{R}\} = \bigcap_{n,m<\omega} \bigcup \{[s] \ : \ s \in R_n \text{ and } |s| > m\}.$$

QED

Note that $H(R_n)$ is a G_δ-set disjoint from $D + E_n$. Our goal is to make the complement of D to be a countable union of G_δ sets in a symmetric submodel of $M[G]$.

We describe the automorphisms of $\mathbb{P}$ which we will use.

Definition 4.13 *1. For $s \in 2^{<\omega}$ define $\hat{\pi}_s : 2^{<\omega} \to 2^{<\omega}$ to be the tree automorphism which swaps $s\hat{\ }\langle 0\rangle$ and $s\hat{\ }\langle 1\rangle$, i.e.,*

$$\hat{\pi}_s(r) = \begin{cases} s\hat{\ }\langle 1-i\rangle\hat{\ }t & \text{if } r = s\hat{\ }\langle i\rangle\hat{\ }t \\ r & \text{if } r \text{ does not extend } s. \end{cases}$$

2. *For each* n *we let* $\mathcal{G}_n$ *be the group of automorphisms of* $\mathbb{P}(D+E_n)$ *which are generated by* $\{\pi_s \;:\; s \in 2^{<\omega}$ *and* $|s| > n\}$.

3. *We take* $\mathcal{G}$ *to be the direct sum of the* $\mathcal{G}_n$, *i.e.,* $\pi \in \mathcal{G}$ *iff* $\pi = (\pi_n : n < \omega)$ *where each* $\pi_n \in \mathcal{G}_n$ *and* π_k *is the identity except for finitely many* k.

4. *We take* $\mathcal{F}$ *to be the filter of subgroups of* $\mathcal{G}$ *which is generated by* $\{H_n : n < \omega\}$ *where*

$$H_n = \{\pi \in \mathcal{G} \;:\; \forall m < n \;\; \pi_m \textit{ is the identity }\}.$$

It is easy to check that $\mathcal{F}$ is a normal filter.

We use the terminology $\hat{\pi}$ (a hatted π) to denote tree automorphisms and unhatted π's to denote the corresponding automorphism of $\mathbb{P}$ and the action on the $\mathbb{P}$-names. We use $\mathcal{N}$ to denote the symmetric model $M \subseteq \mathcal{N} \subseteq M[G]$. We use the terminology $\text{fix}(\tau)$ to denote the subgroup of $\mathcal{G}$ which fixes the $\mathbb{P}$-name τ.

Let

$$\mathring{R}_n = \{(p, \check{s}) \;:\; p \in \mathbb{P} \text{ and } s \in Q_{p_n}\}$$

then $H_{n+1} \subseteq \text{fix}(\mathring{R}_n)$ and so $R_n \in \mathcal{N}$. The following lemma is key:

Lemma 4.14 *Given* $p \in \mathbb{P}$, $\mathring{x}$, *and* $n_0 < \omega$ *such that* $H_{n_0} \subseteq \text{fix}(\mathring{x})$ *and*

$$p \Vdash \mathring{x} \in 2^\omega \setminus (D + E_{n_0})$$

then

$$p \Vdash \exists^\infty k \quad \mathring{x} \restriction k \in \mathring{R}_{n_0} .$$

Proof

If not there exists $q \le p$ and $N > n_0$ such that

$$q \Vdash \forall n > \check{N} \quad \mathring{x} \restriction n \notin \mathring{R}_{n_0} .$$

Claim. There exists $r \le q$ and $s, t_0, t_1 \in 2^{<\omega}$ with

1. $|s| > N > n_0$

2. $\{t \in 2^{<\omega} \;:\; s \subseteq t\} \cap Q_{q_{n_0}} = \emptyset$

3. $[s] \cap F_{q_{n_0}} = \emptyset$.
4. $s \subseteq t_0$, $s \subseteq t_1$, $|t_0| = |t_1|$
5. $t_0 \in Q_{r_{n_0}}$
6. $r \Vdash \check{t}_1 \subseteq \overset{\circ}{x}$

Since p is forcing that x is not in $D + E_{n_0}$ it easy to find $r_1 \leq q$ and s such that

$$r_1 \Vdash \check{s} \subseteq \overset{\circ}{x}$$

and s satisfies 1,2, and 3. Next choose any t_0 with $s \subseteq t_0$ and $t_0 \not\subseteq y$ for all $y \in F_{r_{1,n_0}}$ and put $r_2 = r_1$ except

$$Q_{r_{2,n_0}} = Q_{r_{1,n_0}} \cup \{t_0\}.$$

Finally find $r \leq r_2$ and t_1 with $|t_0| = |t_1|$ and $r \Vdash \check{t}_1 \subseteq \overset{\circ}{x}$. This proves the Claim.

Now find $\hat{\pi}$ a tree automorphism in $\hat{\mathcal{G}}_{n_0}$ such $\hat{\pi}(t_0) = t_1$ and fixes all t except for possibly those extending s. A precise description would be to let:

$$\{n \ : \ t_0(n) \neq t_1(n)\} = \{n_1 < n_2 < \ldots < n_k\}$$

then

$$\hat{\pi} = \hat{\pi}_{s_k} \circ \hat{\pi}_{s_{k-1}} \circ \cdots \circ \hat{\pi}_{s_1}$$

where $s_i = t_1 \restriction n_i$. Note that $\pi \in \mathcal{G}_{n_0}$ because $|t_0| = |t_1| \geq |s| > N > n_0$ so necessarily $n_1 > n_0$. Let $\pi \in \mathcal{G}$ also name the automorphism of $\mathbb{P}$ which is π on the n_0^{th} coordinate and the identity on all other coordinates. Then $\pi \in H_{n_0}$ and hence $\pi(\overset{\circ}{x}) = \overset{\circ}{x}$ and so by (6) of the Claim

$$\pi(r) \Vdash \check{t}_1 \subseteq \overset{\circ}{x} .$$

Note that by (2) and (3) of the Claim, we have $\pi(q) = q$ and so $\pi(r) \leq q$ and thus:

$$\pi(r) \Vdash \forall n > N \quad \overset{\circ}{x} \restriction n \notin \overset{\circ}{R}_{n_0} .$$

By (5) of the Claim and the definition of π we have that $t_1 \in Q_{\pi(r)_{n_0}}$ so we have:

$$\pi(r) \Vdash \check{t}_1 \in \overset{\circ}{R}_{n_0} .$$

But $|t_1| > N$ gives us a contradiction.
QED

Let

$$\mathcal{R}_n = \{\hat{\pi}(R_n) \ : \ \hat{\pi} \in \hat{\mathcal{G}}_n\}.$$

That is we take the set of all images of R_n under the tree automorphisms which determine $\mathcal{G}_n$. Since each R_n is in $\mathcal{N}$ and $\hat{\mathcal{G}}_n$ is in the ground model, it is clear that each $\mathcal{R}_n$ is in $\mathcal{N}$.

Lemma 4.15 *For each $\pi \in \mathcal{G}$ and $n < \omega$:*

$$\pi(\overset{\circ}{R}_n)^G = \hat{\pi}_n^{-1}(R_n).$$

Proof
This amounts to unraveling the definitions. The following are equivalent:

- $s \in \pi(\overset{\circ}{R}_n)^G$
- $\exists p \in G$ such that $(p, \check{s}) \in \pi(\overset{\circ}{R}_n)$ and $(p, \check{s}) = (\pi(q), \check{s})$ where $s \in Q_{q_n}$
- $\exists p \in G$ such that $s \in Q_{\pi_n^{-1}(p_n)}$ (equivalently $\hat{\pi}_n(s) \in Q_{p_n}$)
- $\hat{\pi}_n(s) \in R_n$
- $s \in \hat{\pi}_n^{-1}(R_n)$.

QED

Lemma 4.16 *The sequence $(\mathcal{R}_n : n \in \omega)$ is in $\mathcal{N}$.*

Proof
Letting

$$\overset{\circ}{\mathcal{R}}_n = \{\pi(\overset{\circ}{R}_n) \ : \ \pi \in \mathcal{G}\}$$

we see that $\text{fix}(\overset{\circ}{\mathcal{R}}_n) = \mathcal{G}$ for every n, hence the ω-sequence has a name fixed by every π in $\mathcal{G}$.
QED

Next we show that in the hypothesis of the key lemma (Lemma 4.14) we may assume that the trivial condition **1** is doing the forcing.

Lemma 4.17 *Fix G a $\mathbb{P}$-filter generic over M. Suppose $x \in (2^\omega \backslash D) \cap \mathcal{N}$. Then x has a hereditarily symmetric name $\overset{\circ}{x}$ for which there is an n_0 such that $H_{n_0} \subseteq \text{fix}(\overset{\circ}{x})$ and*

$$\mathbf{1} \Vdash \overset{\circ}{x} \in 2^\omega \backslash (D + E_{n_0}).$$

Proof
Let τ be any hereditarily symmetric name for x, i.e., $\tau^G = x$. Let $p \in G$ and n_0 be such that $H_{n_0} \subseteq \text{fix}(\tau)$ and

$$p \Vdash \tau \in 2^\omega \backslash (D + E_{n_0}).$$

Now work in the ground model M. Fix $z \in M \cap (2^\omega \backslash (D + E_{n_0}))$. In M define $\overset{\circ}{x}$ to the set of all $(q, \langle m, i \rangle)$ such that either

$$q \Vdash \text{“}\tau \in 2^\omega \backslash (D + E_{n_0}) \wedge \tau(m) = i\text{”}$$

or

$$z(m) = i \text{ and } q \Vdash \neg(\tau \in 2^\omega \backslash (D + E_{n_0})).$$

For our particular G, since $p \in G$ the second clause is never invoked when evaluating $\overset{\circ}{x}^G$, hence $x = \tau^G = \overset{\circ}{x}^G$. Clearly, $H_{n_0} \subseteq \text{fix}(\tau) \subseteq \text{fix}(\overset{\circ}{x})$. Finally $\mathbf{1}$ forces what it should because for any generic filter G' either the first clause is invoked and $\tau^{G'} = \overset{\circ}{x}^{G'}$ or the second clause is invoked and $\overset{\circ}{x}^{G'} = z$ where z was chosen to be in $2^\omega \backslash (D + E_{n_0})$.
QED

Lemma 4.18 $\mathcal{N} \models (2^\omega \backslash D) = \bigcup_{n<\omega} \bigcap_{R \in \mathcal{R}_n} H(R)$.

Proof
Recall that $H(R_n)$ is a G_δ-set which is disjoint from $D + E_n$ and hence from D. For any $R \in \mathcal{R}_n$ we have that $R = \hat{\pi}(R_n)$ for some $\pi \in \mathcal{G}_n$. But by Lemma 4.15

$$R = \hat{\pi}(R_n) = \pi^{-1}(\overset{\circ}{R}_n)^G = \overset{\circ}{R}_n^{\pi^{-1}(G)}$$

and so $H(R)$ is disjoint from D.

Conversely suppose in $\mathcal{N}$ that $x \in (2^\omega \backslash D)$. Then by Lemma 4.17 x has a name $\overset{\circ}{x}$ for which there exists n_0 such that $H_{n_0} \subseteq \text{fix}(\overset{\circ}{x})$ and

$$\mathbf{1} \Vdash \overset{\circ}{x} \in 2^\omega \backslash (D + E_{n_0}).$$

By the key Lemma 4.14

$$\mathbf{1} \Vdash \exists^\infty k \quad \overset{\circ}{x} \restriction k \in \overset{\circ}{R}_{n_0}$$

i.e.,

$$\mathbf{1} \Vdash \overset{\circ}{x} \in H(\overset{\circ}{R}_{n_0}).$$

Since $H_{n_0} \subseteq \text{fix}(\overset{\circ}{x})$ we have that

$$\mathbf{1} \Vdash \overset{\circ}{x} \in H(\pi(\overset{\circ}{R}_{n_0}))$$

for all $\pi \in H_{n_0}$ and so it follows from Lemma 4.15 that

$$x \in \bigcap_{R \in \mathcal{R}_n} H(R).$$

QED

It follows from this Lemma that (in $\mathcal{N}$) the complement of D is a $G_{\delta\sigma}$ set and hence D is an $F_{\sigma\delta}$-set. Since $\mathbb{P}$ is σ-centered we have that

$$M[G] \models D \text{ is Dedekind finite}$$

and since $M \subseteq \mathcal{N} \subseteq M[G]$

$$\mathcal{N} \models D \text{ is a Dedekind finite } F_{\sigma\delta}\text{-set.}$$

This concludes the proof of Theorem 1.4.

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