Abstract

A Laver tree is a tree in which each node splits infinitely often. A Hechler tree is a tree in which each node splits cofinitely often. We show that every analytic set is either disjoint from the branches of a Hechler tree or contains the branches of a Laver tree. As a corollary we deduce Silver Theorem that all analytic sets are Ramsey. We show that in Godel’s constructible universe that our result is false for co-analytic sets (equivalently it fails for analytic sets if we switch Hechler and Laver). We show that under Martin’s axiom that our result holds for $\Sigma^1_2$ sets. Finally we define two games related to this property.

Definition 1 A subtree $H \subseteq \omega^{<\omega}$ is Hechler iff $\forall s \in H \forall^{\infty} n \ s n \in H$. A subtree $L \subseteq \omega^{<\omega}$ is Laver iff $\forall s \in L \exists^{\infty} n \ s n \in L$.

These definitions are motivated by well-known forcing notions of Laver [4] and Hechler [3]. In the classical Hechler forcing the cofinite sets on the $n^{th}$ level of the tree would all be the same.

Definition 2 For any subtree $T \subseteq \omega^{<\omega}$ define

$$[T] = \{x \in \omega^\omega : \forall n \ x \upharpoonright n \in T\}$$

Theorem 3 For any $\Sigma^1_1$ set $A \subseteq \omega^\omega$ either there exists a Hechler tree $H$ with $[H] \cap A = \emptyset$ or there exists a Laver tree $L$ with $[L] \subseteq A$.

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Proof
Since analytic sets are projections of closed sets there exists a tree $T$ on $\omega^{<\omega} \times \omega^{<\omega}$ such that

$$A = p[T] = \{ x \in \omega^\omega : \exists y \in \omega^\omega \forall n \ (x \upharpoonright n, y \upharpoonright n) \in T \}.$$ 

Assume that for every Hechler $H$ that $A$ meets $[H]$ and we will show there is a Laver $L$ with $[L] \subseteq A$.

For $s, t \in \omega^{<\omega}$ define

$$A_{s,t} = \{ x \in \omega^\omega \ : \ s \subseteq x \exists y \supseteq t \ (x, y) \in [T] \}.$$ 

Definition 4 We say that $H$ is Hechler with root $s$ if for all $t \in H$ either $s \subseteq t$ or $t \subseteq s$ and beneath $s$ there is cofinite splitting.

Lemma 5 Suppose for every Hechler $H$ with root $s$ that $A_{s,t} \cap [H] \neq \emptyset$. Then there are infinitely many $n$ such that for every Hechler $H$ with root $s_n$ that $A_{s_n,t} \cap [H] \neq \emptyset$.

Proof
Otherwise for all but finitely many $n$ (say $n > N$) there exists a Hechler $H_n$ with root $s_n$ which misses $A_{s_n,t}$. But then the Hechler tree: $H = \bigcup_{n > N} H_n$ misses $A_{s,t}$ and has root $s$.

QED

Lemma 6 Suppose for every Hechler $H$ with root $s$ that $A_{s,t} \cap [H] \neq \emptyset$. Then there exists an infinite well-founded tree $T \subseteq \{ r : s \subseteq r \}$ with root $s$ and terminal nodes $B \subseteq T$ such that

(1) The nonterminal nodes of $T$ are $\omega$-splitting, i.e., if $r \in T \setminus B$, then there are infinitely many $n$ with $rn \in T$, and

(2) For every $r \in B$ there exists $n$ such that for every Hechler tree $H$ with root $r$, $A_{r,tn} \cap [H] \neq \emptyset$.

Proof
For each ordinal $\alpha$ define a set $B_\alpha \subseteq \{ r : s \subseteq r \}$ as follows.

(a) $r \in B_0$ iff there exists $n$ such that for every Hechler tree $H$ with root $r$, $A_{r,tn} \cap [H] \neq \emptyset$.

(b) $B_{\alpha+1} = B_\alpha \cup \{ r : \exists^\infty n \ r n \in B_\alpha \}$

(c) $B_\lambda = \bigcup_{\alpha < \lambda}$ for $\lambda$ a limit ordinal.
Define function rank\((r)\) on \(r \supseteq s\) as follows, \(\text{rank}(r) = \alpha\) if \(\alpha\) is the least ordinal with \(r \in B_\alpha\) and \(\text{rank}(r) = \infty\) if there is no such ordinal.

**Case 1.** \(\text{rank}(s)\) is an ordinal. In this case it is easy to build \(T\) and \(B\) as required.

**Case 2.** \(\text{rank}(s) = \infty\). We show that this is impossible. Note that if \(\text{rank}(r) = \infty\) then for all but finitely many \(n\) we must have that \(\text{rank}(rn) = \infty\). Hence we may construct a Hechler tree \(H\) with root \(s\) such that \(\text{rank}(r) = \infty\) for all \(r \in H\) below the root. For each \(n < \omega\) for each \(r \in \omega^{n+|s|} \cap H\) there exists a Hechler \(H_r\) with root \(r\) such that

\[
[H_r] \cap A_{s,tn} = \emptyset.
\]

Define

\[
K_n = \bigcup \{H_r : r \in \omega^{n+|s|} \cap H\}
\]

Note that \(K_n\) is a Hechler tree with root \(s\) whose \(n + |s|\) level is the same as \(H\). It also true that \(K_n \cap A_{s,tn} = \emptyset\). Because they are so wide \(K = \bigcap_{n<\omega} K_n\) is a Hechler tree with root \(s\) such that \([K] \cap \bigcup_n A_{s,tn} = \emptyset\). This contradicts the hypothesis of the Lemma since \(\bigcup_n A_{s,tn} = A_{s,t}\).

Finally, if \(T\) is trivial, i.e., \(T = B = \{s\}\) just apply Lemma 5 to make \(T\) infinite.

QED

**Proof of Theorem 3:**

Suppose for every Hechler tree \(H\) with trivial root that \(A \cap [H] \neq \emptyset\). Apply Lemma 6 to obtain a non-trivial well-founded tree \(T_0\) with terminal nodes \(B_0\) and witnesses of length one.

Suppose we are given a well-founded tree \(T_n\) with trivial root and terminal nodes \(B_n\) such that for all \(s \in T_n \setminus B_n\) there are infinitely many immediate extensions of \(s\) in \(T_n\) and for each \(s \in B_n\) there is a \(t_s\) of length \(n + 1\) such that for every Hechler tree \(H\) with root \(s\), \(A_{s,t_s} \cap [H] \neq \emptyset\). Apply Lemma 6 to each node \((s, t_s)\) with \(s \in B_n\). Union all these trees together to get \(T_{n+1}\) which end extends \(T_n\). It follows that \(L\) is a Laver where

\[
L = \bigcup_{n<\omega} T_n
\]
Note that although the length of the witnesses grow much slower than the $s$-part, nevertheless, they union up to show that $L \subseteq A$.

\[ \text{QED} \]

**Definition 7** For $\mathcal{F}$ a filter extending the cofinite filter on $\omega$ define $\mathbb{H}_\mathcal{F}$ to be the Hechler trees mod $\mathcal{F}$, i.e., instead of demanding that for each $s \in H$ that $sn \in H$ for cofinitely many $n$, we demand that

\[ \{ n : sn \in H \} \in \mathcal{F}. \]

Analogously define $\mathbb{L}_\mathcal{F}$ the Laver trees mod $\mathcal{F}$ by for each $s \in L$

\[ \{ n : sn \in L \} \in \mathcal{F}^+ \]

where $\mathcal{F}^+$ are the positive $\mathcal{F}$ sets, i.e., sets whose complement is not in $\mathcal{F}$.

**Theorem 8** For any filter $\mathcal{F}$ and any $\Sigma^1_1$ set $A \subseteq \omega^\omega$ either there exists a Hechler tree $H \in \mathbb{H}_\mathcal{F}$ with $[H] \cap A = \emptyset$ or there exists a Laver tree $L \in \mathbb{L}_\mathcal{F}$ with $[L] \subseteq A$.

\[ \text{Proof} \]

The proof of this goes over mutatis mutandis, the proof of Theorem 3.

\[ \text{QED} \]

Any Hechler tree $H$ may be pruned so that every node in it is strictly increasing, i.e., if $(x_0, x_1, \ldots, x_n) \in H$ then $x_0 < x_1 < \ldots x_n$. By the range of $H$ we mean all infinite subsets of $\omega$ which are the image of some branch $f \in H$, i.e.,

\[ \text{range}(H) = \{ \{ f(n) : n < \omega \} : f \in [H] \} \]

**Proposition 9** Suppose $H \in \mathbb{H}_\mathcal{F}$. Then there exists $X \in [\omega]^\omega$ such that $[X]^\omega \subseteq \text{range}(H)$.

\[ \text{Proof} \]

We may suppose that the nodes of $H$ are strictly increasing. Construct a strictly sequence $x_0, x_1, \ldots, x_n$ such that for every $k$ and subsequence

\[ 0 \leq i_1 < i_2 < \cdots < i_k \leq n \]

we have that $(x_{i_1}, \ldots, x_{i_k}) \in H$. To obtain $x_{n+1}$ we need only intersect finitely many elements of the filter $\mathcal{F}$.

\[ \text{QED} \]
Corollary 10 (Silver [5]) Analytic sets have the Ramsey Property. This means that for any \( \Sigma^1_1 \) set \( A \subseteq [\omega]^\omega \) there exists an \( X \in [\omega]^\omega \) with either \( [X]^\omega \subseteq A \) or \( [X]^\omega \cap A = \emptyset \).

Proof
Let \( F \) be a nonprincipal ultrafilter. Note that \( \mathbb{H}_F = \mathbb{L}_F \) for ultrafilters. Define \( B \subseteq \omega^\omega \) by \( f \in B \) iff \( f \) is strictly increasing with range in \( A \). Then \( B \) is \( \Sigma^1_1 \) and so by Theorem 8 there is a Hechler tree \( H \in \mathbb{H}_F \) with \( [H]^\omega \subseteq B \) or \( [H]^\omega \cap B = \emptyset \). By Proposition 9 there is an infinite \( X \) as required.

QED
This gives a proof of Silver’s Theorem which avoids the accept-reject arguments of Galvin-Prikry [2] and Ellentuck [1].

Theorem 11 \((V=L)\) There exists a \( \Pi^1_1 \) set \( A \subseteq \omega^\omega \) such that \( [H] \cap A \neq \emptyset \) for every Hechler tree \( H \) but \( A \) contains no Laver \([L]\).

Proof
Using the definable well-ordering of the reals in \( L \) construct \( B \subseteq \omega^\omega \) a \( \Sigma^1_2 \) set with the following two properties:

1. \( B \) is an \( <^* \) scale, i.e., \( B = \{ g_\alpha \in \omega^\omega : \alpha < \omega_1 \} \) where \( \alpha < \beta \) implies \( g_\alpha <^* g_\beta \) and for every \( f \in \omega^\omega \) there exist \( \alpha \) such that \( f <^* g_\alpha \).
2. \( B \) has the property that for any \( \sigma : \omega^{<\omega} \to \omega \) there exists \( g \in B \) such that for all \( x \in 2^\omega \) if \( f = 2g + x \) then \( \forall n \ f(n) > \sigma(f \upharpoonright n) \).

Let \( C \subseteq \omega^\omega \times 2^\omega \) be \( \Pi^1_1 \) so that \( g \in B \) iff \( \exists x \ (g, x) \in C \).

Given \( f \in \omega^\omega \) define \( Q(f) = (g, x) \) where \( g \in \omega^\omega \) and \( x \in 2^\omega \) are determined by \( f = 2g + x \). Define the \( \Pi^1_1 \) set \( A \) by

\[
A = \{ f \in \omega^\omega : Q(f) \in C \}.
\]

Note that for any Hechler \( H \) we can find \( \sigma : \omega^{<\omega} \to \omega \) such that

\[
H_\sigma = \{ f \in \omega : \forall n \ f(n) > \sigma(f \upharpoonright n) \} \subseteq H
\]

It follows from (2) that \( A \) meets every \([H]\). On the other hand \( A \) cannot contain the branches \([L]\) of a Laver tree. This is because of the scale (1). Take a 3 splitting subtree of \( T \subseteq L \), i.e., for every \( s \in T \) there are exactly 3 immediate extensions of \( s \) in \( T \). For each \( g \in \omega^\omega \) define

\[
C_g = \{ f \in \omega^\omega : \exists x \in 2^\omega \ f = 2g + x \}
\]
and note that $A \subseteq \bigcup_{g \in B} C_g$. If $[T] \subseteq A$ then by the scale property of $B$ there would have to be a countable set $Q \subseteq B$ with such that

$$[T] \subseteq \bigcup_{g \in Q} C_g$$

But the $C_g$ are the branches of a binary splitting tree and since $T$ is 3-splitting, it is easy to construct $f \in [T]$ such that $f \notin C_g$ for every $g \in Q$. QED

**Theorem 12** Assume MA $\Rightarrow CH$. If $A \subseteq \omega^\omega$ is $\Sigma^1_2$, then either there is a Hechler tree $H$ with $[H] \cap A = \emptyset$ or there is a Laver tree $L$ with $[L] \subseteq A$. In fact, this is true for any set $A$ which is the union of $\omega_1$ many Borel sets.

**Proof**

Suppose $A = \bigcup_{\alpha < \omega_1} B_\alpha$ where each $B_\alpha$ is Borel and the union is increasing. Since no $B_\alpha$ contains the branches of a Laver tree we have $H_\alpha$ a Hechler tree with $[H_\alpha] \cap B_\alpha = \emptyset$. Without loss we may assume that

$$[H_\alpha] = \{ f \in \omega^\omega : \forall n \ f(n) > \sigma_\alpha(f \upharpoonright n) \}$$

where $\sigma_\alpha : \omega^{<\omega} \to \omega$. By Martin’s axiom we may find $\sigma : \omega^{<\omega} \to \omega$ which eventually dominates each $\sigma_\alpha$. By a counting argument we can find a single $\sigma : \omega^{<\omega} \to \omega$ which everywhere dominates $\omega_1$ of the $\sigma_\alpha$. But this means that $H_\sigma$ is a Hechler tree disjoint from $A$ since the $B_\alpha$’s are an increasing union. QED

Finally we make some remarks about games.

**Game 1.** Given $A \subseteq \omega^\omega$. Player I and II alternatingly play

$$n_0, m_0 > n_0, \ n_1, \ m_1 > n_1, \ldots$$

with Player I playing $n_k \in \omega$ and Player II responding with $m_k > n_k$. The play of the game is won by Player II iff $(m_i : i < \omega) \in A$.

**Proposition 13** (a) Player II has a winning strategy in Game 1 iff there exists a Laver tree $L$ with $[L] \subseteq A$. (b) Player I has a winning strategy in Game 1 iff there exists a Hechler tree $H$ with $[H] \cap A = \emptyset$. 


Hechler and Laver Trees

Proof
Given the trees it easy to get the strategies. For the other direction:
(a) Use player II’s winning strategy to construct a Laver tree as required.
(b) If $\sigma : \omega^\omega \to \omega$ is Player I’s winning strategy, then for any sequence $(m_i : i < \omega)$ such that $m_{i+1} > \sigma(m_0, \ldots, m_i)$ for every $i$ we have that $(m_i : i < \omega) \notin A$. But this gives a Hechler tree $H$ with $[H]$ disjoint from $A$. QED

Game 2. Given $A \subseteq \omega^\omega$. Player I and II alternatingly play

$$X_0, \ m_0 \in X_1, \ X_1, \ m_1 \in X_1, \ldots$$

with Player I playing $X_k \in [\omega]^\omega$ and Player II responding with $m_k \in X_k$. Player II wins the play of the game iff $(m_i : i < \omega) \in A$.

Proposition 14 (a) Player II has a winning strategy in Game 2 iff there exists a Hechler tree $H$ with $[H] \subseteq A$. (b) Player I has a winning strategy in Game 2 iff there exists a Laver tree $L$ with $[L] \cap A = \emptyset$.

Proof
From right-to-left in both cases is easy. For the other direction:
(a) Let $\sigma$ be a winning strategy for Player II. Consider

$$\{m_0 : \exists X_0 \ \sigma(X) = m_0\}$$

This set must be cofinite, since otherwise consider $\sigma$’s response to its complement. Similarly given any sequence $X_0, X_1, \ldots, X_{n-1}$ the set

$$\{m_n : \exists X_n \ \sigma(X_0, \ldots, X_n) = m_n\}$$

must be cofinite. Construct $X_s$ for $s \in \omega^\omega$ and get a Hechler tree $H$ all of whose branches are plays of the winning strategy and hence are in $A$.

(b) The sequence of $X_s$ played by winning strategy of Player I determine a Laver tree $L$.

QED

Some of the results in this note follow from Zapletal [6].
References


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