ON THE LENGTH OF BOREL HIERARCHIES

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0. Introduction

For any separable metric space $X$ and $\alpha$ with $1 \leq \alpha \leq \omega_1$ define the Borel classes $\Sigma^0_\alpha$ and $\Pi^0_\alpha$. Let $\Sigma^0_1$ be the class of open sets and for $\alpha > 1$ $\Sigma^0_\alpha$ is the class of countable unions of elements of $\bigcup \{ \Pi^0_\beta : \beta < \alpha \}$ where $\Pi^0_\alpha = \{ X - A : A \in \Sigma^0_\alpha \}$. Hence $\Sigma^0_1 = \text{open} = G$, $\Pi^0_1 = \text{closed} = F$, $\Sigma^0_2 = F_\sigma$, $\Pi^0_2 = G_\delta$, etc. Note that $\Sigma^0_\omega = \Pi^0_\omega$ is the set of all Borel in $X$ subsets of $X$. The Baire order of $X$ (ord($X$)) is the least $\alpha \leq \omega_1$ such that every Borel in $X$ subset of $X$ is $\Sigma^0_\alpha$ in $X$. Since the Borel subsets of $X$ are closed under complementation we could equally well have defined ord($X$) in terms of $\Pi^0_\alpha$ in $X$ or $\Delta^0_\alpha = \Pi^0_\alpha \cap \Sigma^0_\alpha$ in $X$. Note also that for $X \subseteq \mathbb{R}$ (the real numbers) ord($X$) is the least $\alpha$ such that for every Borel set $A$ in $\mathbb{R}$ there is a $\Sigma^0_\alpha$ in $\mathbb{R}$ set $B$ such that $A \cap X = B \cap X$. Also note that ord($X$) = 1 iff $X$ is discrete, ord($\mathbb{Q}$) = 2 where $\mathbb{Q}$ is the space of rationals, and in general for $X$ a countable metric space ord($X$) $\leq 2$ since every subset of $X$ is $\Sigma^0_2(\mathbb{R})$ in $X$.

It is a classical theorem of Lebesgue (see [11]) that for any uncountable Polish (separable and completely metrizable) space ord($X$) = $\omega_1$. The same is true for any uncountable analytic ($\Sigma^1_1$) space $X$ since $X$ has a perfect subspace (see [11]) and Borel hierarchies relativize.

The Baire order problem of Mazurkiewicz (see [19]) is: for what ordinals $\alpha$ does there exist $X \subseteq \mathbb{R}$ such that ord($X$) = $\alpha$. Banach conjectured (see [29]) that for any uncountable $X \subseteq \mathbb{R}$ the Baire order of $X$ is $\omega_1$. In Section 3 we review the classically known results of Sierpinski, Szpilrajn, and Popruegenko. We show that it is consistent with ZFC that for each $\alpha \leq \omega_1$ there is an $X \subseteq \mathbb{R}$ with ord($X$) = $\alpha$. In fact, we prove a theorem of Kunen's that CH implies this. We also show that Banach's conjecture is consistent with ZFC.

Given a set $X$ and $R$ a family of subsets of $X$ ($R \subseteq P(X)$) define for every $\alpha \leq \omega_1$ $R_\alpha \subseteq P(X)$ as follows. Let $R_0 = R$ and for each $\alpha > 0$ if $\alpha$ is even (odd) let $R_\alpha$ be the family of countable intersections (unions) of elements of $\bigcup \{ R_\beta : \beta < \alpha \}$. Generalizing Mazurkiewicz's question Kolmogorov (see [8]) asked: for what ordinals $\alpha$ does there exist $X$ and $R \subseteq P(X)$ such that $\alpha$ is the least such

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that \( R_\alpha = R_{\omega_1} \). Kolmogorov’s question can be generalized by replacing \( P(X) \) by an arbitrary \( \sigma \)-algebra (a countably complete boolean algebra). In Section 2 we prove that for any \( \alpha \leq \omega_1 \) there is a complete boolean algebra with the countable chain condition which is countably generated in exactly \( \alpha \) steps. This answers a question of Tarski who had noticed that the boolean algebras \( \text{Borel}(2^n) \) modulo the ideal of meager sets and \( \text{Borel}(2^n) \) modulo the ideal of measure zero sets are countably generated in exactly one and two steps respectively (see [4]). Theorem 12 which is due to Kunen shows that the same answer to Kolmogorov’s problem (every \( \alpha \leq \omega_1 \)) follows from the solution of Tarski’s problem.

Let \( R = \{A \times B : A, B \subseteq 2^n\} \). In Section 4 we show that for any \( \alpha, 2 \leq \alpha < \omega_1 \), it is consistent with ZFC that \( \alpha \) is the least ordinal such that \( R_\alpha \) is the set of all subsets of \( 2^\omega \times 2^\omega \). This answers a question of Mauldin [1].

For \( \alpha \leq \omega_1 \) a set \( X \subseteq 2^n \) is a \( \sigma^* \)-set iff every subset of \( X \) is Borel in \( X \) and \( \text{ord}(X) = \alpha \). It is shown that it is consistent with ZFC that for every \( \alpha < \omega_1 \), there is a \( Q_\alpha \) set. In Section 4 we also show that there are no \( Q_{\omega_1} \) sets. However, we do show that it is consistent with ZFC that there is an \( X \subseteq 2^n \) with \( \text{ord}(X) = \omega_1 \) and every \( X \)-projective set is Borel in \( X \). This answers a question of Ulam [31, p. 10].

Also in Section 4 we show that it is relatively consistent with ZFC that the universal \( \Sigma^1_1 \) set is not in \( R_{\omega_1} \), confirming a conjecture of Mansfield [13] who had shown that the universal \( \Sigma^1_1 \) set is never in the \( \sigma \)-algebra generated by the rectangles with \( \Sigma^1_1 \) sides.

Given \( R \subseteq P(X) \) let \( K(R) \) (the Kolmogorov number of \( R \)) be the least \( \alpha \) such that \( R_\alpha = R_{\omega_1} \). It is an exercise to show that for \( \alpha = 0, 1, \) or \( 2 \) there is an \( R \subseteq P(\{0, 1\}) \) with \( K(R) = \alpha \).

**Proposition 1.** Given \( R \subseteq P(X) \) then (a) if \( R \) is finite or \( X \) is countable, then \( K(R) \leq 2 \), and (b) there exists \( S \subseteq P(Y) \) such that cardinality of \( S \) and \( Y \) is \( \leq 2^\omega \), and \( K(R) = K(S) \).

**Proof.** (a) Note

\[
\bigcup_{\alpha < \omega_1} \bigcap_{\beta < \alpha} A_{\alpha, \beta, \gamma} = \bigcap_{\alpha < \omega_1} \bigcup_{\beta < \alpha} A_{\alpha, \beta, \gamma}
\]

If \( R \) is finite or \( X \) countable, then \( \cap_{\alpha < \omega_1} A_{\alpha, \beta, \gamma} \) can always be taken to be a countable intersection.

(b) Let \( V_\alpha \) be the sets of rank less than \( \alpha \). Choose \( \alpha \) a limit ordinal of uncountable cofinality so that \( R, X \in V_\alpha \). Let \((M, \epsilon)\) be an elementary substructure of \((V_\alpha, \epsilon)\) containing \( R \) and \( X \) such that \( M^\omega \subseteq M \) and \( |M| \leq 2^\omega \). Now let \( Y = X \cap M \) and \( S = \{A \cap Y : A \in R \cap M\} \).

Mazurkiewicz’s problem is equivalent to Kolmogorov’s problem for \( R \) a countable field of sets (that is closed under finite intersection and complementation).
Proposition 2. (Sierpinski [23] also in [30]). Given $R \subseteq P(X)$ a countable field of sets there exists $Y \subseteq 2^\omega$ such that $K(R) = \text{ord}(Y)$. (That is we may reduce to considering subsets $Y$ of $2^\omega$ and relativizing the usual Borel hierarchy on $2^\omega$ to $Y$.)

Proof. Let $R = \{A_n : n \in \omega\}$ and define $F : X \rightarrow 2^\omega$ by $F(x)(n) = 1$ iff $x \in A_n$. Put $Y = F''X$.

Define $K = \{\beta : 2 \leq \beta < \omega_1$ and there is $X \subseteq \omega^\omega$ uncountable with $\text{ord}(X) = \beta\}$. What can $K$ be?

Proposition 3. $K$ is a closed subset of $\omega_1$.

Proof. Given $A \subseteq \omega^\omega$ and $n \in \omega$ define $nA = \{x \in \omega^\omega : x(n) = n\}$ and $\forall y \in A \forall n (x(n + 1) = y(n))$. If $X = \bigcup_{n \in \omega} nX_n$, then it is readily seen that $\text{ord}(X) = \sup \{\text{ord}(X_n) : n \in \omega\}$.

Note that $K$ is the same set of ordinals if we replace $\omega^\omega$ by $\mathbb{R}$ the real numbers or $2^\omega$. This is true for $\mathbb{R}$ because if $X \subseteq \mathbb{R}$ and $\mathbb{R} - X$ is not dense, then $X$ contains a nonempty interval, hence $\text{ord}(X) = \omega_1$; but $\mathbb{R} - X$ dense means we may as well assume $X \subseteq$ irrationals $\cong \omega^\omega$.

In the definition of $K(R) = \omega$ for $R \subseteq P(X)$ we ignored the possibility that the hierarchy on $R$ might have exactly $\omega$ levels, i.e. $R_{\omega_1} = \bigcup \{R_n : n < \omega\}$ but for all $n < \omega$ $R_n \neq R_{\omega_1}$. In fact a Borel hierarchy of length less than $\omega_1$ must have a top level.

Proposition 4. If $R \subseteq P(X)$ is a field of sets, $\lambda$ is a countable limit ordinal, and $R_{\omega_1} = \bigcup \{R_\alpha : \alpha < \lambda\}$, then there is $\alpha < \lambda$ such that $R_\alpha = R_{\omega_1}$.

Proof. Using the proof of Proposition 2 we can assume $X \subseteq 2^\kappa$ for some $\kappa$ and $R = \{[s] \cap X : \exists D \in [\kappa]^{<\omega} (s \in 2^D)\}$ where $[s] = \{f \in 2^\kappa : f \text{ extends } s\}$. For each $A$ in $R_\omega$, there is $T \subseteq \kappa$ countable such that for any $f$ and $g$ in $X$ if $f \upharpoonright T = g \upharpoonright T$, then $f \in A$ iff $g \in A$. In this case we say $T$ supports $A$. Choose $T \subseteq \kappa$ countable so that for any $D \subseteq T$ finite and $s : D \rightarrow 2$ if $\text{ord}(X \cap [s]) = \lambda$, then for any $\alpha < \lambda$ there is an $A \subseteq [s]$ in $R_{\alpha + 1} - R_\alpha$ such that $T$ supports $A$. By taking an autohomeomorphism of $2^\kappa$ we may assume $T = \omega$. Define $L$ to be $\{s \in 2^{<\omega} : \text{ord}([s] \cap X) = \lambda\}$.

Claim. For any $s$ in $L$ there are $t$ and $\hat{t}$ in $L$ incompatible extensions of $s$.

Proof. Without loss of generality assume $s = \emptyset$ and there is $f \in 2^{\omega}$ such that for every $s \in L$ $s \subseteq f$. For each $n < \omega$ define $t_n$ in $2^{n+1}$ by $t_n(m) = f(m)$ for $m < n$ and $t_n(n) = 1 - f(n)$. Then $[f] \cup \bigcup \{[t_n] : n < \omega\}$ is a disjoint union covering $2^\omega$. If there is a $\beta_0 < \lambda$ such that for all $n < \omega$ ord([t_n] \cap X) < \beta_0$, then for all $A$ in $R_{\omega_1}$ supported by $\omega A$ is in $R_{\beta_0 + 1}$. This is because $A \cap [f] = \emptyset$ or $X \cap [f] \subseteq A$. But this contradicts the choice of $\omega$. 

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On the other hand, if there is no such bound $\beta_0$, choose $Z_n \subseteq [\alpha_n]$ with $Z_n \notin R_{\alpha_0}$, so that for every $\beta < \lambda$ there is $n < \omega$ with $Z_n \notin R_{\beta}$. But then $\bigcup \{Z_n : n < \omega\}$ is not in $\bigcup \{R_\beta : \beta < \lambda\}$. This proves the claim and this last argument also proves the proposition from the claim.

**Remark.** If $R \subseteq P(X)$ and $R_{\alpha_0} = \bigcup \{R_n : n < \omega\}$ and there is $n_0 < \omega$ such that $\{X - A : A \in R\} \subseteq R_{n_0}$, then there is $n_1 < \omega$ such that $R_{n_1} = R_{\alpha_0}$. Willard [32] shows that for any $\alpha < \omega_1$ there are $R$ and $X$ with $R \subseteq P(X)$ such that $\alpha$ is the least ordinal such that $\{X - A : A \in R\} \subseteq R\alpha$.

1. Some basic definitions and lemmas

For $T \subseteq \omega^{<\omega}$, $T$ is a well-founded tree iff $T$ is a tree (if $t \supseteq s \in T$, then $t \in T$) and is well-founded (for any $f \in \omega^\omega$ there is an $n < \omega$ such that $f \upharpoonright n \notin T$). For $s \in T$ define $|s|_T$ (the rank of $s$ in $T$) by $|s|_T = \sup \{|t|_T + 1 : s \subseteq t \in T\}$. Often we drop $T$ and let $|s| = |s|_T$. $T$ is normal of rank $\alpha$ means that:

(a) $T$ is a well-founded tree;
(b) $|\beta| = \alpha$ ($\emptyset$ is the empty sequence);
(c) $(s \in T$ and $|s| > 0) \to (\forall i < \omega \; (s \upharpoonright i \in T))$;
(d) $(s \in T$ and $|s| = \beta + 1) \to (\forall i < \omega \; (|s \upharpoonright i| = \beta))$;
(e) $(s \in T$ and $|s| = \lambda$ where $\lambda$ is a limit ordinal) $\to (\forall \beta < \lambda \; \{i : |s \upharpoonright i| < \beta\}$ is finite and $\forall i < \omega \; |s \upharpoonright i| \geq 2$.

Note that for any $n < \omega$ the tree $\omega^{<\omega}$ is normal of rank $n$. If $\alpha_n$ for $n < \omega$ are strictly increasing to $\alpha$ (or $\alpha_n = \beta$ where $\alpha = \beta + 1$) and for each $n < \omega$ $T_n$ is normal of rank $\alpha_n \geq 2$, then $T = \{\emptyset\} \cup \{n - s : n < \omega$ and $s \in T_n\}$ is normal of rank $\alpha$. We often use $T_\alpha$ to denote some fixed normal tree of rank $\alpha$. Let $M$ be the ground model of ZFC. Working in $M$ for any $\alpha < \omega_1$ and $Y \subseteq X \subseteq \omega^\omega$ define the partial order $P_\alpha(Y, X)$ (the order is given by inclusion). Fix some $T$ normal of rank $\alpha$. $p \in P_\alpha(Y, X)$ iff $p \subseteq (T - \{\emptyset\} \times (X \cup \omega^{<\omega})$ and (1) through (5) hold.

(1) $p$ is finite.
(2) $|s| = 0$ implies that if $(s, x) \in p$, then $x \in \omega^{<\omega}$ and if $(s, y) \in p$, then $x = y$. (So if $T^* = \{s \in T : |s| = 0\}$, then $p \setminus (T^* \times (X \cup \omega^{<\omega}))$ is a function from a finite subset of $T^*$ into $\omega^{<\omega}$.)
(3) If $|s| > 0$ and $(s, x) \in p$, then $x \in X$.
(4) If $s$ and $s \upharpoonright i \in T$ and $x \in X$, then not both $(s, x)$ and $(s \upharpoonright i, x)$ are in $p$, or if $|s \upharpoonright i| = 0$, there is no $k \in \omega$ such that both $(s, x)$ and $(s \upharpoonright i, x \upharpoonright k)$ are in $p$.
(5) If $s$ of length one and $(s, x) \in p$, then $x$ is not in $Y$.

Let $G$ be $P_\alpha(Y, X)$-generic over $M$. Working in $M[G]$ define for each $s \in T$, $G_s \subseteq \omega^\omega$. For $|s| = 0$, let

$$G_s = \{x \in \omega^\omega : \exists t \in \omega^{<\omega} \; t \subseteq x \text{ and } \{(s, t)\} \in G\}.$$
For $|s|>0$, let $G_s = \bigcap \{\omega^\omega - G_{s^{-i}} : i < \omega\}$. Note that for each $s \in T$, $G_s \in \Pi_{|s|}^0$.

Lemma 5. For each $x$ in $X$ and $s$ in $T - \{\emptyset\}$ with $|s|>0$ $[x \in G_s \iff \{(s, x)\} \in G]$.

Proof. Case 1. $|s| = 1$. (This is the argument from almost-disjoint-sets forcing.)

If $x \in G_s$, then $x \notin G_{s^{-i}}$ for all $i \in \omega$. Hence for all $k$ and $i$ in $\omega$ $(s^{-i}, x \upharpoonright k) \notin G$.

Let $D = \{p : (s, x) \in p \lor \text{there exist } k \text{ and } i \text{ such that } (s^{-i}, x \upharpoonright k) \in p\}$. $D$ is dense since if $(s, x) \notin p$ if we let $\{x_1, x_2, \ldots, x_n\} \subseteq X$ be all the elements of $\omega^\omega$ mentioned in $p$ other than $x$, we can choose $k$ sufficiently large so that $x \upharpoonright k \neq x_i \upharpoonright k$ for all $i \leq n$. Also we can choose $\alpha$ sufficiently large so that $(s^{-j})$ is not mentioned in $p$ and then $p \cup \{(s^{-j}, x \upharpoonright k)\} \in (\mathcal{P}_\alpha(Y, X) \cap D)$. Since $G \cap D$ is non-empty and $x \notin G_{s^{-i}}$ all $i$; we conclude that $(s, x) \in G$.

If $x \notin G_s$, then $x \in G_{s^{-i}}$ for some $i$. Hence there exist $k$ such that $(s^{-i}, x \upharpoonright k) \in G$ so $(s, x) \notin G$ by clause (4).

Case 2. $|s| > 1$.

If $x \in G_s$, then $x \notin G_{s^{-i}}$ for all $i$, and hence by induction $(s^{-i}, x) \notin G$ for all $i$.

Let $D = \{p : (s, x) \in p \lor \text{there exist } i \text{ such that } (s^{-i}, x) \in p\}$. $D$ is dense hence $(s, x) \in G$.

If $x \notin G_s$, then $(s^{-i}, x) \in G$ for some $i$ (by induction). Hence $(s, x) \notin G$ by clause (4).

Corollary 6. $G_\emptyset \cap X = Y \ (\alpha \geq 2)$.

Proof. If $x \in Y$, then for every $n, ((n), x) \notin G$ (by clause 5). Hence by Lemma 5 for every $n, x \notin G_{(n)}$ and so $x \in G_\emptyset$. If $x \notin Y$, then $\{p : \text{there exists } n \text{ such that } ((n), x) \in p\}$ is dense hence there exists $n$ such that $x \in G_{(n)}$ (by Lemma 5) so $x \notin G_\emptyset$.

Remarks: (1) $\mathcal{P}_0(Y, X)$ is trivial (the empty set).
(2) $\mathcal{P}_1(Y, X)$ has nothing to do with $X$ and $Y$ and is isomorphic as a partial order to the usual Cohen partial order for adding a map from $\omega$ to $\omega$.
(3) $\mathcal{P}_2(Y, X)$ is another way of viewing Solovay's ``almost-disjoint-sets forcing'' (see [6]).

Lemma 7. $\mathcal{P}_\alpha(Y, X)$ has the countable chain condition.

Proof. Suppose by way of contradiction that there exist $F$ included in $\mathcal{P}_\alpha(Y, X)$ of cardinality $\aleph_1$ of pairwise incompatible conditions. Since there are only countably many finite subsets of $T$, we may assume there exist $H \subseteq T - \{\emptyset\}$ finite so that every $p \in F$ is included in $H \times (Y \cup \omega^{<\omega})$. We may also assume that for every $p \in F$ and $q \in F$ and $s \in H$ with $|s| = 0$ and $t \in \omega^{<\omega}$ that $[(s, t) \in p \iff (s, t) \in q]$. Now let
(x_\beta : \beta < \aleph_1) be all the elements of X occurring in members of F. For each p in F let p^*: G_p \rightarrow P(H) be defined by G_p = \{ \beta : \text{there exists } s, (s, x_\beta) \in p \} and for \beta \in G_p, p^*(\beta) = \{ s : (s, x_\beta) \in p \}. \{ p^*: p \in F \} is a family of \aleph_1 incompatible conditions in the partial order Q, where Q = \{ p : \text{domain of } p \text{ is a finite subset of } \aleph_1 \text{ and range of } p \text{ is } P(H) \}, ordered by inclusion. Since it is well-known that Q has the countable chain condition we have a contradiction.

Remarks: (1) If P = \mathbb{P}_\alpha(Y, X) for any \alpha, X, and Y, then P is absolutely c.c.c. That is to say if \mathbb{P} \in M \vDash \text{"ZFC"}, then M \vDash \text{"P has c.c.c."}. It follows that the direct sum of any combination of the \mathbb{P}_\alpha's has the c.c.c.

(2) We assume the fact that iterated c.c.c. forcing is c.c.c. (Solovay-Tennenbaum [26]) and occasionally use notation and facts from [26].

I would like to prove next an heuristic proposition. Roughly, if we add a generic \Pi_2 set, then it will not be \Sigma_0. This is a special case of more difficult arguments later with generic \Pi_3 sets.

Define \mathbb{P} a partial order: p \in \mathbb{P} iff p is a finite consistent set of sentences of the form "[s] \subseteq G_n", "x \notin G_n", or "x \in \bigcap_{n \in \omega} G_n" (where s \in \omega^{<\omega} and x \in \omega^\omega). Order \mathbb{P} by inclusion. Any G \mathbb{P}-generic determines a \Pi_3 set \bigcap_{n \in \omega} G_n.

Proposition. If G is \mathbb{P}-generic over M (transitive countable model of ZFC). then

\[ M[G] \vDash \forall F \in F_n \left( F \cap M \neq \bigcap_{n \in \omega} G_n \cap M \right). \]

Proof. Suppose not and let p \in G and C_n be names such that p \vDash "C_n is closed" and such that

\[ p \vDash \bigcup_{n \in \omega} C_n \cap M = \bigcap_{n \in \omega} G_n \cap M. \]

It is easily seen that \mathbb{P} has c.c.c. (see the proof of Lemma 7). Thus working in M we can find Q \subseteq \mathbb{P} countable such that for any \hat{G} \mathbb{P}-generic, n \in \omega, and s \in \omega^{<\omega}, if M[\hat{G}] \vDash "[s] \cap C_n = \emptyset", then \exists q \in Q \cap \hat{G} such that q \vDash "[s] \cap C_n = \emptyset". Since Q is countable, we can find z \in \omega^\omega \setminus \{ f \} not mentioned in p or any condition in Q. Since

\[ p \cup \left\{ z \in \bigcap_{n \in \omega} G_n \right\} \vDash "z \in \bigcup_{n \in \omega} C_n". \]
we can find \( \tilde{n} \in \omega \) and \( \dot{p} \supseteq p \) and not mentioning \( z \) so that

\[
\dot{p} \cup \left\{ z \in \bigcap_{n \in \omega} G_n \right\} \vDash "z \in C_{\tilde{n}} ",
\]

because the only other way to mention \( z \) is "\( z \notin G_n \)". By taking \( \tilde{n} \) large enough \( \dot{p} \cup \{ z \notin G_n \} \) will be consistent, and since it extends \( p \) it forces "\( z \notin C_n \)". Let \( G \) be \( \mathbb{P} \)-generic with \( \dot{p} \cup \{ z \notin G_{\tilde{n}} \} \) in \( G \). Let \( k \in \omega \) and \( q \in G \) be so that \( q \vDash "[z \upharpoonright k] \cap C_n = \emptyset " \). But \( \dot{p} \cup q \cup \{ z \in \bigcap_{n \in \omega} G_n \} \) is consistent because \( q \in Q \) and so doesn’t mention \( z \). This is a contradiction since \( q \vDash "z \notin C_n \)" and

\[
\dot{p} \cup \left\{ "z \in \bigcup_{n \in \omega} G_n " \right\} \vDash "z \in C_{\tilde{n}} " .
\]

Define for \( F \subseteq \omega^\omega \) and \( p \in \mathbb{P} = \mathbb{P}_\omega(Y, X) \),

\[
|p|(F) = \max \left( \{ |s| : \text{there is } x \notin F \text{ with } (s, x) \in p \} \right).
\]

This is called the rank of \( p \) over \( F \).

**Lemma 8.** For all \( \beta \geq 1 \) and \( p \in \mathbb{P} \) there is \( \dot{p} \in \mathbb{P} \) compatible with \( p \) and \( |\dot{p}|(F) < \beta + 1 \) so that for any \( q \in \mathbb{P} \) with \( |q|(F) < \beta \), if \( \dot{p} \) and \( q \) are compatible, then \( p \) and \( q \) are compatible.

**Proof.** First find an extension \( p_0 \supseteq p \) so that for all \( (s, x) \in p \) and \( i < \omega \) if \( |s| = \lambda \) is a limit ordinal and \( |s \upharpoonright i| \leq \beta + 1 < \lambda \) (there are only finitely many such \( s \upharpoonright i \)), then there is a \( j < \omega \) such that \( (s \upharpoonright i, x) \in p_0 \). Now let \( \dot{p} = \{(s, x) \in p_0 : |s| < \beta + 1 \) or \( x \in F \} \). We check that \( \dot{p} \) has the requisite property. Suppose \( p \) and \( q \) are incompatible, \( \dot{p} \) and \( q \) are compatible, and \( |q|(F) < \beta \). Since \( \beta \geq 1 \) for all \( (s, x) \in p \) if \( |s| \leq 1 \), then \( (s, x) \in \dot{p} \), hence since \( \dot{p} \) and \( q \) are compatible there are \( s, t \in \omega^{< \omega} \), \( i < \omega \), and \( x \in \omega^\omega \) such that \( (s, x) \in p \), \( (t, x) \in q \), and \( s = t \upharpoonright i \) or \( t = s \upharpoonright i \).

**Case 1.** If \( x \in F \) or \( |s| < \beta + 1 \), then \( (s, x) \in \dot{p} \) and so \( \dot{p} \) and \( q \) are incompatible.

**Case 2.** If \( x \notin F \) and \( |s| \geq \beta + 1 \), then by definition of \( |q|(F) < \beta \), \( |t| < \beta \). So \( t = s \upharpoonright i \). If \( |s| = \gamma + 1 \) for some \( \gamma \), then \( |t| = \gamma \geq \beta \), contradiction. If \( |s| = \lambda \) is an infinite limit ordinal, then by the construction of \( p_0 \) there is \( j < \omega \) with \( (t \upharpoonright j, x) \in p_0 \) and hence \( (t \upharpoonright j, x) \in \dot{p} \) and so \( q \) and \( \dot{p} \) are incompatible.
2. Boolean algebras

For \( B \) a complete boolean algebra, \( C \) included in \( B \), and \( \alpha \geq 1 \) define \( \Sigma_\alpha(C) \), \( \Pi_\alpha(C) \):

\[
\Sigma_1(C) = \left\{ \sum S : S \subseteq C \right\}, \quad \Sigma_\alpha(C) = \left\{ \sum S : S \subseteq \bigcup_{\beta < \alpha} \Pi_\beta(C) \right\} \text{ for } \alpha > 1,
\]

and

\[
\Pi_\alpha(C) = \{-a : a \in \Sigma_\alpha(C)\}
\]

Define \( K(B) \) to be the least ordinal \( \alpha \) such that there exists a countable \( C \) included in \( B \) with \( \Sigma_\alpha(C) = B \).

**Theorem 9.** For each \( \alpha \leq \omega \), there exists a complete boolean algebra \( B \) with countable chain condition and \( K(B) = \alpha \).

**Proof.** For \( \alpha = 0 \) take \( B \) to be any finite boolean algebra. For \( \alpha = 1 \) take \( B \) to be \((P(\omega), \cap, \cup)\) (or more appropriately the regular open subsets of \( \omega^\omega \) since this corresponds to Cohen real forcing).

For \( \alpha, 2 \leq \alpha < \omega_1 \), \( B \) will be the complete boolean algebra associated with \( \Pi_\alpha \)-forcing. Let \( P = P_\alpha(\beta, X) \). Given a partial order \( P \) there is a canonical way of constructing a complete boolean algebra \( B \) in which \( P \) is densely embedded (see \( \textbf{[5]} \)). Let \( [p] \) denote the image of \( p \in P \) under this embedding. If \( p \geq q \), then \( [p] \leq [q] \). For every \( a \in B \) if \( a \neq 0 \), then there is a \( p \in P \) such that \( [p] \leq a \).

**Lemma 10.** Suppose \( F \subseteq X \) and \( C = \{[p] : p \in P \text{ and } |p|(F) = 0\} \). For any \( \beta \geq 1 \), \( p \in P \), and \( a \in \Sigma_\alpha(C) \), if \( [p] \leq a \), then there is \( q \in P \) such that \( |q|(F) < \beta \) and \( q \) and \( p \) are compatible, and \( [q] \leq a \).

**Proof.** The proof is by induction on \( \beta \).

**Case 1.** \( \beta = 1 \). Suppose \( a = \sum \{[q] : q \in \Gamma \} \) for some \( \Gamma \subseteq C \). If \( [p] \leq a \), then for some \( q \in \Gamma \), \( p \) and \( q \) are compatible.

**Case 2.** \( \beta \) a limit ordinal. Suppose \( a = \sum \{b : b \in \Gamma \} \) for some \( \Gamma \subseteq \bigcup \{ \Sigma_\alpha(C) : \alpha < \beta \} \). Then there is \( \dot{p} \geq p \) and \( b \in \Gamma \cap \Sigma_\alpha(C) \) for some \( \alpha < \beta \) so that \( [\dot{p}] \leq b \). Now apply the inductive hypothesis to \( \dot{p} \).

**Case 3.** \( \beta + 1 \). Suppose \( [p] \leq \sum \{b : b \in \Gamma \} \) for some \( \Gamma \subseteq \Pi_\beta(C) \). Choose \( \dot{p} \leq p \) so that for some \( b \in \Gamma \), \( [\dot{p}] \leq b \). By Lemma 8 of Section 1, there exists \( q \) compatible with \( \dot{p} \) with \( |q|(F) < \beta + 1 \) and for any \( r \) with \( |r|(F) < \beta \), if \( r \) and \( q \) are compatible, then \( r \) and \( \dot{p} \) are compatible. This \( q \) works since if \( [q] \neq b \), then there exists \( q_0 \geq q \) with \( [q_0] \leq -b \). Since \( -b \in \Sigma_\beta(C) \) by induction there is \( q_1 \) compatible with \( q_0 \) with
\[|q_1|(F) < \beta \text{ and } [q_1] \leq -b. \] But then \(q_1\) would be compatible with \( \hat{\rho}\), contradicting \([\hat{\rho}] \leq b.\)

Now if \(X = \omega^\omega\), for example, the lemma shows that \(B\) cannot be generated by a set of size less than the continuum in fewer than \(\alpha\) steps. For suppose \(D \subseteq B\) has cardinality less than \(|\omega^\omega|\), then there exists \(F \supseteq \omega^\omega\) with \(X - F \neq \emptyset\) and \(D \subseteq \Sigma_1\{[p]: |p|(F) = 0\}\). Let \(\beta < \alpha, z \in X - F, \) and \(s \in T - \{\emptyset\}\) with \(|s|_\tau = \beta\) (where \(T\) is the normal \(\alpha\)-tree used in the definition of \(\mathcal{P}_\alpha(\emptyset, X)\)). \([((s, z))]\) is not in \(\Sigma_\beta(D)\). Because if it were it would be in \(\Sigma_\beta(C)\) and so by the lemma there exists \(q\) with \(|q|(F) < \beta\) and \([q] \subseteq [((s, z))]\). But since \(|s|_\tau = \beta\) and \(z \notin F\) we know \((s, z) \notin q\). Thus there are \(n\) (and \(m\)) such that \(q \cup \{(s - n, z)\} (q \cup \{(s - n, z \upharpoonright m)\})\) in case \(|s|_\tau = 1\) is in \(\mathcal{P}\), but this is a contradiction.

Next we show \(B\) is countably generated in \(\alpha\) steps. Let \(\hat{C} = \{[p]: |p|(\emptyset) = 0\}\).

Claim. For all \(x \in X\) and \(s \in T - \{\emptyset\}\) if \(|s|_\tau = \beta \geq 1\), then \([((s, x))]\) is in \(\Pi_\beta(\hat{C})\).

Proof. If \(|s|_\tau = 1\), then
\[[((s, x))] = \prod \{-[((s - n, x \upharpoonright m))] : n, m \in \omega\}.
\]
If \(|s| > 1\), then
\[[((s, x))] = \prod \{-[((s - n, x))] : n \in \omega\}.
\]
For \(A \in B\), \(-A = \{p \in \mathcal{P} : [p] \cap A = \emptyset\}\). If \((s, x) \in p\), then \([p] \cap \{((s - n, x))]\} = \emptyset\) all \(n\). On the other hand if \([p] \cap \{((sn, x))]\} = \emptyset\) for all \(n\), then easily \((s, x) \in p\).

Now for any \(p \in \mathcal{P}\) \([p] = \prod \{[((s, x))] : (s, x) \in p\}\), so \([p] \in \Sigma_\alpha(\hat{C})\). For any \(A \in B\) \(A = \sum \{[p] : p \in A\}\) so \(A \in \Sigma_\alpha(\hat{C})\). Thus \(K(B) \leq \alpha\).

We are now ready to consider the case of \(\alpha = \omega_1\). Let \(\mathcal{P} = \sum_{\omega_1}, \mathcal{P}_\alpha(\emptyset, \omega^\omega)\). Now the complete boolean algebra associated with \(\mathcal{P}\) does take \(\omega_1\) steps to close (for suitable generators), however, \(\mathcal{P}\) is not countably generated. So we do as follows: Let \((x_\alpha : \alpha < \omega_1)\) be any set of \(\omega_1\) distinct elements of \(\omega^\omega\). Let \(*: \omega^{<\omega} \times \omega^{<\omega} \to \omega\) be a 1–1 map. Let \(T_\alpha\) be the normal tree of rank \(\alpha\) used in the construction of \(\mathcal{P}_\alpha(\emptyset, \omega^\omega)\). Any \(G\) which is \(\mathcal{P}_\alpha\)-generic is determined by \(G \cap \{(s, t) \in T_\alpha : |s|_{\tau_\alpha} = 0\}\). That is a map \(y\) from \(T_\alpha^* = \{s \in T_\alpha : |s|_{\tau_\alpha} = 0\}\) to \(\omega^{<\omega}\). Now imagine \(G\) \(\mathcal{P}\)-generic and let \(y_\alpha : T_\alpha^* \to \omega^{<\omega}\) be the maps determined by \(G\). Let \(Y = \{(\hat{s}(s, t) \upharpoonright \tau_\alpha) : y_\alpha(s) = t \text{ and } \alpha < \omega_1\}\). Form in the generic extension \(\mathcal{P}_2(\omega^\omega - Y, \omega^\omega) = Q\) (in both cases we mean \(\omega^\omega\) formed in the ground model). The partial order we are interested in is \(R = \mathcal{P} \star Q, \mathcal{P} \star Q = \{(p, q) : p \in \mathcal{P}\) and \(p \vdash \text{"}q \in Q\"\} \langle p, q, \hat{p}, \hat{q}\rangle \rangle\) iff \((\hat{p} \vdash p \text{ and } \hat{q} \vdash q)\). \(p \vdash \text{"}q \in Q\"\) just in case whenever \((n), (\hat{s}(s, t) \upharpoonright \tau_\alpha)\) is in \(q\), then \((s, t) \in p(\alpha)\). Now let \(B\) be the complete boolean algebra associated with \(R\). Since \(R\) has the countable chain condition so does \(B\).
Claim. $\mathcal{B}$ is countably generated.

Proof. The idea is that once you know what the real is gotten by $Q$ you know all the reals gotten by $P$ — and hence everything. Let $C = \{(|\emptyset, q|): |q| (\emptyset) = 0\}$. Then $C$ is countable and generates $\mathcal{B}$.

For $C \subseteq \omega^\omega$ and $(p, q) \in R$ define

$$(p, q)(C) = \max \{|s|_{\tau_\alpha}: \text{there exists } x \notin C, (s, x) \in p(\alpha) \text{ and } \alpha < \omega_1\}$$

Lemma 11. Given $F \subseteq \omega^\omega \forall p \in R \forall \beta \geq 1 \exists \hat{\beta} \in R$ compatible with $p$, $|\hat{\beta}| (F) < \beta + 1$ and $\forall q |q| (F) < \beta$ (if $\hat{p}, q$ compatible, then $p, q$ are compatible).

Proof. This is proved similarly to Lemma 8. Given $p = \langle p_0, p_1 \rangle$ extend each $p_0(\alpha) \subseteq p_0(\alpha)$ as in Lemma 8, then take $\hat{p} = \langle \hat{p}_0, \hat{p}_1 \rangle$, $\hat{p}_1 = p_1$, $\hat{p}_0(\alpha) = \{\langle s, x \rangle \in p_0(\alpha): |s| < \beta + 1 \text{ or } x \in C\}$. Note that $\hat{p}_0 \Vdash \langle \hat{\beta}_1 \in Q \rangle$ because requirements in $Q$ are decided by rank zero condition in $P$.

From this lemma it is easily shown as before that $K(\mathcal{B}) \geq \omega_1$. Since $\mathcal{B}$ is countably generated and has the countable chain condition we have $K(\mathcal{B}) \leq \omega_1$, hence $K(\mathcal{B}) = \omega_1$.

For any $\sigma$-complete boolean algebra $\mathcal{B}$ the Sikorski–Loomis theorem [25, p. 93] says that $\mathcal{B}$ is isomorphic to a $\sigma$-field of subsets of some $X$ modulo a $\sigma$-ideal of subsets of $X$.

Theorem 12 (Kunen). $\forall \alpha \leq \omega_1 \exists X, R$ with $R \subseteq P(X)$ such that $K(R) = \alpha$.

Proof. By the Sikorski–Loomis theorem and Theorem 9 we can find $\hat{R}, X, \text{ and } I$ with $\hat{R} \subseteq P(X)/I$ where $I$ is a $\sigma$-ideal and $\alpha$ is the least ordinal such that $\hat{R}_\alpha = \hat{R}_\alpha$. Define $R \subseteq P(X)$ by $(A \in R \text{ iff } A/I \in \hat{R})$. It is easily shown by induction on $\beta \leq \omega_1$ that $(A \in R_\beta \text{ iff } A/I \in \hat{R}_\beta)$. Hence we have $K(R) = \alpha$.

Let $\mathcal{B}_M$ be the complete boolean algebra Borel($2^\omega$) modulo the ideal of meager sets.

Theorem 13. For any $\alpha$, $1 \leq \alpha < \omega_1$, there is a countable $C \subseteq \mathcal{B}_M$ which is closed under finite conjunction and complementation so that $\alpha$ is the least ordinal such that $\Sigma_\alpha(C) = \mathcal{B}_M$.

Proof. Let $x \in \omega^\omega$ be arbitrary and $\mathcal{B}$ be the complete boolean algebra associated with $P_\alpha(\emptyset, \{x\})$. Note that if $|p| (\emptyset) = 0$, then $-|p| = \sum \{|q|: |q| (\emptyset) = 0 \text{ and } q \text{ is incompatible with } p\}$. Let $C$ be the closure of $\{|p|: |p| (\emptyset) = 0\} = \hat{C}$ under finite boolean combinations. Note that since $\hat{C}$ is closed under finite intersections and
On the length of Borel hierarchies

$-[p]$ is in $\Sigma_1(\hat{C})$ for any $p$ in $\hat{C}$, we have that $\Sigma_\beta(C) = \Sigma_\beta(\hat{C})$ for all $\beta \geq 1$. By Lemma 10 $\alpha$ is the least such that $\Sigma_\alpha(\hat{C}) = B$. Since $\mathbb{P}_\alpha(\emptyset, \{x\})$ is countable and separative, $B$ is separable and nonatomic and hence isomorphic to $BM$.

**Remark.** The theorem above is false for $\alpha = \omega_1$ since for any countable $C$ which generates $BM$, at some countable stage every clopen set is generated and after one more step all of $BM$.

### 3. Countably generated Borel hierarchies

A set $X \subseteq 2^\omega$ is called a Luzin set iff $X$ is uncountable and for every meager $M$, $M \cap X$ is countable. The analogous definition with measure zero in place of meager is of a Sierpinski set [30]. For $I$ a $\sigma$-ideal in $\text{Borel}(2^\omega)$ say $X$ is $I$-Luzin iff $[\forall A \in \text{Borel}(2^\omega)] (|A \cap X| < 2^\kappa$, iff $A \in I)$. The following theorem was first proved by Luzin [12] assuming $I$ is the ideal of meager sets and CH.

**Theorem 14.** (MA). If $I$ is an $\omega_1$ saturated $\sigma$-ideal in $\text{Borel}(2^\omega)$ containing singletons, then there exists an $I$-Luzin set.

**Proof.** Let $\kappa = |2^\omega|$, $\{A_\alpha : \alpha < \kappa\} = I$, and $\{B_\alpha : \alpha < \kappa\} = \text{Borel}(2^\omega) - I$ each set repeated $\kappa$-many times. Choose $x_\alpha$ for $\alpha < \kappa$, so that for every $\alpha$ $x_\alpha$ is in $B_\alpha - (\bigcup \{A_\beta : \beta < \alpha\} \cup \{x_\beta : \beta < \alpha\})$. Clearly if this can be done, then $X = \{x_\alpha : \alpha < \kappa\}$ is $I$-Luzin. If $\kappa = \omega_1$, then it is trivial, and if MA, then this follows from [14, Lemma 1, p. 158].

The next theorem was proved by Poprougenko [19] and Sierpinski (see [29]).

**Theorem 15.** If $X \subseteq 2^\omega$ is a Luzin set, then $\text{ord}(X) = 3$.

**Proof.** Since every Borel set $B$ has the property of Baire, $B = G \Delta M$ where $G$ is open and $M$ is meager. But $M \cap X = F$ is countable hence $F_{\alpha_n}$ so $B \cap X = (G \Delta F) \cap X$ showing $\text{ord}(X) \leq 3$. Now choose $s \in 2^{<\omega}$ so that $[s] \cap X$ is uncountable and dense in $[s]$. If $D \subseteq [s] \cap X$ is countable and dense in $[s]$, then $D \neq G \cap X$ for all $G \in G_\kappa$, so $\text{ord}(X) \geq 3$.

A modern example of a Luzin set arises when one adds an uncountable (in $M$) number of product generic Cohen reals $X$ to $M$ a countable transitive model of ZFC. $M[X] \models " X$ is a Luzin set". See also Kunen [10] for more on Luzin sets and MA.

In contrast to the boolean algebras Szpilrajn [29] showed:

**Theorem 16.** If $X \subseteq 2^\omega$ is a Sierpinski set, then $\text{ord}(X) = 2$. 
Proof. The proof is similar except note that any measurable set is the union of an $F_\sigma$ set and a set of measure zero.

The following theorem generalizes these classical results using a lemma of Silver (see [14, p. 162]) that assuming MA every $X \subseteq 2^\omega$ with $|X| < |2^\omega|$ is a $Q$ set, i.e. every subset of $X$ is an $F_\sigma$ in $X$.

**Theorem 17.** (MA). There are uncountable $X, Y \subseteq 2^\omega$ such that ord $(X) = 3$ and ord $(Y) = 2$.

**Proof.** Let $X$ be $I$-Luzin where $I$ is the ideal of meager Borel sets. For any meager set $M$ choose $F$ a meager $F_\sigma$ with $M \subseteq F$. By Silver's Lemma there exists $F_0$ an $F_\sigma$ set such that $F_0 \cap F \cap X = M \cap F \cap X = M \cap X$. Thus every meager set intersected with $X$ is an $F_\sigma$ set intersected with $X$ and this shows as before ord $(X) = 3$. For $I$ the ideal of measure zero sets analogous arguments work.

After $I$ had shown that it is consistent with ZFC that $\forall \alpha \leq \omega_1 \exists X \subseteq \omega^\omega$ ord $(X) = \alpha$, Kunen showed that in fact CH implies $\forall \alpha \leq \omega_1 \exists X \subseteq \omega^\omega$ ord $(X) = \alpha$. The following theorem sharpens his result slightly.

**Theorem 18.** If there exists a Luzin set, then for any $\alpha$ such that $2 < \alpha \leq \omega_1$ there is an $X \subseteq 2^\omega$ such that ord $(X) = \alpha$.

**Proof.** Let $Y$ be a Luzin set with the property that for every Borel set $A \subseteq 2^\omega$ ($A \cap Y$ is countable iff $A$ is meager). Such a set always exists if a Luzin set does. By Theorem 13 there is a $C \subseteq B_M$ countable such that $C$ generates $B_M$ in exactly $\alpha$ steps and $C$ is closed under finite Boolean combinations. Let $C = \{[C_n] : n \in \omega\}$ where the $C_n$ are Borel subsets of $2^\omega$ and $[C_n]$ is the equivalence class modulo meager of $C_n$. For $x, y \in 2^\omega$ define $x \sim y$ iff for all $n < \omega$ ($x \in C_n$ iff $y \in C_n$). We claim that for each $x \in 2^\omega$ the $\sim$ equivalence class $[x]$ is meager. Note that any element of the $\sigma$-algebra generated by $\{C_n : n < \omega\}$ is a union of $\sim$ equivalence classes. If some $\sim$ equivalence class $E$ is not meager, then there are $K_0$ and $K_1$ disjoint nonmeager Borel sets such that $E = K_0 \cup K_1$. Since $\{[C_n] : n < \omega\}$ generates $B_M$ there are $L_0$ and $L_1$ in the $\sigma$-algebra generated by $\{C_n : n < \omega\}$ such that $[L_0] = [K_0]$ and $[L_1] = [K_1]$. For some $i, L_i$ is disjoint from $E$, but then $L_i$ is meager, contradiction. By shrinking $Y$ if necessary we may assume that for all $x, y \in Y$ ($x = y$ iff $x \sim y$). Let $R = \{C_n \cap Y : n < \omega\}$, then $R$ contains every countable subset of $Y$. It is easily seen that $K(R) = \alpha$, so by Proposition 2, we are done.

**Theorem 19.** (MA). For any $\alpha < \omega_1$ there is an $X \subseteq \omega^\omega$ such that $\alpha \leq \text{ord } (X) \leq \alpha + 2$.

**Proof.** For $\alpha < \omega_1$ let $P_\alpha$ be the partial order $P_\alpha(\emptyset, \omega^\omega)$. Let $T_\alpha$ be the normal
tree of rank $\alpha$ used in the definition of $\mathbb{P}_\alpha$. $T^*_\alpha = \{ s \in T_\alpha : |s|_{T_\alpha} = 0 \}$. If $G$ is $\mathbb{P}_\alpha$-generic, then $G$ is completely determined by the real $y_G : T^*_\alpha \to \omega^{\omega \omega}$ defined by $y_G(s) = t$ iff $(s, t) \in G$. Each condition $p \in \mathbb{P}_\alpha$ can be thought of as a statement about $y_G$. Let $C_p = \{ y \in \omega^{\omega} : y$ codes a map $\hat{y} : T^*_\alpha \to \omega^{\omega \omega}$ and $p(\hat{y})$ is true $\}$. It is easily seen that for any $p \in \mathbb{P}_\alpha$ there is $\beta < \alpha$ such that $C_p$ is $\Pi^0_\beta$.

**Lemma 20.** If $B_\alpha$ is the complete boolean algebra associated with $\mathbb{P}_\alpha$ and $X_\alpha$ is $\omega^{\omega}$ with the topology generated by basic open sets $\{ C_\alpha : p \in \mathbb{P}_\alpha \}$, then $B_\alpha$ is isomorphic to the boolean algebra of regular open subsets of $X_\alpha$.

**Proof.** Given $A \subseteq X_\alpha$ a regular open set let $D_A = \{ p \in \mathbb{P}_\alpha : C_\alpha \cap A \neq \emptyset \}$. The map $A \to D_A$ is an isomorphism.

Define $I_\alpha$ to the $\sigma$-ideal generated by $\Pi^0_\beta$ sets of the form $\omega^{\omega \omega} - \bigcup \{ C_\alpha : p \in D \}$ where $D$ is a maximal antichain in $\mathbb{P}_\alpha$.

**Lemma 21.** $\alpha$ is the least ordinal such that for every Borel $A$ there is a $\Sigma^0_\beta$ $B$ such that $A \Delta B \in I_\alpha$.

**Proof.** Note first that $I_\alpha$ is the ideal of meager subsets of $X_\alpha$. If $D$ is a maximal antichain in $\mathbb{P}_\alpha$, then $\bigcup \{ C_\alpha : p \in D \}$ is open dense in $X_\alpha$, so every element of $I_\alpha$ is meager in $X_\alpha$. If $C$ is closed nowhere dense in $X_\alpha$, then let $Q = \{ p \in \mathbb{P} : C_\alpha \cap C = \emptyset \}$. Since $Q$ is open dense in $\mathbb{P}_\alpha$, we can pick $D \subseteq Q$ a maximal antichain. Thus $C \subseteq \omega^{\omega \omega} - \bigcup \{ C_\alpha : p \in D \}$ and every meager subset of $X_\alpha$ is in $I_\alpha$.

Since $A$ is Borel in $X_\alpha$ there is a regular open set $B$ in $X_\alpha$ such that $(A \Delta B) \in I_\alpha$. Let $Q = \{ p \in \mathbb{P}_\alpha : C_\alpha \cap B \neq \emptyset \}$. Pick $D \subseteq Q$ an antichain which is maximal with respect to being contained in $Q$. Since $B$ is regular open, $B = \bigcup \{ C_\alpha : p \in D \}$, so $B$ is $\Sigma^0_\beta$ in $\omega^{\omega \omega}$. To see that $\alpha$ is minimal note that for $s \in T_\alpha$ with $|s|_{T_\alpha} = \beta$ there is no $B \subseteq C$ in $\omega^{\omega \omega}$ with $(C_{s,s} \Delta B) \in I_\alpha$.

Now let $X \subseteq \omega^{\omega \omega}$ be $I_\alpha$-Luzin. Then ord $(X) \geq \alpha$ since for any $A$ and $B$ Borel in $\omega^{\omega \omega}$ ($(A \Delta B) \in I_\alpha$ iff $|(A \Delta B) \cap X| < |X|$). But ord $(X) \leq \alpha + 2$ follows from the fact that for all $B$ in $I_\alpha$ there exists $C$ in $I_\alpha \cap \Sigma^0_{\alpha+1}$ with $B \subseteq C$, just as in the proof of Theorem 17. This concludes the proof of Theorem 19.

**Remarks.** (1) If $V = L$, then using the $\Delta^1_2$ well-ordering of $L \cap 2^{\omega}$ we can get $X \subseteq 2^{\omega}$ a $\Delta^1_2$ set with ord $(X) = \alpha$ for any $\alpha \leq \omega_1$. If $X$ is $\Pi^1_\alpha$ (or $\Sigma^1_\alpha$), then $X = A \Delta M$ where $A$ is $\Pi^0_\alpha$ and $M \in I_\alpha$, so $X$ cannot be $I_\alpha$-Luzin.

(2) A finer index can be given to a set $X \subseteq \omega^{\omega}$ by considering the classical Hausdorff difference hierarchies. A set $C \subseteq \omega^{\omega}$ is a $\beta - \Pi^0_\alpha$ set iff there exists $D_\gamma \in \Pi^0_\alpha$ for $\gamma < \beta$ such that the $D_\gamma$'s are decreasing and $D_\lambda = \bigcup_{\gamma < \lambda} D_\gamma$ for $\lambda$ limit and $C = \bigcup \{ D_\gamma : \gamma < \beta$ and $\gamma$ even $\}$. It is a theorem of Hausdorff that $\Delta^0_{\alpha+1} = \bigcup \{ \beta - \Pi^0_\alpha : \beta < \omega_1 \}$ (see [11, pp. 417, 448]). It is also not hard to show,
using a universal set argument, that there exists a properly $\beta - \Pi^0_\omega$ set for all $\alpha, \beta < \omega_1$. Accordingly define $H(X)$ to be the lexicographical least pair $(\alpha, \beta) \in \omega^2$ such that for any Borel set $A$ there exists $B$ a $\beta - \Pi^0_\omega$ set such that $A \cap X = B \cap X$. If $X$ is a Luzin set (Sierpinski set), then $H(X) = (2, 2)$ ($H(X) = (2, 1)$). It is easily shown that in Theorem 22 $N \vDash "H(X_{\alpha+1}) = (\alpha + 1, 1)"$. It is not hard to see that for $C$ a countable closed set $H(C) = (1, \alpha)$ where $\alpha$ is the Cantor-Bendixson rank of $C$.

**Theorem 22.** It is relatively consistent with ZFC that for any uncountable $X \subseteq 2^\omega$ ord $(X) = \omega_1$. This can be generalized to show that for any successor ordinal $\beta_0$ such that $2 \leq \beta_0 < \omega_1$, it is consistent that

$$\{\beta : \exists X \subseteq 2^\omega \text{ uncountable ord } (X) = \beta\} = \{\beta : \beta_0 \leq \beta \leq \omega_1\}.$$ 

**Remark.** It is true in the model obtained that for any uncountable separable metric space $X$ the Borel hierarchy on $X$ has length $\omega_1$. This is true, since if $|X| = \omega_1$, then since $|2^\omega| \geq \omega_2$ and $X$ can be embedded into $\mathbb{R}^\omega$, $X$ must be zero dimensional. But any zero dimensional space can be embedded into $2^\omega$.

To prove Theorem 22 let $M$ be a countable transitive model of ZFC+GCH. Choose $(\alpha_1 : \lambda < \omega_2)_M$ so that for all $\beta < \omega_1$ $(\lambda : \alpha_1 = \beta)$ is unbounded in $\omega_2$.

\begin{itemize}
    \item Define $P^\gamma$ for $\gamma < \omega_2$ by induction $P^0 = P_{\alpha_1}(\phi, 2^\omega \cap M)$, $P^{\gamma+1} = P^\gamma * Q^\gamma$ where $Q^\gamma$ is a term in the forcing language of $P^\gamma$ denoting $P_{\alpha}(\emptyset, M[G_\gamma] \cap 2^\omega)$ for any $G_\gamma$ $P^\gamma$-generic over $M$ and at limits take the direct limit.
    \item Call $p \in P^\gamma$ nice if it has the following properties for all $\gamma < \beta$.
        \begin{enumerate}
            \item $p(\gamma)$ is a canonical name for $p^* \cup \{(s, \tau) : s \in F\}$ where $p^*$ is a function from some finite subset of $\{s \in T_\alpha : |s| = 0\}$, $F$ is some finite subset of $\{s \in T_\alpha : |s| > 0\}$, and each $\tau$ is forced with value one to be an element of $2^\omega$.
            \item For each $(s, \tau) \in p(\gamma)$ $\exists t \in 2^{<\omega}$ such that $p \upharpoonright \gamma \vDash "t \subseteq \tau"$ and if $(s, \tau), (s \setminus n, \tau')$ are in $p(\gamma)$ (or $(s \setminus n, t) \in p^*), then t$ and $t'(t)$ are incompatible.
        \end{enumerate}
    \item It is not hard to see by induction on $\beta$ that the nice $p$ are dense. For the rest of the proof we assume all $p$ are nice.
\end{itemize}

For $Q \subseteq P$ and $\theta$ a sentence we say that $Q$ decides $\theta$ iff $\{p \in Q : there is a q \in Q such that p \vDash q$ and $(q \vDash "\theta"$ or $q \vDash "\neg \theta")\}$ is dense in $Q$. For any $H \subseteq 2^\omega$ define $|p| (H)$ and $|\tau| (H, p)$ for $p \in Q^\gamma$ and $\tau$ a $Q^\gamma$ term for an element of $2^\omega$ by induction on $\gamma$.

\begin{itemize}
    \item For $p \in Q^0 = P_{\alpha_1}(\emptyset, 2^\omega \cap M)$ define
        $$|p| (H) = \max \{|s|_{T_{\alpha_1}} : \exists x \in 2^\omega - H (s, x) \in p\}.$$ 
    \item For $p \in Q^{\gamma+1}$ define
        $$|p| (H) = \max \{|p \upharpoonright \gamma| (H), |\tau| (H, p \upharpoonright \gamma) : (s, \tau) \in p(\gamma)\}.$$ 
\end{itemize}
(3) For $p \in \mathbb{P}^\alpha$ define

$$|p|(H) = \sup \{|p \upharpoonright \gamma| : \gamma < \lambda\}.$$ 

(4) Define $|\tau|(H, p)$ is the least $\beta$ such that for any $n \in \omega \{q \in \mathbb{P}^\gamma : q$ incompatible with $p$ or $|q|(H) \leq \beta\}$ decides "$\tau(n) = 0$"

$\mathbb{P}^\omega := \mathbb{P}$ is not a lattice, however, it does have one similar property:

**Lemma 23.** Suppose $G$ is $\mathbb{P}^\alpha$-generic over $M$ and for $i < n < \omega$ $q_i \in G$ and $|q_i|(H) < \beta$, then there is a $q \in G$ with $|q|(H) < \beta$ and $q \geq q_i$ for all $i < n$.

**Proof.** The proof is by induction on $\alpha$. For $\alpha = 0$ or a $\alpha$ a limit it is easy. So suppose $\alpha = \beta + 1$ and $G_\beta \times G^\beta$ where $G_\beta$ is $\mathbb{P}^\beta$-generic over $M$. Find $\Gamma \subseteq G_\beta$ finite so that for any $q \in \Gamma$ with $|q|(H) < \beta$ and for any $i$ and $j$ less than $n$ if $(s, \tau) \in q_i(\beta)$ and $(s - k, \tilde{\tau}) \in q_j(\beta)$ (or $(s - k, t) \in q_i(\beta)$ where $t \in 2^\omega$), then there is $r \in \Gamma$ such that $r \vdash \tau \neq \tilde{\tau}$. By induction there is $q$ in $G_\beta$ such that $|q|(H) < \beta$, for all $\tilde{q} \in \Gamma$ $q \geq \tilde{q}$, and for all $i < n$ $q \geq q_i \upharpoonright \beta$. Define $q(\beta)$ to be equal to $\bigcup \{q_i(\beta) : i < n\}$.

**Lemma 24.** Given $P_0$ a countable subset of $\mathbb{P}^\alpha$ and $Q_0$ a countable set of $\mathbb{P}^\alpha$ terms for elements of $2^\omega$, there exists $H$ countable such that for every $p \in P_0$ and $\tau \in Q_0$ $|p|(H) = |\tau|(H, \emptyset) = 0$.

**Proof.** This is easy using c.c.c. of $\mathbb{P}^\alpha$.

Let $|p| = p(H)$ and $|\tau|(p) = |\tau|(H, p)$, for some fixed $H$.

**Lemma 25.** For each $p \in \mathbb{P}^\alpha$ and $\beta$ there exists $\hat{p} \in \mathbb{P}^\alpha$ compatible with $p$, $|\hat{p}| < \beta + 1$, and for every $q \in \mathbb{P}^\alpha$ with $|q| < \beta$, if $\hat{p}$ and $q$ are compatible, then $p$ and $q$ are compatible.

**Proof.** The proof is by induction on $\alpha$. For $\alpha = 0$ this is just Lemma 8 of Section 1. For $\alpha$ limit it is easy. From now on assume the lemma is true for $\alpha$.

Define for $x, y \in 2^\omega$, $x$ is lexicographically less than $y$ iff

$$\exists n \forall m < n \ (x(m) = y(m) \text{ and } x(n) < y(n)).$$

This is the lexicographical order. For $C \subseteq 2^\omega$ a nonempty closed set let $x_C$ be the lexicographically least element of $C$.

**Claim 1.** Let $\hat{C}$ be a term in $\mathbb{P}^\alpha$ and $p_0 \in \mathbb{P}^\alpha$ with $|p_0| < \beta + 1$ such that $p_0 \upharpoonright \\hat{C}$ is a nonempty closed subset of $2^\omega$'. Suppose for every $G \mathbb{P}^\alpha$-generic with $p_0 \in G$, and
Proof. First we show that given any $p \in \mathbb{P}^\omega$ with $p \geq p_0$, if $s \in 2^{\omega_0}$, $p \Vdash \langle[s] \cap \dot{C} \neq \emptyset \rangle$, then there exist $\dot{p} \in \mathbb{P}^\omega$ compatible with $p$, $|\dot{p}| < \beta + 1$, and $\dot{p} \Vdash \langle[s] \cap \dot{C} \neq \emptyset \rangle$. Let $p'$ be as from Lemma 25 for $p$. By using Lemma 23 obtain $\dot{p}$ compatible with $p$, $\dot{p} \geq p'$, $\dot{p} \geq p_0$, and $|\dot{p}| < \beta + 1$. I claim $\dot{p} \Vdash \langle[s] \cap \dot{C} \neq \emptyset \rangle$. Suppose not then there exists $G \in \mathbb{P}^\omega$-generic, $\beta \in G$, and $M[G] \not\Vdash \langle[s] \cap \dot{C} = \emptyset \rangle$. So there exists $q \in G$, $|q| < \beta$, and $q \Vdash \langle[s] \cap \dot{C} = \emptyset \rangle$. But then since $q$ is compatible with $\dot{p}$ it is compatible with $p'$ and hence with $p$, contradiction. In order to show $|x_\alpha| (p_0) < \beta + 1$ it suffices to show for every $p \geq p_0$ and $n \in \omega$ there exist $\dot{p} \in \mathbb{P}^\omega$ compatible with $p$, $|\dot{p}| < \beta + 1$, and there exists $s \in 2^n$ such that $\dot{p} \Vdash \langle x_\alpha \upharpoonright n = s \rangle$. So given $p$ and $n$ find $r \geq p$ and $s \in 2^n$ such that $r \Vdash \langle x_\alpha \upharpoonright n = s \rangle$. We have just shown there exists $\dot{r}$ compatible with $r$ with $|\dot{r}| < \beta + 1$ and $\dot{r} \Vdash \langle[s] \cap \dot{C} = \emptyset \rangle$. Let $G$ be $\mathbb{P}^\omega$-generic containing $r$ and $\dot{r}$. For each $t \in 2^{m+1}$ with $m + 1 \leq n$ and for all $k < m$ $(t(k) = s(k))$ and $t(m) < s(m)$, choose $q, \in G$ with $|q| < \beta$ and $q \Vdash \langle[s] \cap \dot{C} = \emptyset \rangle$. (There are only finitely many such $t$). Choose $q \in G$ with $|q| < \beta + 1$, $q \geq \dot{r}$, and $q \geq q_i$ for each such $t$ ($q$ exists by Lemma 23). Then $q \Vdash \langle x_\alpha \upharpoonright n = s \rangle$.

For $p$ and $q$ compatible define $p \cup q \Vdash \langle\theta \rangle$ to mean that for every $r$, if $r \geq p$ and $r \geq q$, then $r \Vdash \langle\theta \rangle$. For $\tau$ a $\mathbb{P}^\omega$ term for an element of $2^\omega$ and $p \in \mathbb{P}^\omega$, define $C(\tau, p)$ a $\mathbb{P}^\omega$ term so that for any $G$ which is $\mathbb{P}^\omega$-generic (it need not contain $p$) $C^G(\tau, p) = \bigcap \{D_\tau : \text{there exist } q \in G, |q| < \beta, |\dot{\tau}| (q) < \beta, q \Vdash \langle\dot{\tau} \in 2^\omega \rangle, p \text{ and } \alpha \text{ are compatible, and } p \cup q \Vdash \langle\tau \in D_\tau \rangle\}$. $D$ is a universal $\Pi_1^0$ subset of $2^\omega \times 2^\omega$ ($\forall K \in \mathbb{P}^\omega_1 \exists x \in 2^\omega \exists K = D = \{y : (x, y) \in D\}$).

Claim 2. Let $\dot{p}$ be given by Lemma 25 for $p \in \mathbb{P}^\omega$ (i.e. for all $q \in \mathbb{P}^\omega$ if $|q| < \beta$, then if $q$ and $\dot{p}$ are compatible, then $q$ and $p$ are compatible). Then $\dot{p}$ and $C(\tau, p)$ satisfy the hypothesis of Claim 1 for $p_0$ and $\dot{C}$.

Proof. Suppose $M[G] \Vdash \langle[s] \cap C(\tau, p) = \emptyset \rangle$. By compactness there exists $n < \omega$, $q_i \in G$, $\tau_i$ for $i < n$ with $|q_i| < \beta$ and $\tau_i (q_i) < \beta$ so that $p \cup q_i \Vdash \langle\tau_i \in D_{\tau_i} \rangle$ and $M[G] \Vdash \langle \bigcap \{D_{\tau_i} : i < n\} \cap [s] = \emptyset \rangle$. Let $\dot{\tau}$ be a term for an element of $2^\omega$ so that $D_{\tau} = \bigcap \{D_{\tau_i} : i < n\}$ and $q \in G$ with $q \geq q_i$ for $i < n$ and $|q| < \beta$. ($\dot{\tau}$ can be chosen so that $|\dot{\tau}| (q) < \beta$ assuming some nice properties of $D$). Since $q$ and $\dot{p}$ are compatible, $q$ and $\dot{p}$ are compatible and $q \cup \dot{p} \Vdash \langle\tau \in D_{\tau}'\rangle$. Since $M[G] \Vdash \langle D_{\tau}' \cap [s] = \emptyset \rangle$ by compactness there exists $m \in \omega$ so that if $t = \dot{\tau}' \upharpoonright m$ then for every $x \geq t$, $x \in 2^\omega$, $D_{\tau} \cap [s] = \emptyset$. Since $|\dot{\tau}| (q) < \beta$ there exists $\dot{q} \equiv q$ an element of $G$, $|\dot{q}| < \beta$, and $\dot{q} \Vdash \langle\dot{\tau} \upharpoonright m = t \rangle$; hence $\dot{q} \Vdash \langle[s] \cap C(\tau, p) = \emptyset \rangle$. The fact that $\dot{p} \Vdash \langle C(\tau, p) \neq \emptyset \rangle$ follows from this since if not there exists $q$ compatible with $\dot{p}$, $|q| < \beta$, and $q \Vdash \langle[s] \cap C(\tau, p) = \emptyset \rangle$. But then $q$ is compatible with $p$, contradiction.

We now return to the proof of the $\alpha + 1$ step of Lemma 25.

Assume $p \in \mathbb{P}^\omega$ is nice. Let $(s_i, \tau_i)$ for $i < n$ be all $(s, \tau) \in p(\alpha)$ with $|s| \geq 1$ and
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Let $\tau = (\tau_0, \tau_1, \ldots, \tau_{n-1})$ (where $(\ldots, \ldots)$: $(2^\omega)^n \rightarrow 2^\omega$ is some recursive coding). Let $\hat{\beta} \upharpoonright_\alpha$ be as given from Lemma 25 for $p \upharpoonright_\alpha$. Let $\tilde{\tau}'$ be the lexicographical least element of $C(\tilde{\tau}, p \upharpoonright_\alpha)$. By Claim 1 and 2 $|\tilde{\tau}'| (\hat{\beta} \upharpoonright_\alpha) < \beta + 1$. Now let

$$\hat{\beta}(\alpha) = \{(s, t) \in p(\alpha) : |s| = 0\} \cup \{(s, \tau_i) : i < n\}$$

($\tilde{\tau}' = (\tau_0, \ldots, \tau_{n-1})$). Since $\emptyset \models "C(\tilde{\tau}, p_\alpha)"$ is included in $\prod_{n<\alpha} [s_n]^\omega$, $\hat{\beta}$ is a condition, $\hat{\beta}$ and $p$ are compatible, also $|\hat{\beta}| < \beta + 1$. Now suppose $q \in \mathbb{P}^{\alpha+1}$ compatible with $\hat{\beta}, |q| < \beta$, and $q$ and $p$ are not compatible. Let $G$ be $\mathbb{P}^\omega$-generic with $\hat{\beta} \upharpoonright_\alpha$ and $q \upharpoonright_\alpha$ elements of $G$ and $M[G] \models "\hat{\beta}(\alpha)$ and $q(\alpha)$ are compatible". If we think of $p(\alpha)$ as a statement about $\tilde{\tau}$ i.e. $p(\alpha)(\tilde{\tau})$, then $\hat{\beta}(\alpha) = p(\alpha)(\tilde{\tau}')$. Since $p$ and $q$ are incompatible but $p_\alpha$ and $q_\alpha$ are compatible ($p \upharpoonright_\alpha \cup q \upharpoonright_\alpha \models "p(\alpha)$ and $q(\alpha)$ are incompatible". $D(\tilde{\tau}) \equiv "p(\alpha)(\tilde{\tau})$ and $q(\alpha)$ are incompatible" is a $\Pi^0_1$ statement with parameters from $q(\alpha)$ about $\tilde{\tau}$. Thus we conclude that $M[G] \models "p(\alpha)(\tilde{\tau}')$ and $q(\alpha)$ are incompatible", contradiction. This concludes the proof of Lemma 25.

From now on let $\mathbb{P} = \mathbb{P}^\omega$.

**Lemma 26.** Suppose $|\tau| = 0$, $B(\nu)$ is a $\Sigma^0_3$ predicate, $\beta \geq 1$, with parameters from $M$, and $p \in \mathbb{P}$ is such that $p \models "B(\tau)"$; then there exists $q \in \mathbb{P}$ compatible with $p$, $|q| (H) < \beta$ and $q \models "B(\tau)"$.

**Proof.** The proof is by induction on $\beta$.

**Case 1.** $\beta = 1$.

Suppose $p \models "\exists n R(x \mid n, \tau \mid n)"$ for $R$ recursive and $x \in M$. Let $G$ be $\mathbb{P}$-generic with $p \in G$. Choose $n \in \omega$ and $s \in 2^n$ so that $M[G] \models "R(\mid n, \tau \mid n)$ and $\tau \mid n = s"$.

Choose $q \in G$ with $|q| = 0$ and $q \models \tau \mid n = s$.

**Case 2.** $\beta$ is a limit ordinal.

If $p \models "\exists n B(n, \nu)"$, then $\exists \check{\nu} \models p \hat{\nu} \models "B(n_0, \tau)"$ and $B(n_0, \nu) \Sigma^0_\gamma$ for $\gamma < \beta$, so apply induction hypothesis to $\hat{\nu}$.

**Case 3.** $\beta + 1$.

Suppose $p \models "\exists n B(n, \tau)"$ where $B(n, \nu)$ is $\Pi^0_\beta$ with parameters from $M$. Choose $r \models p$ and $n_0 \in \omega$ so that $r \models "B(n_0, \tau)"$. By Lemma 25 there is $q$ compatible with $r, |q| < \beta + 1$, and for every $s, |s| < \beta$, if $q$ and $s$ are compatible, then $r$ and $s$ are compatible. $q \models "B(n_0, \tau)"$ because if not, then there is $q' \models q$ such that $q' \models "B(n_0, \tau)"$, and so by induction there is $s$ with $|s| < \beta$ compatible with $q'$ and $s \models "B(n_0, \tau)"$; but then $s$ is compatible with $r$, contradiction.

Now let us prove the first part of Theorem 22. Let $G$ be $\mathbb{P}$-generic over $M$. We claim $M[G] \models "\text{for every } X \subseteq 2^\omega \text{ and } \alpha < \omega_1 \text{ if } |X| = \omega_1, \text{ then } \text{ord}(X) \models \alpha + 1"$. But since any such $X$ is in some $M[G_\beta]$ for $\beta < \omega_2$, we may as well assume $X \in M$, $\alpha_0 = \alpha + 1$, and we must show $M[G] \models "\text{ord}(X) \models \alpha + 1"$. Let $G_{\alpha_0}$ be the $\Pi^0_\alpha$ set created by $G \cap \mathbb{P}_{\alpha_0}(\emptyset, 2^\omega \cap M)$. Suppose that $M[G] \models "\text{there is } K \text{ a } \Sigma^0_\alpha \text{ set such that}"$.
Let $\tau$ be a term for the parameter of $K$. Choose $p \in G$ such that $p \Vdash \forall \alpha \in \omega_1 \exists z \in X (z \in K \iff z \in G)$. By Lemma 24 there exists $H$ in $M$ countable so that $|\tau| (H, \emptyset) = |p| (H) = 0$. Let $z \in X - H$. Define $\hat{p} \in \mathcal{P}$ by $\hat{p} (0) = p (0) \cup \{((0), z)\}$ and $\hat{p} (\alpha) = p (\alpha)$ for $\alpha > 0$. Since $\hat{p}$ says $z \in G (\alpha)$, $\hat{p} \Vdash \forall \alpha \in \omega_1 \exists z \in X (z \in K \iff \alpha > 0 \iff z \in K)$. By Lemma 26 there exists $q$ compatible with $\hat{p}$, $|q| (H) < \beta$, and $q \Vdash \forall \alpha \in \omega_1 \exists z \in K$. By Lemma 23 there exists $\hat{q}$ with $|\hat{q}| (H) < \beta$, $\hat{q} \geq q$, and $\hat{q} \geq p$. Since $|(0)| = \alpha$, $((0), z) \notin \hat{q} (0)$, there exists $m \in \omega$ such that $r$ defined by $r (0) = q (0) \cup \{(m, z)\}$ and $r(\alpha) = \hat{q} (\alpha)$ for $\alpha > 0$ is a condition. But this is a contradiction since $r \Vdash \forall \alpha \in \omega_1 \exists z \in X (z \in K)$ and $z \in K$ and $z \notin G (\alpha)$.

Now we prove the second sentence of Theorem 22. Let $X = \bigcup \{X_\alpha : \beta_0 \leq \alpha < \omega_1 \text{ and } \alpha \text{ a successor}\}$ where each $X_\alpha$ is a set of $\omega_1$ product generic Cohen reals. Let $M_0 = M [X]$. Define in $M_0$ the partial order $\mathcal{P}_\gamma$ for $\gamma \leq \omega_2$ so that $\mathcal{P}_\gamma + 1 = \mathcal{P}_\gamma \times \mathcal{Q}_\gamma$, where $\mathcal{Q}_\gamma$ is a term denoting:

**Case 1.** $\mathcal{P}_\beta (0, M_0 [G_\gamma] \cap 2^\omega)$ or

**Case 2.** $\mathcal{P}_\beta (Y_\gamma, X_\beta \cup F)$ where $Y_\gamma$ is a Borel subset of $X_\beta$ in $M_0 [G_\gamma]$ and $F = \{x \in 2^\omega : x \text{ eventually zero}\}$.

Case 1 is done cofinally in $\omega_2$ and Case 2 is done in such a way as to insure: $M_0 [G_\omega] \Vdash \forall \beta_0 \leq \beta < \omega_1 \text{ and } Y \text{ Borel in } X_\beta \text{ there is a } \gamma \text{ such that } Y = Y_\gamma$. First we show that essentially the same arguments as before show that $M_0 [G_\omega] \Vdash \forall X \subseteq 2^\omega \text{ uncountable ord } (X) \equiv \beta_0$. This will not use that the $X_\alpha$ are made up of Cohen reals, hence any of the intermediate models would serve as the ground model. So suppose Case 1 occurs on the first step, $Y \in M_0$ is uncountable, $\beta_0 = \gamma + 1$, and $M_0 [G_\omega] \Vdash \forall X \subseteq 2^\omega \text{ uncountable ord } (X) \equiv \beta_0$. This will not use that the $X_\alpha$ are made up of Cohen reals, hence any of the intermediate models would serve as the ground model. So suppose Case 1 occurs on the first step, $Y \in M_0$ is uncountable, $\beta_0 = \gamma + 1$, and $M_0 [G_\omega] \Vdash \forall X \subseteq 2^\omega \text{ uncountable ord } (X) \equiv \beta_0$. This will not use that the $X_\alpha$ are made up of Cohen reals, hence any of the intermediate models would serve as the ground model. So suppose Case 1 occurs on the first step, $Y \in M_0$ is uncountable, $\beta_0 = \gamma + 1$, and $M_0 [G_\omega] \Vdash \forall X \subseteq 2^\omega \text{ uncountable ord } (X) \equiv \beta_0$.

For $\alpha \in L$:

**Case 1.** $\mathcal{P}_L^{\alpha + 1} = \mathcal{P}_L^\alpha \ast \mathcal{P}_\beta (\emptyset, M [G_\alpha] \cap 2^\omega)$ where $G_\alpha$ is $\mathcal{P}_L^\alpha$-generic over $M_0$.

**Case 2.** $\mathcal{P}_L^{\alpha + 1} = \mathcal{P}_L^\alpha \ast \mathcal{P}_\beta (Y_\alpha, X_\beta \cup F)$ (where we assume $L$ has the property that when Case 2 happens for $\alpha \in L$ then $Y_\alpha$ is a Borel subset of $X_\beta$ coded by some term $\tau_\alpha$ in $\mathcal{P}_L^\alpha$).

For $\alpha \notin L$:

$$\mathcal{P}_L^{\alpha + 1} = \mathcal{P}_L^\alpha \ast \mathcal{P}_\gamma$$

(singleton partial order).

Note that by using c.c.c. of $\mathcal{P}_\gamma$: we can find $L \subseteq \omega_2$ countable, so that the Borel code for the above $J$ is a $\mathcal{P}_L^\alpha$ term and $L$ has the property mentioned under Case 2. For $\alpha$ a limit $\mathcal{P}_L^\alpha$ is the direct limit of $(\mathcal{P}_L^\beta : \beta < \alpha)$.

**Lemma 27.** If $N \supseteq M$ is a model of ZFC and $G$ is $\mathcal{P}_\beta (\emptyset, N \cap 2^\omega)$ generic over $N$, then $G \cap \mathcal{P}_\beta (\emptyset, M \cap 2^\omega)$ is $\mathcal{P}_\beta (\emptyset, M \cap 2^\omega)$ generic over $M$.

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1 I would like to thank the referee for suggesting this proof of Lemma 27 and thus eliminating the need for Lemma 28. A similar argument is utilized by J. Truss, "Sets having calibre $\mathcal{N}_1$", in: Logic Colloquium 76, Studies in Logic, Vol. 87 (North-Holland, Amsterdam, 1977).
Proof. It is sufficient to show that if $A \in M$ and $A$ is a maximal antichain in $\mathbb{P}_\beta(0, M \cap 2^\omega)$ (where $\beta < \omega^M$), then $A$ is also a maximal antichain in $\mathbb{P}_\beta(0, N \cap 2^\omega)$ for any $N \supseteq M$ which is a transitive model of ZFC. But by c.c.c. (in $M$), $A$ is countable in $M$, so this result is immediate by absoluteness of $\Pi^1_1$ predicates.

Given any $G \mathbb{P}^\omega$-generic let $G_L$ be the subset of $\mathbb{P}_L$ generated by the rank zero conditions in $G$. The preceding lemma enables us to prove:

Lemma 29. For any $\alpha$ if $G_\alpha$ is $\mathbb{P}_\alpha$-generic over $M_\alpha$, then $G_{\alpha+1}$ is $\mathbb{P}_{\alpha+1}$-generic over $M_\alpha$.

Proof. This is proved by induction on $\alpha$. For $\alpha + 1 \notin L$ it is immediate. For $\alpha + 1 \in L$ Case 1 is handled by Lemma 27 and the product lemma. Case 2 is easy as $\mathbb{P}_\beta(Y_\alpha - F, X_\beta \cup F)$ is the same partial order in either case. For $\alpha$ limit ordinal let $\Delta \subseteq \mathbb{P}_\beta$ be dense, we show $\{q \in \mathbb{P}_\alpha : p \leq q\} \subseteq \mathbb{P}_\alpha$. If $q \in \mathbb{P}_\alpha$, then $q \in \mathbb{P}_\beta$ for some $\beta < \alpha$. Let $\Delta_\beta = \{p \uparrow \beta : p \in \Delta\}$, then $\Delta_\beta$ is dense in $\mathbb{P}_\beta$. Hence if $G_\alpha$ is $\mathbb{P}_\alpha$-generic with $q \in G_\alpha$, then since $G_\beta$ is $\mathbb{P}_\beta$-generic it meets $\Delta_\beta$—say at $p \uparrow \beta$. But then $q$ and $p$ are compatible.

Define for $H \subseteq 2^\omega \upharpoonright \alpha \upharpoonright (H), |\tau| (H, p)$ for $p \in \mathbb{P}_\alpha$ and $\tau$ a $\mathbb{P}_\alpha$-term for a subset of $\omega$ by induction on $\alpha$.

Case 1. $\mathbb{P}_\alpha^{\alpha+1} = \mathbb{P}_\alpha \ast \mathbb{P}_\beta(\emptyset, M[G_\beta^\alpha] \cap 2^\omega)$.

$$|p\upharpoonright (H) = \max \{|p \uparrow \gamma| (H), |p(\gamma)| (H, p \uparrow \gamma)| \} \quad \text{(same as before).}$$

Case 2. $\mathbb{P}_\alpha^{\alpha+1} = \mathbb{P}_\alpha \ast \mathbb{P}_\beta(Y_\alpha - F, X_\alpha \cup F)$.

$$|p\upharpoonright (H) = \max \{|p \uparrow \alpha| (H), |s|_{\tau_\alpha} : x \notin H \upharpoonright (s, x) \in p(\alpha)| \}.$$
Lemma 30. For any $\alpha \leq \beta$ if $G^n$ is $P^n$-generic over $M_\alpha$, then $G^n_H$ is $P^n_H$-generic over $M[H]$.

Proof. The proof is like Lemma 29 except on $\alpha + 1$ under Case 2. $P_1 = P_1(Y_\alpha - F, X_\beta \cup F)$ in $M[X ||G^n||] = M_1$, $P_2 = P_2((Y_\alpha - F) \cap H, (X_\beta \cap H) \cup F)$ in $M[H][G^n_H] = M_2$. Again suppose $\Delta \in M_2$ is dense in $P_2$, we show $\{p \in P_1 : \exists q \in \Delta, q \Vdash \varphi\}$ is dense in $P_1$. Given $p \in P_1$, let $p = r \cup \{(s_n, x_n) : n < N\}$ where $x_n \in X_\alpha - H$, $N < \omega$, and $r \in P_2$. Let $Q_n$ be the partial order for adding $N$ Cohen reals. By the product lemma $\{x_n : n < N\}$ is $Q_n$-generic over $M_2$, and also $p \in P_2\{x_n : n < N\}$. Hence if $\forall q \in \Delta p$ and $q$ are incompatible in

\[ P_3 = P_\beta ((Y_\alpha - F) \cap (H \cup \{x_n : n < N\}), (X_\beta \cap (H \cup \{x_n : n < N\})) \cup F), \]

then $\exists \beta \in Q_n \beta \Vdash \forall q \in \Delta p$ and $q$ are incompatible in $P_3$. Choose $y_n \in F$ for $n < N$ so that $p_n = r \cup \{(s_n, y_n) : n < N\} \in P_2$ and $\forall m < \omega \exists \beta' \supseteq \beta \forall n < N \exists \beta' \beta \Vdash y_n \Vdash \gamma \Vdash x_n \Vdash m"$. Since $\exists q \in \Delta p_0$ and $q$ are compatible, then as before $p$ and $q$ can be forced compatible by an extension of $\beta$. So $p$ and $q$ are compatible in $P_3$ and hence in $P_1$.

Lemma 31. Given $\hat{\varphi}$ a term in forcing language of $P^n_H$ if $p \in P^n$ $p \Vdash \varphi(n) \rightarrow \varphi(n)$ where $B(\varphi)$ is a $\Sigma^1_1$ predicate with parameters in $M[H]$, then $\exists q \in P^n_H$ compatible with $p$ such that $q \Vdash \varphi(n) \rightarrow \varphi(n)$.

Proof. Let $G$ be $P^n$-generic over $M_\alpha$ with $p \in G$. Then by Lemma 9 $G^n_H$ is $P^n_H$-generic over $M[H]$. Since $\Sigma^1_1$ sentences are absolute and $M_\alpha[G] \Vdash \varphi<\beta$ we have $M[H][G_H] \Vdash \varphi < \beta$. So $\exists q \in G_H q \Vdash P^n_H \varphi < \beta$. But for any $G$ $P^n$-generic containing $q$, $M[H] \Vdash \varphi < \beta$ whence by absoluteness $M_\alpha[G] \Vdash \varphi < \beta$. We conclude $q \Vdash \varphi < \beta$.

Lemma 32. Given $H = X - \{z\}$ where $z \in X_\alpha \cup \gamma$, $\gamma \leq \beta$, $1 \leq \beta < \alpha$, $p \in P^n$, then $\exists \hat{\varphi} \in P^n, \hat{\varphi} \Vdash (M[H] \cap 2^\omega) < \beta + 1, \hat{\varphi}$ compatible with $p$, and $\forall q \in P^n$ if $|q| (M[H] \cap 2^\omega) < \beta$, then $(\hat{\varphi}, q$ compatible $\Rightarrow p, q$ compatible).

Proof. This is proved by induction on $\gamma$. For $\gamma$ limit it is easy, also for $\gamma + 1$ in which Case 1 occurs the proof is the same as Lemma 25. So we only have to do $\gamma + 1$ in Case 2.

$p \in P^n \exists \hat{\varphi}_1 (Y_\alpha - F, X_\beta \cup F)$. Extend $p(\gamma)$ if necessary so that $\forall (s, x) \in p(\gamma) \forall i < \omega$ if $|s| = \lambda$ infinite limit $|s - i| < \beta + 1 < \lambda$, then $\exists j < \omega \langle s - i - j, x \rangle \in p(\gamma)$. Let $\hat{\varphi}(\gamma) = \langle (s, x) \in p(\gamma) : |s| < \beta + 1 \text{ or } x \neq z \rangle$. If $\hat{\varphi}(\gamma) \Vdash (M[H] \cap 2^\omega) < \beta + 1$, then $\hat{\varphi}(\gamma)$ were a condition, then just as in Lemma 8, $\hat{\varphi}$ would have the required properties. To be a condition we need to know that whenever $\langle \langle n \rangle, x \rangle \in \hat{\varphi}(\gamma) \hat{\varphi} \Vdash "x \notin (Y_\alpha - F)"$.

Note that none of these $x$'s are equal to $z$ because $z \in X_{\alpha + 1}$ so $\langle \langle n \rangle, z \rangle \in p(\gamma) \rightarrow \langle \langle n \rangle \rangle = \alpha \geq \beta + 1$ so $\langle \langle n \rangle, z \rangle \notin \hat{\varphi}(\gamma)$. Let $G$ be $P^n$-generic containing $p \nvdash \gamma$, and $\hat{\varphi} \nvdash \gamma$. By Lemma 31 $\exists q \in P^n_H \cap G$ (so $|q| (M[H] \cap 2^\omega) = 0$) such that $\forall x \forall n$ if
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\[ \langle n, x \rangle \in \hat{p}(\gamma), \text{ then } q \Vdash \lnot "x \not\in Y_\gamma - F". \] By Lemma 23, \( \exists p_0 \equiv q, \hat{p} \upharpoonright \gamma \) so that \( |p_0| (M[H] \cap 2^\omega) < \beta + 1 \). So \( \langle p_0, \hat{p}(\gamma) \rangle \) works.

Immediate from Lemma 32 we get that: If \( J \) is any \( \Sigma^{\omega+1}_\alpha \) predicate with parameters \((H = X - \{z\}, z \in X_{\alpha+1}, \) and \( \tau \) is in the forcing language of \( \mathcal{P}(\tau) \), then \( \forall p \in \mathcal{P} \) if \( p \Vdash \lnot "z \in J" \), then \( \exists q \in \mathcal{P} [q] (M[H] \cap 2^\omega) < \beta, q \) and \( p \) are compatible, and \( q \Vdash \lnot "z \in J" \). So we get our result \( \text{ord} (X_{\alpha+1}) = \alpha + 1 \) in \( M_\alpha[G_{\omega_1}] \).

Remark. Assuming large amounts of the axiom of determinacy and therefore getting more absoluteness in inner models (see [7]) it is easy to produce an inner model \( N \) such that \( N \Vdash \) "For every \( \alpha < \omega_1 \), there exist \( X \subseteq 2^\omega \) such that \( \text{ord} (X) = \alpha \) and for every \( n < \omega \) and \( A \mathbin{\Pi}_n, A \cap X \) is Borel in \( X \)." Similar improvements for Theorem 43 are possible.

4. The \( \sigma \)-algebra generated by the abstract rectangles

For any cardinal \( \lambda \) let \( R^\lambda = \{A \times B : A, B \subseteq \lambda\} \). If \( R^\alpha_\alpha \) (the \( \sigma \)-algebra generated by \( R^\alpha \)) is the set of all subsets of \( \lambda \times \lambda \), then \( \lambda \leq |2^\omega| \) (see [9, 21]).

**Theorem 33.** If \( \alpha_0 < \omega_1 \) and there is an \( X \subseteq \omega^\omega \) such that \( |X| = \kappa \leq \omega \) and every subset of \( X \) of cardinality less than \( \kappa \) is \( \Pi^\omega_\omega \) in \( X \), then \( R^\alpha_\alpha = P(\kappa \times \kappa) \). The same is true if every subset of \( X \) of cardinality less than \( \kappa \) is \( \Sigma^\omega_\alpha \) in \( X \).

**Proof.** Consider \( A \subseteq \kappa \times \kappa \) and suppose \( (\alpha, \beta) \in A \) implies \( \alpha \leq \beta \). It is enough to show such sets are in \( R^\alpha_\alpha \), since every subset of \( \kappa \times \kappa \) can be written as the union of a set above the diagonal and a set below the diagonal. Let \( T \) be a normal \( \alpha_0 \) tree and \( T^* = \{s \in T : |s|_T = 0\} \). For any \( y : T^* \rightarrow \omega^\omega \) define \( G^y_\beta \) as follows. If \( s \in T^* \), then \( G^y_\beta = [y(s)] \), otherwise \( G^y_\beta = \bigcap \{\omega^\omega - G^{\omega - n}_\gamma : n < \omega\} \). Let \( X = \{x_\alpha : \alpha < \kappa \} \) and for each \( \beta < \kappa \) choose \( \beta \) so that for all \( \alpha \) \( ((\alpha, \beta) \in A \) iff \( x_\alpha \in G^\alpha_\beta \)). For \( s \in T \) define \( B_s \subseteq \kappa \times \kappa \) as follows. If \( s \in T^* \), then \( B_s = \bigcup \{\alpha : t \subseteq x_\alpha \times \beta : y_\beta (s) = t \} : t \in \omega^\omega \} \), otherwise \( B_s = \bigcap \{\kappa \times \kappa - B_{s - n} : n < \omega\} \). Clearly \( B_0 = A \) and \( B_\alpha \) is \( "\Pi^\omega_\alpha \) in \( R^\alpha \), and so every subset of \( \kappa \times \kappa \) is \( "\Pi^\omega_\alpha \) in \( R^\alpha \). Note that \( (\kappa \times \kappa) - (A \times B) = ((\kappa - A) \times \kappa) \cup (\kappa \times (\kappa - B)) \) and thus if \( \alpha_0 \) is even (odd), then \( R^\alpha_\alpha \) is the class of sets \( "\Pi^\omega_\alpha \) (\( "\Sigma^\omega_\alpha \) in \( R^\alpha \). By passing to complements if necessary we have that \( R^\alpha_\alpha = P(\kappa \times \kappa) \). The second sentence of the theorem is proved similarly.

**Corollary** (Kunen [9]; Rao [21]). If there is an \( X \subseteq \omega^\omega \) such that \( |X| = \omega_1 \), then \( R^\omega_\alpha = P(\omega_1 \times \omega_1) \).

The converse of this corollary is also true. Suppose \( R \subseteq P(\omega_1) \) is a countable
field of sets and \( \{ (\alpha, \beta) : \alpha < \beta < \omega_1 \} \in \{ A \times B : A, B \in R \}_{\omega_1} \). Since this set is antisymmetric we conclude that the map given in Proposition 2 is a 1-1 embedding of \( \omega_1 \) into \( 2^\omega \).

**Corollary** (Kunen [9]; Silver). (MA). If \( \kappa = \vert 2^\omega \vert \), then \( R_\kappa = P(\kappa \times \kappa) \).

**Proof.** If \( X \) is \( I \)-Luzin where \( I \) is the ideal of meager sets, then every subset of \( X \) of smaller cardinality is \( \Sigma^0_2 \) in \( X \) (see proof of Theorem 17).

For any \( \alpha \leq \omega_1 \), \( X \subseteq \omega^\omega \) is a \( Q_\alpha \) set iff \( \text{ord}(X) = \alpha \) and every subset of \( X \) is Borel in \( X \).

**Theorem 34.** If \( M \) is countable transitive model of ZFC, \( 1 \leq \alpha_0 < \omega_1^M \), and \( X = M \cap \omega^\omega \), then there is a Cohen extension \( M[G] \) such that \( M[G] \models \text{"} X \text{ is a } Q_{\alpha_0+1} \text{ set} \text{"} \).

**Remark.** This shows that the Baire order of the constructible reals can be any countable successor ordinal greater than one. In fact the argument shows that in \( M[G] \) for any uncountable \( Y \subseteq 2^\omega \) with \( Y \subseteq M \), \( Y \) is a \( Q_{\alpha_0+1} \) set. Thus, for example, if \( M \) models \( V = L \), then in \( M[G] \) there are \( \Pi_1^1 Q_{\alpha_0+1} \) sets. In Theorem 55 we show that it is consistent with ZFC that for every \( \alpha < \omega_1 \) there is a \( Q_\alpha \) set (in that model the continuum is \( \aleph_{\alpha_0+1} \)).

**The proof of Theorem 34.** \( M[G] \) is gotten by iterated \( \Pi^0_{\alpha_0+1} \)-forcing. Let \( \kappa = \vert 2^\omega \vert \). Suppose we are given \( P^\kappa \) for some \( \alpha < \kappa \) and \( Y_\alpha \) a term in the forcing language of \( P^\alpha \) for a subset of \( X \) \((\mathcal{B} \models \text{"} Y_\alpha \subseteq X \text{"})\), then let \( P^{\alpha+1} = P^\alpha * P_{\alpha+1}^\alpha (Y_\alpha, X) \). At limit ordinals take direct limits. \( P^\kappa \) may be viewed as a sub-lower lattice of \( \Sigma_k P_{\alpha_0+1}^\alpha (\emptyset, X) \). We may assume that for every set \( B \subseteq X \) in \( M[G] \) \((G \models \text{P}^\kappa \text{-generic over } M)\) there exists \( \alpha \) such that \( Y_\alpha = B \). This is because \( P^\kappa \) has c.c.c. It follows from Corollary \( \ell \) that \( M[G] \models \text{"} \text{ord}(X) \leq \alpha_0 + 1 \text{ and every subset of } X \text{ is Borel in } X \text{"} \).

We assume \( P^0 = P_{\alpha_0+1}(\emptyset, X) \). Let \( G_{(0)} \) be one of the \( \Pi^0_{\alpha_0} \) set determined by \( G \cap P^0 \). We want to show that \( M[G] \models \text{"} \text{For every } K \text{ in } \Sigma^0_{\alpha_0} \text{, } K \cap X \neq G_{(0)} \cap X \text{"} \). To this end we make the following definition: For \( H \subseteq \omega^\omega \), \( \vert p \vert (H) = \max \{ \vert s \vert : \text{there exists } x \notin H \text{ \( s, x \in p \) for some } \alpha < \kappa \} \). Let \( \text{supp}(p) = \{ \alpha < \kappa : p(\alpha) \neq \emptyset \} \). Given \( \tau \) a term in the forcing language of \( P^\kappa \) denoting a subset of \( \omega \), we can find \( H \) included in \( \omega^\omega \) and \( K \) included in \( \kappa \) with the following properties:

(a) \( H \) and \( K \) are countable;
(b) for each \( n \in \omega \) \( \{ p \in P^\kappa : \text{supp}(p) \subseteq K, \vert p \vert (H) = 0 \} \), decides "\( n \in \tau \);"
(c) \( \forall x \in H \forall \alpha \in K \{ p \in P^\kappa : \text{supp}(p) \subseteq K, \vert p \vert (H) = 0 \} \) decides "\( x \in Y_\alpha \)."

\( H \) and \( K \) can be found by repeatedly using the c.c.c. of \( P^\kappa \).
Lemma 35. If H and K have property (c), then for any $p \in \mathbb{P}$ and $\beta$ with $1 \leq \beta < \alpha_\omega$, there exists $\hat{p} \in \mathbb{P}$ compatible with $p$, $|\hat{p}|(H) < \beta + 1$, supp$(\hat{p}) \subseteq K$, and for any $q \in \mathbb{P}$ if $|q|(H) < \beta$ and supp$(q) \subseteq K$, then [if $\hat{p}$ and $q$ are compatible, then $p$ and $q$ are compatible].

Proof. The proof of this is like Lemma 8. Let $G$ be $\mathbb{P}$-generic over $M$ with $p \in G$. Choose $\Gamma \subseteq G$ finite with the properties:

1. $\forall q \in \Gamma$ ($|q|(H) = 0$ and supp$(q) \subseteq K$).
2. If $((n), x) \in p(\alpha)$ for some $n < \omega$, $\alpha \in K$, and $x \in H$ (so $p \models \langle \alpha \rangle \vdash \forall x \neg \exists y F(x, y)$), then there is $q \in \Gamma \cap \mathbb{P}$ such that $q \vdash \forall x \neg \exists y F(x, y)$.
3. If $(s, x) \in p(\alpha)$, $\alpha \in K$, and $|s| = \lambda$ is an infinite limit ordinal, and $|s^i| < \beta + 1 < \omega$, then there is a $j \in \omega$ such that $\langle (s^i, j), x \rangle \in p$.

Now let $\tilde{p} \in \mathbb{P}$ be defined by

$$\tilde{p}(\alpha) = \bigcup \{r(\alpha) : r \in \Gamma\} \cup \{(s, x) \in p(\alpha) : |s| < \beta + 1 \text{ or } x \in H\}$$

when $\alpha \in K$ and $\tilde{p}(\alpha) = \emptyset$ for $\alpha \notin K$. Note if $((n), x) \in \beta(\alpha)$, then $x \in H$ since $|\langle n \rangle| = \alpha_\omega \geq \beta + 1$. By choice of $\Gamma$, $\tilde{p}$ is a condition and also $|\tilde{p}|(H) < \beta + 1$ and is compatible with $p$ since $\hat{p}, p \in G$. It is easily checked as in Lemma 8 that $\tilde{p}$ has the required property.

Lemma 36. Let H and K have properties (b) and (c) for $\tau$. Let $B(v)$ be a $\Sigma^0_\beta$ ($1 \leq \beta < \alpha_\omega$) predicate with parameters from $M$ and $p \in \mathbb{P}$ such that $p \models "B(\tau)"$. Then there exists $q \in \mathbb{P}$ compatible with $p$, $|q|(H) < \beta$, $q \models "B(\tau)"$, and supp$(q) \subseteq K$.

Proof. The proof is by induction on $\beta$.

$\beta = 1$: $p \models "\exists n R(n, \tau \uparrow n, x \uparrow n)"$, $x \in M$, and $R$ primitive recursive. Let $G$ be $\mathbb{P}$-generic over $M$ with $p \in G$. There exist $n \in \omega$ and $s \in 2^\omega$ such that $M[G] \models "R(n, \tau \uparrow n, x \uparrow n)"$ and $\tau \uparrow n = s"$. By property (b) there exists $q \in G$ such that $q \models "\tau \uparrow n = s"$, supp$(q) \subseteq K$, and $|q|(H) = 0$. $q$ does it.

$\beta$ limit: $p \models "\exists n B^n_n(\tau)\", B^n_n \in \Sigma^0_\beta$, $\beta_n < \beta$. Choose $r \geq p$ such that $r \models "B^n_n(\tau)\"$ for some $n$. By induction there exist $q$ such that $q \models "B^n_n(\tau)\", q$ is compatible with $r$ (and hence with $p$), and $|q|(H) < \beta$, supp$(q) \subseteq K$. $q$ does it.

$\beta + 1$: If $p \models "\exists n B^n_n(\tau)\"$ we could extend $p$ to force $B^n_n(\tau)$ for some particular $n$. So we may as well assume $p \models "B(\tau)\"$ where $B(\tau)$ is $\Pi^0_\beta$ with parameter in $M$. Since $1 \leq \beta < \alpha_\omega$ by Lemma 35 there is $\hat{p}$ compatible with $p$, $|\hat{p}|(H) < \beta + 1$, etc. Then $\hat{p} \models "B(\tau)\"$ because otherwise there is $p_0 \models \neg B(\tau)$, and so by induction there is $q$ compatible with $p_0$ (hence with $p$) $|q|(H) < \beta$, supp$(q) \subseteq K$, and $q \models \neg B(\tau)$. By our assumption on $\hat{p}$, since $\hat{p}$ and $q$ are compatible, $p$ and $q$ are compatible, but $p \models "B(\tau)\"$.
We now use Lemma 36 to show that for any $G \mathbb{P}^\leq$-generic over $M$, $M[G] \vDash \text{"For every } L, \text{ a } \Sigma^0_\alpha \text{ set with parameter } \tau, \text{ and } p \in G \text{ such that } p \Vdash \text{"for every } x \in X, x \in L \iff x \in G(\tau),\text{"}. Choose } H \text{ and } K \text{ with properties (a), (b), and (c) with respect to } \tau \text{ and also so that supp}(p) \subseteq K \text{ and } |p|(H) = 0. \text{ Since } H \text{ is countable there exists } x \in X - H. \text{ Let } r = p \cup \{(0, ((0), x))\} \text{ (so } r \Vdash x \in G(\tau)). \text{ Since } r \Vdash \text{"}x \in L\text{"}, \text{ by Lemma 36 there exists } q \text{ compatible with } r, |q|(H) < \alpha_0, \text{ and } q \Vdash \text{"}x \in L\text{"}. \text{ Since } |q|(H) < \alpha_0, (0, (0), x) \notin q(0). \text{ Let } \hat{q} \text{ be defined by:}

$$
\hat{q}(\alpha) = \begin{cases} 
p(\alpha) \cup q(\alpha) & \text{if } \alpha > 0, 
n(0) \cup q(0) \cup \{(0, m), x\} & \text{otherwise} (m \text{ sufficiently large so that } \hat{q}(0) \text{ is condition}).
\end{cases}
$$

$\hat{q} \Vdash \text{"}x \in L \text{ and } x \notin G(\tau) \text{ and } (x \in L \iff x \in G(\tau))\text{"}. \text{ This a contradiction and concludes the proof of Theorem 34.}$

**Theorem 37.** For any $\alpha_0$ a successor ordinal such that $2 \leq \alpha_0 < \omega_1$, it is relatively consistent with ZFC that $|2^n| = \omega_2$ and $\alpha_0$ is the least ordinal such that $R^\omega_{\alpha_0} = P(\omega_2 \times \omega_2)$.

**Remark.** In Theorem 52 we remove the restriction that $\alpha_0$ is a successor (but the continuum in that model is $\aleph_\omega$). In [1] it is shown that $\alpha_0$ cannot be $\omega_1$.

**Proof.** Let $M$ be a countable transitive model of $\text{"ZFC} + |2^n| = |2|_\omega^\omega = \omega_2\text{"}$. Let $X = \omega_\omega \cap M$ and define $\mathbb{P}_\alpha$ for $\alpha \leq \omega_2$ so that $\mathbb{P}_\alpha \mathbb{P}^{\leq \alpha} = \mathbb{P}_\alpha \mathbb{P}_\alpha(\alpha_\alpha, X)$ where $\alpha_\alpha$ is a $\mathbb{P}^\alpha$ term for a subset of $X$, and at limits take the direct limit. Dovetail so that in $M[G_{\alpha_0}]$ for every $Y \subseteq X$ such that $|Y| \leq \omega_1$ there are $\omega_2$ many $\alpha < \omega_2$ such that $\alpha_\alpha = Y$. By Theorem 33 $R^\omega_{\alpha_0} = P(\omega_2 \times \omega_2)$.

Now comes the difficulty: we must show some subset of $\omega_2 \times \omega_2$ is not in $R^\omega_{\alpha_0-1}$. For the remainder of the proof let $(A_s : s \in \omega^{<\omega})$ and $(B_s : s \in \omega^{<\omega})$ be fixed terms in the forcing language of $\mathbb{P}^\omega$, such that for every $s \in \omega^{<\omega}$ $\emptyset \vDash \text{"}A_s \subseteq X \text{ and } B_s \subseteq \omega_2\text{"}$. For $p \in \mathbb{P}^\omega$ define $\text{supp}(p) = \{\alpha < \omega_2 : p(\alpha) \neq \emptyset\}$ and $\text{trace}(p) = \{x \in X : \exists \alpha \exists t, (t, x) \in p(\alpha)\}$. By using the c.c.c. of $\mathbb{P}^\omega$ choose for each $x \in X$ countable sets $I_x \subseteq X$ and $J_x \subseteq \omega_2$ so that:

(1) for each $s \in \omega^{<\omega}$ $\{p \in \mathbb{P}^\omega : \text{trace}(p) \subseteq I_x \text{ and supp}(p) \subseteq J_x\} \text{ decides } \text{"}x \in A_s\text{"}$, and

(2) for each $y \in I_x$ and $\alpha \in J_x$ $\{p \in \mathbb{P}^\omega : \text{trace}(p) \subseteq I_x \text{ and supp}(p) \subseteq J_x\} \text{ decides } \text{"}y \in A_\alpha\text{"}$.

Similarly for $\alpha < \omega_2$ we can pick countable sets $I_\alpha \subseteq X$ and $J_\alpha \subseteq \omega_2$ having properties (1) and (2) with $A_s, B_s, I_x, I_\alpha$ in place of $x, A_s, I_x, I_\alpha$.

For $x \in X$ and $\alpha < \omega_2$ let $L(x, \alpha) = (I_x \times J_x) \cup (I_\alpha \times J_\alpha)$ and define for $p \in \mathbb{P}^\omega$,

$$|p|(x, \alpha) = \max \{|s|_{\tau_\alpha} : (s, u) \in p(\gamma) \text{ and } (u, \gamma) \notin L(x, \alpha)\}.$$
Lemma 38. Fix $x \in X$ and $\alpha < \omega_2$ and let $|p| = |p| (x, \alpha)$. For any $\beta \geq 1$ and $p \in \mathbb{P}^{\omega_2}$, there is a $\hat{p} \in \mathbb{P}^{\omega_2}$ with $|\hat{p}| < \beta + 1$, $\hat{p}$ compatible with $p$, and for any $q \in \mathbb{P}^{\omega_2}$. if $|q| < \beta$ and $\hat{p}$ and $q$ are compatible, then $p$ and $q$ are compatible.

Proof. The proof of this is like that of Lemma 35. Let $p_0 \geq p$ so that if $(s, x) \in p(\gamma)$ with $|s| = \lambda$ a limit ordinal greater than $\beta$ and $|s\setminus i| < \beta + 1$, then there is $j < \omega$ so that $(s\setminus i \setminus j, x) \in p_0(\gamma)$. Let $G$ be $\mathbb{P}^{\omega_2}$-generic with $p_0 \in G$. Choose $\Gamma \subseteq G$ finite so that if $(n, u) \in p_0(\gamma)$ (so $p_0 \models \langle u \notin Y_\gamma \rangle$) and $(u, \gamma) \in L(x, \alpha)$, then there is a $q \in \Gamma$ such that $q \models \langle u \notin Y_\gamma \rangle$. Define $\hat{p}$ by

$$\hat{p}(\gamma) = \bigcup \{q(\gamma) : q \in \Gamma \} \cup \{(s, u) \in p_0(\gamma) : |s| < \beta + 1 \text{ or } (u, \gamma) \in L(x, \alpha)\}.$$

For any well-founded tree $\hat{T}$ define $C_s(\hat{T})$ for $s \in \hat{T}$ as follows. If $|s|_{\hat{T}} = 0$, then $C_s(\hat{T}) = A_s \times B_s$, otherwise

$$C_s(\hat{T}) = \bigcup \{(X \times \omega_2) \setminus C_{s \setminus \iota}(\hat{T}) : i < \omega\}.$$

Lemma 39. If $x \in X$, $\alpha \in \omega_2$, $\hat{T} \in M$ is a well-founded tree, $s \in \hat{T}$ with $|s|_{\hat{T}} = \beta$ where $1 \leq \beta \leq \alpha_0 - 1$, and $p \in \mathbb{P}^{\omega_2}$ such that $p \models \langle (x, \alpha) \notin C_s(\hat{T}) \rangle$, then there exist $q$ compatible with $p$, $|q| (x, \alpha) < \beta$, and $q \models \langle (x, \alpha) \notin C_s(\hat{T}) \rangle$.

Proof. The proof is by induction on $\beta$.

Case 1. $\beta = 1$: Suppose $p \models \langle (x, \alpha) \notin \bigcup_{i \in \omega} (A_{s \setminus \iota} \times B_{s \setminus \iota}) \rangle$.

So there exists $i_0 \in \omega$ and $\hat{p}$ and $\hat{q}$ elements of $\mathbb{P}^{\omega_2}$ so that $(p \cup \hat{p} \cup \hat{q}) \in \mathbb{P}^{\omega_2}$, and using (1) above,

$$(t, u) \in \hat{p}(\gamma) \rightarrow (u, \gamma) \in I_x \times J_x$$

and

$$(t, u) \in \hat{q}(\gamma) \rightarrow (u, \gamma) \in I_\alpha \times J_\alpha$$

and

$$\hat{p} \models \langle x \in A_{s \setminus i_0} \rangle, \quad \hat{q} \models \langle y \in B_{s \setminus i_0} \rangle.$$

So $\hat{p} \cup \hat{q} = q$ does the job.

Case 2. $\beta$ a limit ordinal: Suppose

$$p \models \langle (x, \alpha) \in \bigcup_{i \in \omega} C_{s \setminus \iota}(\hat{T}) \rangle$$

where $|s|_T = \beta$. Find $q \equiv p$ and $i_0 \in \omega$ such that $q \models "(x, y) \in C_{s - i_0}(\hat{T})"$. Let

$$T_0 = \{t \in \hat{T} : s - i_0 \leq t \text{ or } t \leq s - i_0\}.$$

Then

$$|s|_{T_0} = |s - i|_T + 1 < \beta, \quad \text{and} \quad C_s(T_0) = (X \times \omega_2) - C_{s - i_0}(T),$$

hence $q \models "(x, \alpha) \notin C_s(T_0)"$ where $|s|_{T_0} < \beta$; so by induction hypothesis there exists $r$ compatible with $q$ (and hence with $p$), $|r| (x, \alpha) < \beta$, and $r \models "(x, \alpha) \in C_{s - i_0}(T)"$. $r$ does the trick.

Case 3. $\beta + 1$: Since $\beta + 1 < \alpha_0$, let $q$ be as from Lemma 38.

Define $D \subseteq X \times \omega_2$ by $D = \{(x, \alpha) : x \in G^\alpha_{i_0}\}$ where $G^\alpha_{i_0}$ is one of the $\Pi^0_{\alpha - 1}$ sets created on the $\alpha$th step. $D$ is $\Pi^0_{\alpha - 1}$ in the rectangles on $X \times \omega_2$. We want to show it is not $\Sigma^0_{\alpha - 1}$ in the rectangles on $X \times \omega_2$ in $M[G^i_{i_0}]$.

Define: $(x, \alpha)$ is free (with respect to $(A_\alpha : s \in \omega^{-\omega})$, $(B_\alpha : s \in \omega^{-\omega})$) iff $x \notin I_\alpha$ and $\alpha \notin J_\alpha$.

**Lemma 40.** If $T \subseteq \omega^{-\omega}$ is well-founded and $T \in M$, $s \in T$ with $|s|_T = \alpha_0 - 1$, $(x, \alpha)$ is free, and $Y_\alpha = \emptyset$; then for every $p \in \mathcal{P}^{\omega^\omega}$ such that $|p| (x, \alpha) = 0$ it is not the case that $p \models "(x, \alpha) \in D \text{ iff } (x, \alpha) \notin C_s(T)"$.

**Proof.** Let $\hat{p} \equiv p$ by defining $\hat{p}(\gamma) = p(\gamma)$ for $\gamma \neq \alpha$ and $\hat{p}(\alpha) = p(\alpha) \cup \{((0), x)\}$. Then $\hat{p} \models "(x, \alpha) \in D"$ so by Lemma 39 there exists $q$ compatible with $\hat{p}$, $|q| (x, \alpha) < \alpha_0$, and $q \models "(x, \alpha) \notin C_s(T)"$. But $(x, \alpha)$ free implies that $(x, \alpha) \notin L(x, \alpha)$ so $q$ does not say "$x \in G^\alpha_{i_0}$". Thus for a sufficiently large $m < \omega$ $r$ defined by $r(\gamma) = p(\gamma) \cup q(\gamma)$ for $\gamma \neq \alpha$ and $r(\alpha) = p(\alpha) \cup q(\alpha) \cup \{((0, m), x)\}$ is a member of $\mathcal{P}^{\omega^\omega}$. But $r \models "(x, \alpha) \notin D \text{ and } (x, \alpha) \notin C_s(T)"$, a contradiction since $r$ extends $p$.

Since the terms $(A_\alpha : s \in \omega^{-\omega})$ and $(B_\alpha : s \in \omega^{-\omega})$ were arbitrary to start with it will complete the proof of the theorem to find lots of $(x, \alpha)$ free.

The next lemma generalized Kunen [9, p. 74].

**Lemma 41.** Given $|I_\alpha| < \kappa$ for $\alpha < \kappa^+$, there exists $G \subseteq \kappa^+$ with $|G| = \kappa^+$ and there is $S$ with $|S| = \kappa$ so that for any $\alpha, \beta \in G$ if $\alpha \neq \beta$, then $I_\alpha \cap I_\beta \subseteq S$.

**Proof.** We can assume $I_\alpha \subseteq \kappa^+$.

Define $\mu_\alpha, z_\alpha < \kappa^+$ for $\alpha < \kappa^+$ nondecreasing so that:

1. $\mu_\alpha = \sup \{\mu_\beta : \beta < \alpha\}$ for $\lambda$ limit;
2. $z_\alpha$'s are strictly increasing;
3. for $\alpha$ a successor and for distinct $\beta, \gamma < \alpha$ $I_\alpha \cap I_\gamma \subseteq \mu_\alpha$;
4. if $\mu_{\alpha + 1} > \mu_\alpha$, then for any $z > z_\alpha$ $\mu_\alpha \notin I_z \cup \{I_{z_\alpha} : \beta \leq \alpha\}$ and $\cup \{I_{z_\alpha} : \beta \leq \alpha\} \subseteq \mu_{\alpha + 1}$.
Let \( G = \{ z_\alpha : \alpha < \kappa^+ \} \) and \( S = \sup \{ \mu_\alpha : \alpha < \kappa^+ \} \). To see that \( S < \kappa^+ \) note that for any \( \alpha < \kappa^+ \) \( \| \beta : \mu_{\beta+1} > \mu_\beta \text{ and } \beta < \alpha \| < \kappa \). This is because \( I_{z_\alpha} \cap (\mu_{\beta+1} - \mu_\beta) \neq \emptyset \) for all \( \beta < \alpha \) such that \( \mu_{\beta+1} > \mu_\beta \).

**Lemma 42.** There exists \( \Sigma_0 \subseteq X \) \( \Sigma_1 \subseteq \omega_2 \) with \( |\Sigma_0| = |\Sigma_1| = \omega_2 \), for every \( \alpha \in \Sigma_1 \), \( Y_\alpha = \emptyset \), and for every \( (x, \alpha) \in \Sigma_0 \times \Sigma_1 \), \( (x, \alpha) \) is free.

**Proof.** By Lemma 41 there exists \( \hat{\Sigma}_0 \subseteq X \) \( \hat{\Sigma}_1 \subseteq \omega_2 \) with \( |\hat{\Sigma}_0| = \omega_2 \) and \( |\hat{\Sigma}_1| < \omega_2 \) so that for every distinct \( x, y \in \hat{\Sigma}_0 \) \( J_x \cap J_y \subseteq S \). Since \( \{ J_x - S : x \in \hat{\Sigma}_0 \} \) is a disjoint family, we can cut down \( \hat{\Sigma}_0 \) (maintaining \( |\hat{\Sigma}_0| = \omega_2 \)) and find \( \hat{\Sigma}_1 \subseteq \omega_2 \) so that \( |\hat{\Sigma}_1| = \omega_2 \), for every \( \alpha \in \hat{\Sigma}_1 \), \( Y_\alpha = \emptyset \), and for every \( x \in \hat{\Sigma}_0 \) \( J_x \subseteq \hat{\Sigma}_1 \). Applying Lemma 41 again find \( \hat{\Sigma}_1 \subseteq \hat{\Sigma}_1 \) with \( |\hat{\Sigma}_1| = \omega_2 \) and \( T \subseteq X \) with \( |T| < \omega_2 \) so that for every distinct \( \alpha, \beta \in \Sigma_1 \) \( I_\alpha \cap I_\beta \subseteq T \). Since \( \{ I_\alpha - T : \alpha \in \Sigma_1 \} \) are disjoint by cutting down \( \Sigma_1 \) (maintaining \( |\Sigma_1| = \omega_2 \)) we can assume \( \Sigma_0 \) defined to be equal to \( \hat{\Sigma}_0 - (T \cup \bigcup \{ I_\alpha : \alpha \in \Sigma_1 \}) \) has cardinality \( \omega_2 \). \( \Sigma_0 \) and \( \Sigma_1 \) do the job.

Lemma 42 finishes the proof of Theorem 37.

**Remark.** There is nothing special about \( \omega_2 \) in the above theorem; we could have replaced it by any larger cardinal \( \kappa \) with \( \kappa^\kappa = \kappa \).

Now we turn to a slightly different problem. For \( X \) a topological space a set \( A \subseteq X^n \) is projective iff it is in the smallest class containing the Borel sets (in the product topology on \( X^n \) for any \( m \in \omega \)) and closed under complementation and projection (\( B \subseteq X^n \) is the projection of \( C \subseteq X^{n+1} \) iff \( (y \in B \iff \exists x \in X x \hat{y} \in C) \)).

**Theorem 43.** If \( M \) is a countable transitive model of ZFC, then there exists \( N \) a c.c.c. Cohen extension of \( M \) such that if \( M \cap \omega^\omega = X \), then \( N \models " \text{Every projective set in } X \text{ is Borel and the Borel hierarchy of } X \text{ has } \omega_1 \text{ distinct levels } (\text{ord } (X) = \omega_1)". \)

This shows the relative consistency of an affirmative answer to a question of Ulam [31, p. 10]. Note that since \( X \times X \) is homeomorphic to \( X \) (take any recursive coding function), if for every \( B \subseteq X \times X \) Borel \( \{ x : \exists y(x, y) \in B \} \) is Borel in \( X \), then every projective set in \( X \) is Borel in \( X \).

**Proof.** The proof is slightly simpler if we assume that CH holds in \( M \). We give the proof in that case and then later indicate the necessary modifications. In any case \( |2^\omega|^M = |2^\omega|^N \).

Construct a sequence \( M = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_{\omega_1} = N \), by iterated forcing so that \( M_{\alpha+1} \) is obtained from \( M_\alpha \) by \( \Pi^\alpha_{\alpha+1} \)-forcing. On the \( \alpha \)th stage we are presented with a term \( \tau_\alpha \) in the forcing language of \( P^\alpha \) denoting a real. Then letting \( Y_\alpha \) be the projective set (over \( X \)) determined by \( \tau_\alpha \) we let \( P^{\alpha+1} = \text{proj}_{P^{\alpha+1}}(Y_\alpha, X) \). What is being done is that at stage \( \alpha \) we make \( Y_\alpha \) a \( \Pi^\alpha_{\alpha+1} \) set intersected with \( X \). The reason this will work is that after the \( \alpha \)th stage our forcing will not interfere.
with the Borel hierarchy on $X$ up to the $\alpha$th level. Since this is c.c.c. forcing we can imagine that each $X$-projective set in $N$ is eventually caught by some $\tau_\alpha$ for $\alpha < \omega_1$. So it is clear that $N \vdash \text{"Every X-projective set is Borel in X"}$, for any $N = M[G]$, where $G$ is $\mathbb{P}^\omega$-generic over $M$. Define for $H \subseteq X$ and $p \in \mathbb{P}$, $|p|\langle H \rangle = \max\{\{|s|_{\tau_\alpha} : \text{there exist } \alpha < \omega_1 \text{ and } x \notin H, (s,x) \in p(\alpha)\}\}$. Given $\tau$ a term in the forcing language of $\mathbb{P}^\gamma$ denoting a subset of $\omega$ ($\gamma < \omega_1$), there exists $H \subseteq X$ such that:

(a) $H$ is countable;
(b) $\forall n \in \omega, \{p \in \mathbb{P}^\gamma : |p|\langle H \rangle = 0\}$ decides "$n \in \tau$";
(c) $\forall \beta < \gamma$ and $x \in H$, $\{p \in \mathbb{P}^\gamma : |p|\langle H \rangle = 0\}$ decides "$x \in Y_\beta$".

**Lemma 44.** (Write $|p|\langle H \rangle = |p|\langle (H) \rangle$). "Exactly statement of Lemma 38" for $\mathbb{P}^\gamma$.

**Proof.** Extend $p \leq p_0$ as before. Let $G$ be $\mathbb{P}^\gamma$-generic with $p_0 \in G$. Choose $\Gamma \subseteq G$ finite so that:

1. $q \in \Gamma \rightarrow |q|\langle H \rangle = 0$;
2. if $(\langle n \rangle, x) \in p_0(\alpha)$ (so $p_0\langle \alpha \rangle \Vdash \text{"}x \notin Y_\alpha\text{"}$), then $\exists q \in \Gamma \cap \mathbb{P}^\alpha$ such that $q \Vdash \text{"}x \notin Y_\alpha\text{"}$.

Define $\hat{\rho}(\alpha) = \bigcup \{r(\alpha) : r \in \Gamma\} \cup \{(s,x) \in p_0(\alpha) : |s|_{\tau_\alpha} < \beta + 1 \text{ or } x \in H\}$.

$\hat{\rho}$ is a condition because if $(\langle n \rangle, x) \in p(\alpha)$ and $|\langle n \rangle|_{\tau_\alpha} < \beta + 1$, then $\hat{\rho}\langle \alpha \rangle \supseteq p\langle \alpha \rangle$ (so $\hat{\rho}\langle \alpha \rangle \Vdash \text{"}x \notin Y_\alpha\text{"}$ as required).

The $r \in \Gamma$ take care of such requirements about $x \in H$. The rest of the proof is the same.

**Lemma 45.** If $\tau, H, \gamma$ are as above, $B(\nu)$ is a $\Sigma^\gamma_\beta$ predicate for some $\beta \geq 1$ with parameter from $M$, and $p \in \mathbb{P}^\gamma$ such that $p \Vdash \text{"}B(\tau)\text{"}$, then there is a $q \in \mathbb{P}^\gamma$ compatible with $p$, $|q|\langle H \rangle < \beta$ and $q \Vdash \text{"}B(\tau)\text{"}$.

**Proof.** The proof is the same as before.

We can assume that for unboundedly many $\alpha < \omega_1$, $Y_\alpha = \emptyset$. Let $G_\alpha(G^{\alpha}_{(0)})$ be one of the $\Pi^\alpha_1$ sets determined by $G \cap P_{\alpha+1}(\emptyset, X)$ where $Y_\alpha = \emptyset$.

**Claim.** $M[G] \Vdash \text{"}for any } L \in \Sigma^\alpha_\alpha \text{ (} L \cap X \neq G_\alpha \cap X \text{)"}$.

**Proof.** Otherwise let $\tau$ be a term for a real in the forcing language $\mathbb{P}^\gamma$ for some $\gamma < \omega_1$ such that for some $L$ a $\Sigma^\alpha_\alpha$ set with parameter $\tau$ and some $p \in \mathbb{P}^\gamma$ $p \Vdash \text{"}L \cap X = G_\alpha \cap X\text{"}$. Choose $H$ with properties (a), (b), and (c) with respect to $\tau$, and also $|p|\langle (H) \rangle = 0$. Let $x \in X - H$. Define $r(\alpha) = p(\alpha) \cup \{(0), x\}$ and for $\beta \neq \alpha$ $r(\beta) = p(\beta)$. Note that $r \Vdash \text{"}x \in G_\alpha\text{"}$ hence $r \Vdash \text{"}x \in L\text{"}$. By Lemma 45 there exists $q \in \mathbb{P}^\gamma$ compatible with $r$, $|q|\langle (H) \rangle < \beta$, and $q \Vdash \text{"}x \in L\text{"}$. Since $x \notin H$ we know
Define \( \hat{q} \in \mathcal{P}^{\omega_1} \) by \( \hat{q}(\beta) = \hat{p}(\beta) \cup q(\beta) \) for \( \beta \neq \alpha \) and \( \hat{q}(\alpha) = \hat{p}(\alpha) \cup q(\alpha) \cup \{(0, n), x\} \) where \( n \) is picked sufficiently large so \( \hat{q}(\alpha) \) is a condition. But then \( \hat{q} \models \langle x \in L \text{ and } x \notin G_\alpha \rangle \) and \( (x \in L \iff x \in G_\alpha) \) and this is a contradiction. This concludes the proof of Theorem 43.

When the continuum hypothesis does not hold in \( M \) the construction of \( N \) still has \( \omega_1 \) steps but at each step we must take care of all reals in the ground model. That is \( \mathcal{P}^{\omega_1} = \mathcal{P}^\omega \ast Q_\alpha \) where \( Q_\alpha \) is a term denoting \( \sum \{ \mathcal{P}_{\alpha+1}(H_x, X) : x \in \omega^\omega \cap M[G_\alpha] \} \) for \( G \) \( \mathcal{P}^\omega \)-generic over \( M \). This works since all reals in \( N = M[G] \) for \( G \) \( \mathcal{P}^{\omega_1} \)-generic over \( M \) are caught at some countable stage.

Remark. It is easy to see that if \( V = L \) there is an \( X \subseteq \omega^\omega \) uncountable \( II_1 \) set such that \( X \in L \) and \( X \times X \) is homeomorphic to \( X \). Also by absoluteness it is possible to make sure that for every \( A \Sigma_1^1 \) in \( \omega^\omega \), \( A \cap X \) is Borel in \( X \). This family of sets includes those obtained by the Souslin operation from Borel sets in \( X \).

**Theorem 46.** (MA). \( \exists X \subseteq 2^\omega \text{ ord } (X) = \omega_1 \) and \( \forall A \in \Sigma_1^1 \in 2^\omega \exists B \text{ Borel(} 2^\omega \) \( A \cap X = B \cap X \).

**Proof.** Let \( \mathbb{B} \) be the c.c.c. countably generated boolean algebra of Theorem 9 with \( K(\mathbb{B}) = \omega_1 \). \( \mathbb{B} = \text{Borel}(2^\omega)/J \) for some \( J \) an \( \omega_1 \)-saturated \( \sigma \)-ideal in the Borel sets.

**Lemma 47.** If \( I \) is an \( \omega_1 \)-saturated \( \sigma \)-ideal in \( \text{Borel}(2^\omega) \), then \( B_I = \{ A \subseteq 2^\omega : \exists B \text{ Borel } \exists C \in I (A \Delta B) \subseteq C \} \) is closed under the Souslin operation.

For a proof the reader is referred to [11, p. 95].

By Theorem 14 MA implies there is \( X \subseteq 2^\omega \) a \( J \)-Luzin set. For any \( \alpha < \omega_1 \) there is \( A \Pi_1^\alpha \) so that for every \( B \Sigma_\alpha^\alpha \), \( (A \Delta B) \notin J \), hence \( |(A \Delta B) \cap X| = 2^\omega \), so \( A \cap X \neq B \cap X \), and thus \( \text{ord } (X) = \omega_1 \). If \( A \) is \( \Sigma_1^1 \), then by Lemma 47 there is \( B \) Borel and \( C \) in \( J \) with \( A \Delta B \subseteq C \). Since \( |C \cap X| < 2^\omega \) by MA \( \exists D \in \text{Borel}(2^\omega) (A \Delta B) \cap X = D \cap X \). So \( A \cap X = (B \Delta D) \cap X \).

This suggests the following question:

Can you have \( X \subseteq 2^\omega \) such that every subset of \( X \) is Borel in \( X \) and the Borel hierarchy on \( X \) has \( \omega_1 \) distinct levels? The answer is no.

**Theorem 48.** If \( X \subseteq 2^\omega \) and every subset of \( X \) is Borel in \( X \), then \( \text{ord } (X) < \omega_1 \).

**Proof.** Let \( X = \{ x_\alpha : \alpha < \kappa \} \) and \( X_\alpha = \{ x_\beta : \beta < \alpha \} \).

**Lemma 49.** If \( |X| \leq \kappa \). every subset of \( X \) is Borel in \( X \), and \( R^*_\omega = \mathcal{P}(\kappa \times \kappa) \), then \( \text{ord } (X) < \omega_1 \).
Proof. Since every rectangle in $X 	imes X$ is Borel in $X 	imes X$ and $R^x_{\omega_1} = P(\kappa \times \kappa)$, every subset of $X \times X$ is Borel in $X \times X$. Suppose for contradiction $\forall \alpha < \omega_1 \exists H_\alpha \subseteq X$ not $\Pi^0_\alpha$ in $X$. Let $H = \bigcup_{\alpha < \omega_1} \{x_\alpha\} \times H_\alpha$. For some $\alpha < \omega_1$, $H$ is $\Pi^0_\alpha$ in $X \times X$. But then every cross section of $H$ is $\Pi^0_\alpha$ in $X$ contradiction.

The proof of the theorem is by induction on $|X| = \kappa$.

For $\kappa = \omega_1$ it follows from Lemma 49 since $R^x_{\omega_1} = P(\omega_1 \times \omega_1)$.

For cof ($\kappa$) = $\omega_1$ it is trivial.

For cof ($\kappa$) > $\omega_1$: $\forall \alpha < \kappa$ choose $\beta_\alpha$ minimal $< \omega_1$ so that every subset of $X_\alpha$ is $\Pi^0_{\beta_\alpha}$ in $X$ (we can do this since $X_\alpha$ is $\Pi^0_\beta$ in $X$ some $\beta < \omega_1$). Since cof ($\kappa$) > $\omega_1$ there exists $\alpha_0 < \omega_1$ such that for a final segment of ordinal less than $\kappa$, $\beta_\alpha = \alpha_0$.

By Theorem 33 $R^x_{\omega_1} = P(\kappa \times \kappa)$ so by Lemma 49 ord ($X$) < $\omega_1$.

For cof ($\kappa$) = $\omega_1$: Let $\eta_\alpha \uparrow \kappa$ for $\alpha < \omega_1$ be an increasing continuous cofinal sequence.

Lemma 50. $\exists \beta_0 < \omega_1 \forall \alpha < \omega_1 X_\alpha$ is $\Pi^0_{\beta_0}$ in $X$.

Proof. If $G \subseteq \kappa \times \kappa$ is the graph of a partial function, then $G \in R^x_\omega$ (Rao [21]). This is because if $f: D \rightarrow \kappa$ where $D \subseteq \kappa$, then viewing $x \in$ irrational real numbers we have: $(f(\alpha) = \beta)$ iff $(\alpha \in D$ and $\forall r \in Q(r < x_{(\alpha)}$ iff $r < x_\beta))$ where $Q$ is the set of rational numbers.

Then $D = \{ (\alpha, \beta): \alpha < \omega_1 \land \beta < \eta_\alpha \}$ is the complement in $\omega_1 \times \kappa$ of a countable union of graphs of functions from $\kappa$ into $\omega_1$. Hence the set $\bigcup_{\alpha < \omega_1} \{x_\alpha\} \times X_\alpha$ is Borel in $X \times X$. Say it is $\Pi^0_{\beta_0}$. It follows that each $X_\alpha$ is $\Pi^0_{\beta_0}$.

For all $\lambda < \omega_1$ let $\beta(\lambda)$ be minimal so that every subset of $X_\lambda$ is $\Pi^0_{\beta(\lambda)}$ in $X$. If the hypothesis of Theorem 33 fails, then $\exists f: \omega_1 \rightarrow \omega_1$ increasing so that for all $\lambda < \omega_1$ $\beta(f(\lambda)) < \beta(f(\lambda + 1))$. So for all $\lambda < \omega_1$ there is some $H_\lambda \subseteq X_{\eta(\alpha_{\lambda} + 1)}$ which is not $\Pi^0_{\beta(f(\lambda))}$ in $X$. Since every subset of $X_{\eta(\alpha)}$ is $\Pi^0_{\beta(f(\lambda))}$ in $X$ we can assume $H_\lambda \subseteq (X_{\eta_{\alpha_{\lambda} + 1}} - X_{\eta_{\alpha_{\lambda}}})$. Let $H = \bigcup_{\lambda < \omega_1} H_\lambda$. Then $H$ is $\Pi^0_{\alpha_0}$ in $X$ for some $\alpha_0 < \omega_1$. But for each $\lambda$, $H_\lambda = H \cap (X_{\eta_{\alpha_{\lambda} + 1}} - X_{\eta_{\alpha_{\lambda}}})$. so each $H_\lambda$ is $\Pi^0_{\max(\alpha_0, \beta_0 + 1)}$ in $X$, contradiction. This ends the proof of Theorem 48.
Remark. Kunen has noted that Theorem 48 may be generalized to nonseparable metric spaces. Let $\mathcal{B}$ be a $\sigma$-discrete basis for $X$ and assume that every subset of $X$ is Borel in $X$. By using $\sigma$-discreteness it is easily seen that $\exists \mathcal{H} \subseteq \mathcal{B} \exists \beta < \omega_1$ so that $\mathcal{B} - \mathcal{H}$ is countable and $\forall U \in \mathcal{H} \operatorname{ord}(U) = \beta$. But $Y = \{x \in X : \forall U \in \mathcal{B} (x \in U \rightarrow U \notin \mathcal{H})\}$ is separable and hence by the theorem $\operatorname{ord}(Y) < \omega_1$, and so $\operatorname{ord}(X) < \omega_1$.

As a partial converse of Theorem 33 we have:

**Theorem 51.** If $\kappa = |2^\omega|$, $\kappa^+ = \kappa$, and $R^\kappa_{\alpha_\beta} = P(\kappa \times \kappa)$, then there is $X \subseteq 2^\omega$ with $|X| = \kappa$ and every subset of $X$ of cardinality less than $\kappa$ is $\Pi^0_\alpha$ in $X$.

**Proof.** Let $Z_\alpha$ for $\alpha < \kappa$ be all the subsets of $\kappa$ of cardinality less than $\kappa$. Put $Z = \bigcup_{\alpha < \kappa} \{\alpha\} \times Z_\alpha$ and $W = \{(\alpha, \beta) : \alpha < \beta < \kappa\}$. Let $\{A_n : n < \omega\}$ be closed under finite boolean combinations and $Z, W \subseteq \{A_n \times A_m : n, m < \omega\}_{\alpha, \beta}$. The map $F : \kappa \rightarrow 2^\omega$ defined by $(F(\alpha)(n) = 1$ iff $\alpha \in A_n$) is 1-1 and the set $X = F^{-1}\kappa$ has the required property.

For any cardinal $\kappa$ let $R(\kappa)$ be the least $\beta < \omega_1$ such that $R^\kappa_\beta = P(\kappa \times \kappa)$ or $\omega_1$ if no such $\beta$ exists.

**Theorem 52.** It is relatively consistent with ZFC that $|2^\omega| = \omega_{\omega_1 + 1}$, for every $n \leq \omega$ $R(\omega_n) = 1 + n$, and $R(\omega_{\omega_1}) = \omega$. This can be generalized to show that for any $\lambda < \omega_1$ a limit ordinal it is consistent with ZFC that $R(|2^\omega|) = \lambda$.

**Proof.** Let $M \models \text{""}'\text{ZFC+MA+}[2^\omega] = \omega_{\omega_1 + 1}'\text{""}$ be countable and transitive. Let $\kappa = \omega_{\omega_1 + 1}$ and define $\mathcal{P}^\alpha$ for $\alpha \leq \kappa$ so that $\mathcal{P}^{\alpha + 1} = \mathcal{P}^\alpha + \mathcal{P}_{2 + \beta + 1}(X_\alpha, Y_\alpha)$ where $Y_\alpha \subseteq 2^\omega$, $Y_\alpha \in M$, $|Y_\alpha| = \omega_{\beta + 1}$, and $\emptyset \Vdash \text{""}X_\alpha \subseteq Y_\alpha\text{""}$. At limits take the direct limit. By dovetailing arrange that for any $G$ $\mathcal{P}^\omega$-generic over $M$, $M[G] \models \text{""}$If $Y \subseteq 2^\omega$, $Y \in M$, and $|Y| = \omega_{\beta + 1}$ for some $\beta < \omega$, then every subset of $Y$ is $\Pi^0_{\beta + 1}$ in $Y"$.

As in the proof of Theorem 34 given any $\tau$ a term for a subset of $\omega$, find in $M, H \subseteq 2^\omega$, $K \subseteq \kappa$ so that: Let $Q = \{p \in \mathcal{P}^\kappa : \supp(p) \subseteq K, |p|(H) = 0\}$:

1. $|H| \leq \omega_{\beta_0}, |K| \leq \omega_{\beta_0}$.
2. $\forall n \in \omega Q$ decides ""$n \in \tau$"".
3. $\forall \beta \in K \forall x \in H Q$ decides ""$x \in X_\beta$"".
4. If $\alpha \in K$ and $|Y_\alpha| \leq \omega_{\beta_0}$, then $Y_\alpha \subseteq H$.

**Lemma 53.** If $H, K$ have property (3), (4) above, then for any $p \in \mathcal{P}^\kappa$ and $\beta$ with $1 \leq \beta < 2 + \beta_0$ there is $\dot{p}$ compatible with $p$, $|\dot{p}|(H) < \beta + 1$, supp $(\dot{p}) \subseteq K$, and for any $q$ if $|q|(H) < \beta$, supp $(q) \subseteq H$, and $\dot{p}$ and $q$ are compatible, then $p$ and $q$ are compatible.

**Proof.** The proof of this is just like the proof of Lemma 35. To check that the $\dot{p}$
gotten there is an element of $\mathbb{P}^\kappa$, note that if $((n), x) \in \beta(\alpha)$, then $x \in H$. Because if $x \notin H$ and $\alpha \in K$, then $|Y_\alpha| \geq \omega_{\mu_\alpha + 1}$ because of (4). Say $|Y_\alpha| = \omega_{\gamma + 1}$, so $\mathbb{P}^{\alpha + 1} = \mathbb{P}^\alpha \ast \mathbb{P}_{\omega + 1}(X_\alpha, Y_\alpha)$ and $|(n)|_{\omega_\gamma + 1} = 2 + \gamma \geq 2 + \beta_0 \geq \beta + 1$, but then it was thrown out, contradiction.

**Lemma 54.** Suppose $H$ and $K$ have properties (2), (3), and (4) for $\tau \subseteq \omega$. Suppose $1 \leq \beta \leq 2 + \beta_0$ and $B(\nu)$ is a $\Sigma^0_\beta$ predicate with parameters from $M$, $p \in \mathbb{P}^\kappa$ and $p \vDash \langle \text{"B(\tau)"} \rangle$. Then $\exists q \in \mathbb{P}^\kappa$ compatible with $p$, $|q|(H) \leq \beta$, supp$(q) \subseteq K$ and $q \vDash \langle \text{"B(\tau)"} \rangle$.

**Proof.** This follows from Lemma 53 just as in Theorem 34.

From Lemma 54 we have that:

(A) For any $Y \subseteq 2^\omega$ with $Y \in M$ and $n$ with $1 \leq n \leq \omega$ ($|Y| = \omega_n$ if $Y$ is a $G_{2, n}$-set). We claim that:

(B) For any $n < \omega$ there are $X, Y \subseteq 2^\omega$ with $|X| = |Y| = \omega_{n + 2}$ so that if $U$ is the usual $\Pi^{n + 2}_n$ set universal for $\Pi^0_{n + 2}$ sets, then $U \cap (X \times Y)$ is not $\Sigma^0_{n + 2}$ in the abstract rectangles on $X \times Y$.

To prove (B) just generalize the argument of Theorem 37, for $n = 0$ the argument is the same. Let $X \subseteq 2^\omega$ be in $M$ with $|X| = \omega_{n + 2}$. Choose $K \subseteq \kappa$, $|K| = \omega_{n + 2}$, and $K \in M$, so that for any $\alpha \in K$ $Y_\alpha = X$ and $\emptyset \vDash \langle \text{"X_\alpha = \emptyset"} \rangle$. Let $Y = \{y_\alpha : \alpha \in K\}$ where $y_\alpha$ is the $\Pi^{n + 2}_n$ code (with respect to $U$) for $G(\alpha)$. To generalize the argument allow $I_\gamma, J_\gamma, I_\alpha, J_\alpha$ to have cardinality $\leq \omega_n$ and also whenever $\gamma \in J_\gamma (\gamma \in J_\alpha)$ and $|Y_\gamma| \leq \omega_n$, then $Y_\gamma \subseteq I_\gamma (Y_\gamma \subseteq I_\alpha)$.

In $M[G]$ for any $n < \omega$ $R(\omega_n) = 1 + n$. To see this, let $Y \subseteq 2^\omega$ with $Y \in M$ and $|Y| = \omega_{n + 1}$. If $X \subseteq Y$ and $|X| \leq \omega_n$, then there is $Z \in M$ with $|Z| \leq \omega_n$ and $X \subseteq Z$. Because $\mathbb{M}$ is "MA" $Z$ is $\Pi^0_2$ in $Y$ and since $X$ is $\Pi^0_2$ in $Z$ by (A), we have $X$ is $\Pi^0_{2 + n}$ in $Y$. By Theorem 33 $R^0_{2 + n} = P(\omega_{n + 1} \times \omega_{n + 1})$. By (B) $n + 2$ is the least which will do.

Thus $R(\omega) = \omega$. To see that $R(\kappa) = \omega$ let $Y \subseteq 2^\omega$ with $Y \in M$ $|Y| = \kappa$, and every subset $Z \subseteq Y$ such that $|Z| < \kappa$ and $Z \in M$ is $\Sigma^0_2$ in $Y$ (see Theorem 17). In $M[G]$ every $Z \subseteq Y$ with $|Z| < \kappa$ is $\Sigma^0_2$ in $Y$, so by Theorem 33 $R^\kappa_\omega = P(\kappa \times \kappa)$.

**Remark.** It is easy to generalize Theorem 52 to show that for any $\lambda < \omega_1$ a limit ordinal and $\kappa > \omega$ of cofinality $\omega$, it is consistent that $|2^\omega| = \kappa^+$ and $R(\kappa^+) = \lambda$.

**Theorem 55.** It is relatively consistent with ZFC that

(a) $|2^\omega| = \omega_{\omega_1 + 1}$,
(b) for any $\alpha < \omega_1$ there is a $Q_\alpha$ set,
(c) $R(\omega_n) = n + 1$ for $n < \omega$,
(d) $R(\omega_\omega) = \lambda$ for $\lambda < \omega_1$ a limit ordinal,
(e) $R(\omega_{\lambda + n + 1}) = \lambda + n$ for $\lambda < \omega_1$ a limit ordinal and $n < \omega$. 

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The proof of this is an easy generalization of Theorem 52 and is left to the reader.

A set \( U \subseteq 2^\omega \times 2^\omega \) is universal for the Borel sets iff for every \( B \subseteq 2^\omega \) there exists \( x \in 2^\omega \) such that \( B = U_x = \{ y : (y, x) \in U \} \).

**Theorem 56.** It is relatively consistent with ZFC that no set universal for the Borel sets is in the \( \sigma \)-algebra generated by the abstract rectangles in \( 2^\omega \times 2^\omega \).

**Proof.** Let \( M \models \text{"ZFC} \rightarrow \neg \text{CH}" \) and let
\[
Q = \sum_{\beta < \omega_1} \left( \sum \{ P_\alpha(\emptyset, 2^\omega \cap M) : \alpha < \omega_1 \} \right).
\]
Let \( G \) be \( Q \)-generic over \( M \), then in \( M[G] \) there is no set \( U \) universal for the Borel sets in the \( \sigma \)-algebra generated by the rectangles. Suppose \( G \) is given by \((y_\beta : T^*_\alpha + 1 \rightarrow 2^{<\omega} : \alpha < \omega_1 \) and \( \beta < \omega_2 \)) where \( T^*_\alpha + 1 \) is the normal \( \alpha + 1 \) tree used in the definition of \( P_\alpha + 1 \) and \( G^{(0)}_v \) are the \( \Pi^0_\beta \) sets determined by \( y^\beta_\gamma \). Then as before we can easily get for each \( \alpha < \omega_1 \) that \( V^\alpha = \{(x, \beta : x \in G^{(0)}_v) \} \) is not \( \Sigma^0_\alpha \) in the abstract rectangles on \((2^\omega \times 2^\omega)\). Now suppose such a \( U \) existed and were \( \Sigma^0_\alpha \) in the abstract rectangles on \( 2^\omega \times 2^\omega \). Choose \( F : \omega_2 \rightarrow 2^\omega \) (necessarily \( 1 \)-1) so that \( \forall \beta < \omega_2 \forall x \in 2^\omega (x, \beta) \in V^\alpha \leftrightarrow (x, f(\beta)) \in U \). If \( U \) is \( \Sigma^0_\alpha \) in \( \{ A_n \times B_n : n < \omega \} \), then \( V^\alpha \) is \( \Sigma^0_\alpha \) in \( \{ A_n \times f^{-1}(\beta_n) : n < \omega \} \), contradiction.

**Remarks.** (1) In [9] Kunen shows that if one adds \( \omega_2 \) Cohen reals to a model of GCH, then no well-ordering of \( \omega_2 \) is in \( R^\omega_{\omega_1} \).

(2) In [1] it is shown that if \( G \) is a countable field of sets with \( \text{Borel}(2^\omega) \subseteq G_{\omega_1} \), the order of \( G \) is \( \omega_1 \).

In the model of Theorem 56 for any countable \( G \) and \( \alpha < \omega_1 \), \( \text{Borel}(2^\omega) \) is not included in \( G_{\alpha} \). This can be seen as follows. Let \( G = \{ A_n : n < \omega \} \) and let \( \{ s_n : n < \omega \} = T^* \) where \( T \) is a normal \( \alpha \) tree. Define for any \( y \in \omega^\omega \) and \( s \in T \) the set \( G^s \) as follows. For \( s = s_n \) let \( G^s = A_{\pi(s)} \), otherwise \( G^s = \bigcap \{ \omega^\omega - G^y : n < \omega \} \). If \( U = \{(x, y) : x \in G^0 \} \), then \( U \) is \( \Pi^0_\alpha \) in the abstract rectangles and universal for all Borel sets, contradicting Theorem 56.

5. Problems

Show:

(1) If \( |X| = \omega_1 \), then \( X \) is not a \( Q_\omega \) set.

(2) If \( R^\omega_{\omega_1} = P(\omega_2 \times 2^\omega) \), then there is \( n < \omega \) with \( R^\omega_{\omega_1} = P(\omega_2 \times \omega_2) \).

(3) If there exists a \( Q_\omega \) set, then there exists a \( Q_n \) set for some \( n < \omega \).

(4) If \( R^\omega_{\omega_1} = P(\omega_2 \times 2^\omega) \) and \( |2^\omega| = \omega_2 \), then \( |2^\omega| = \omega_2 \).

(5) If there is a \( Q_2 \) set of size \( \omega_1 \), then every subset of \( 2^\omega \) of size \( \omega_1 \) is a \( Q_2 \) set.

(6) If $X$ is a $Q_\alpha$ set and $Y$ is a $Q_\beta$ set, then $2^\alpha < \alpha < \beta$ implies $|X| < |Y|$.

Show consistency of:

(7) $\{\alpha : X \subseteq \mathbb{N} : \text{ord}(X) = \alpha\} = \{1\} \cup \{\alpha < \omega_1 : \alpha \text{ is even}\}$.

(8) $2^\omega = \omega_1$ and for any $X \subseteq \mathbb{N}$ if $|X| = \omega_1$, then $X$ is a $Q_1$ set, if $|X| = \omega_2$, then $X$ is a $Q_{\omega+3}$ set, and if $|X| = \omega_3$, then $\text{ord}(X) = \omega_1$.

(9) For any $\alpha < \omega_1$ there is a $\Pi_1^1 X$ with $\text{ord}(X) = \alpha$.

(10) For any $X \subseteq \mathbb{N}$ if $|X| > \omega_1$ then there is an $X$-projective set not Borel in $X$.

(11) There is no $G$ countable with $\Sigma_1^1 \subseteq G_{\omega_1}$ (This is a problem of Ulam, see Fund. Math. 30 (1938) 365.)

References


