Uniquely Universal Sets

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Abstract

We say that $X \times Y$ satisfies the Uniquely Universal property (UU) iff there exists an open set $U \subseteq X \times Y$ such that for every open set $W \subseteq Y$ there is a unique cross section of $U$ with $U_x = W$. Michael Hrušák raised the question of when does $X \times Y$ satisfy UU and noted that if $Y$ is compact, then $X$ must have an isolated point. We consider the problem when the parameter space $X$ is either the Cantor space $2^\omega$ or the Baire space $\omega^\omega$. We prove the following:

1. If $Y$ is a locally compact zero dimensional Polish space which is not compact, then $2^\omega \times Y$ has UU.
2. If $Y$ is Polish, then $\omega^\omega \times Y$ has UU iff $Y$ is not compact.
3. If $Y$ is a $\sigma$-compact subset of a Polish space which is not compact, then $\omega^\omega \times Y$ has UU.

For any space $Y$ with a countable basis there exists an open set $U \subseteq 2^\omega \times Y$ which is universal for open subsets of $Y$, i.e., $W \subseteq Y$ is open iff there exists $x \in 2^\omega$ with

$$U_x = \{ y \in Y : (x, y) \in U \} = W.$$ 

To see this let $\{ B_n : n < \omega \}$ be a basis for $Y$. Define

$$(x, y) \in U \text{ iff } \exists n (x(n) = 1 \text{ and } y \in B_n).$$

More generally if $X$ contains a homeomorphic copy of $2^\omega$ then $X \times Y$ will have a universal open set.

In 1995 Michael Hrušák mentioned the following problem to us. Most of the results in this note were proved in June and July of 2001.

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Hrušák’s problem.

Let $X$, $Y$ be topological spaces, call $X$ the parameter space, and $Y$ the base space. When does there exists $U \subseteq X \times Y$ which is uniquely universal for the open subsets of $Y$? This means the $U$ is open and for every open set $W \subseteq Y$ there is a unique $x \in X$ such that $U_x = W$.

Let us say that $X \times Y$ satisfies UU (uniquely universal property) if there exists such an open set $U \subseteq X \times Y$ which uniquely parameterizes the open subsets of $Y$. Note that the complement of $U$ is a closed set which uniquely parameterizes the closed subsets of $Y$.

**Proposition 1** (Hrušák) $2^\omega \times 2^\omega$ does not satisfy UU.

**proof:**

The problem is the empty set. Suppose $U$ is uniquely universal for the closed subsets of $2^\omega$. Then there is an $x_0$ such that $U_{x_0} = \emptyset$ but all other cross sections are nonempty. Take $x_n \to x_0$ but distinct from it. Since all other cross sections are non-empty we can choose $y_n \in U_{x_n}$. But then $y_n$ has a convergent subsequence, say to $y_0$, but then $y_0 \in U_{x_0}$.

QED

More generally:

**Proposition 2** (Hrušák) Suppose $X \times Y$ has UU and $Y$ is compact. Then $X$ must have an isolated point.

**proof:**

Suppose $U \subseteq X \times Y$ witnesses UU for closed subsets of $Y$ and $U_{x_0} = \emptyset$. For every $y \in Y$ there exists $U_y \times V_y$ open containing $(x_0, y)$ and missing $U$. By compactness of $Y$ finitely many $V_y$ cover $Y$. The intersection of the corresponding $U_y$ isolates $x_0$.

QED

Hence, for example, $2^\omega \times (\omega + 1)$, $\omega^\omega \times (\omega + 1)$, and $\omega^\omega \times 2^\omega$ cannot have UU.
Proposition 3 Let $2^{\omega} \oplus 1$ be obtained by attaching an isolated point to $2^{\omega}$. Then $(2^{\omega} \oplus 1) \times 2^{\omega}$ has UU.

proof: Define $T \subseteq 2^{<\omega}$ to be a nice tree iff

(a) $s \subseteq t \in T$ implies $s \in T$ and

(b) if $s \in T$, then either $s \cdot \langle 0 \rangle$ or $s \cdot \langle 1 \rangle$ in $T$.

Let $NT \subseteq \mathcal{P}(2^{<\omega})$ be the set of nice trees. Define the universal set $U$ by

$$U = \{(T,x) \in NT \times 2^{\omega} : \forall n \ x|n \in T\}.$$

Note that the empty tree $T$ is nice and parameterizes the empty set. Also $NT$ is a closed subset of $\mathcal{P}(2^{<\omega})$ with exactly one isolated point (the empty tree), and hence it is homeomorphic to $2^{\omega} \oplus 1$.

QED

Question 4 Does $(2^{\omega} \oplus 1) \times [0,1]$ have UU?

Remark 5 $2^{\omega} \times \omega$ has the UU property. Just let $(x,n) \in U$ iff $x(n) = 1$.

Question 6 Does either $\mathbb{R} \times \omega$ or $[0,1] \times \omega$ have UU? Or more generally, is there any example of UU for a connected parameter space?

Recall that a topological space is Polish iff it is completely metrizable and has a countable dense subset. A set is $G_\delta$ iff it is the countable intersection of open sets. The countable product of Polish spaces is Polish. A $G_\delta$ subset of a Polish space is Polish (Alexandrov). A space is zero-dimensional iff it has a basis of clopen sets. All compact zero-dimensional Polish spaces without isolated points are homeomorphic to $2^{\omega}$ (Brouwer). A zero-dimensional Polish space is homeomorphic to $\omega^\omega$ iff compact subsets have no interior (Alexandrov-Urysohn). For proofs of these facts see Kechris [5] p.13-39.

Proposition 7 Suppose $Y$ is a zero dimensional Polish space. If $Y$ is locally compact but not compact, then $2^{\omega} \times Y$ has UU. So, for example, $2^{\omega} \times (\omega \times 2^{\omega})$ has UU.
proof:
Let $\mathcal{B}$ be a countable base for $Y$ consisting of clopen compact sets. Define $G \subseteq \mathcal{B}$ is good iff
\[ G = \{ b \in \mathcal{B} : b \subseteq \bigcup G \} \]

Let $\mathcal{G} \subseteq \mathcal{P}(\mathcal{B})$ be the family of good subsets of $\mathcal{B}$. We give the $\mathcal{P}(\mathcal{B})$ the topology from identifying it with $2^\mathcal{B}$. Since $\mathcal{B}$ is an infinite countable set $\mathcal{P}(\mathcal{B})$ is homeomorphic to $2^\omega$. A sequence $G_n$ for $n < \omega$ converges to $G$ iff for each finite $F \subseteq \mathcal{B}$ we have that $G_n \cap F = G \cap F$ for all but finitely many $n$. Hence $\mathcal{G}$ is a closed subset of $\mathcal{P}(\mathcal{B})$ since by compactness $b \subseteq \bigcup G$ iff $b \subseteq \bigcup F$ for some finite $F \subseteq G$.

There is a one-to-one correspondence between good families and open subsets of $Y$: Given any open set $U \subseteq Y$ define
\[ G_U = \{ b \in \mathcal{B} : b \subseteq U \} \]
and given any good $G$ define $U_G = \bigcup G$. (Note that the empty set is good.)

We claim that no $G_0 \in \mathcal{G}$ is an isolated point. Suppose for contradiction it is. Then there must be a basic open set $N$ with $\{G_0\} = G \cap N$. A basic neighborhood has the following form
\[ N = N(F_0, F_1) = \{ G \subseteq \mathcal{B} : F_0 \subseteq G \text{ and } F_1 \cap G = \emptyset \} \]
where $F_0, F_1 \subseteq \mathcal{B}$ are finite.

For each $b \in F_1$ since $G_0$ is good, $b$ is not a subset of $\bigcup G_0$, and since $\bigcup F_0 \subseteq \bigcup G_0$, we can choose a point $z_b \in b \setminus \bigcup F_0$. Since $Y$ is not compact, $Y \setminus (\bigcup (F_0 \cup F_1))$ is nonempty. Fix $z \in Y \setminus (\bigcup (F_0 \cup F_1))$.

Now let $U_1 = Y \setminus \{ z_b : b \in F_1 \}$ and let $U_2 = U_1 \setminus \{ z \}$. Then $G_{U_1}, G_{U_2}$ are distinct elements of $N \cap \mathcal{G}$.

Hence $\mathcal{G}$ is a compact zero-dimensional metric space without isolated points and therefore it is homeomorphic to $2^\omega$.

To get a uniquely universal open set $U \subseteq \mathcal{G} \times Y$ define:
\[ (G, y) \in U \text{ iff } \exists b \in G \ y \in b. \]

QED

Example 26 is a countable Polish space $Z$ such that $2^\omega \times Z$ has UU, but $Z$ is not locally compact.
Lemma 8 Suppose $f : X \rightarrow Y$ is a continuous bijection and $Y \times Z$ has UU. Then $X \times Z$ has UU.

proof: Given $V \subseteq Y \times Z$ witnessing UU, let

$$U = \{(x, y) : (f(x), y) \in V\}.$$

QED

Many uncountable standard Borel sets\(^2\) are the bijective continuous image of the Baire space $\omega^n$. According to the footnote on page 447 of Kuratowski [6] Sierpinski proved in a 1929 paper that any standard Borel set in which every point is a condensation point is the bijective continuous image of $\omega^n$. We weren’t able to find Sierpinski’s paper but we give a proof of his result in Lemma 21.

We first need a special case for which we give a proof.

Lemma 9 There is a continuous bijection $f : \omega^n \rightarrow 2^n$.

proof: Let $\pi : \omega \rightarrow \omega + 1$ be a bijection. It is automatically continuous. It induces a continuous bijection $\pi : \omega^n \rightarrow (\omega + 1)^n$. But $(\omega + 1)^n$ is a compact Polish space without isolated points, hence it is homeomorphic to $2^n$.

QED

Remark. If $C \subseteq 2^n \times \omega^n$ is the graph of $f^{-1}$, then $C$ is a closed set which uniquely parameterizes the family of singletons of $\omega^n$.

Corollary 10 If $2^n \times Y$ has UU, then $\omega^n \times Y$ has UU.

Question 11 Is the converse of Corollary 10 false? That is: Does there exist $Y$ such that $\omega^n \times Y$ has UU but $2^n \times Y$ does not have UU?

Lemma 12 Suppose $X$ is a zero-dimensional Polish space without isolated points. Then there exists a continuous bijection $f : \omega^n \rightarrow X$.

\(^2\)A standard Borel set is a Borel subset of a Polish space.
proof:

Construct a subtree $T \subseteq \omega^\omega$ and $(C_s \subseteq X : s \in T)$ nonempty clopen sets such that:

1. $C_0 = X$,

2. if $s \in T$ is a terminal node, then $C_s$ is compact, and

3. if $s \in T$ is not terminal, then $s^{\langle n \rangle} \in T$ for every $n \in \omega$ and $C_s$ is partitioned by $(C_{s^{\langle n \rangle}} : n < \omega)$ into nonempty clopen sets each of diameter$^3$ less than $\frac{1}{|s|+1}$.

For each terminal node $s \in T$ choose a continuous bijection $f_s : [s] \to C_s$ given by Lemma 9. Define $f : \omega^\omega \to X$ by $f(x) = f_{x|n}(x)$ if there exists $n$ such that $x|n$ is a terminal node of $T$ and otherwise determine $f(x)$ by the formula:

$$\{f(x)\} = \bigcap_{n<\omega} C_{x|n}$$

Checking that $f$ is a continuous bijection is left to the reader.

QED

Remark. An easy modification of the above argument shows that any zero-dimensional Polish space is homeomorphic to a closed subspace of $\omega^\omega$. It also gives the classical result that if no clopen sets are compact, then $X$ is homeomorphic to $\omega^\omega$. A different proof of Lemma 12 is given in Moschovakis [8] p. 12.

Definition 13 We use $cl(X)$ to denote the closure of $X$.

Proposition 14 Suppose $Y$ is Polish but not compact. Then $\omega^\omega \times Y$ has the UU. So for example, $\omega^\omega \times \omega^\omega$ and $\omega^\omega \times \mathbb{R}$ both have UU.

proof:

We assume that the metric on $Y$ is complete and bounded. Let $\mathcal{B}$ be a countable basis for $Y$ of nonempty open sets which has the property that no finite subset of $\mathcal{B}$ covers $Y$.

For $s, t \in \mathcal{B}$ define $t \triangleleft s$ iff $cl(t) \subseteq s$ and $diam(t) \leq \frac{1}{2} diam(s)$.

$^3$This is with respect to a fixed complete metric on $X$. 

Lemma 15 Suppose \( G \subseteq B \) has the following properties:

(1) for all \( t, s \in B \) if \( t \subseteq s \in G \), then \( t \in G \) and

(2) \( \forall s \in B \) if \( (\forall t < s \ t \in G) \), then \( s \in G \),

then for any \( s \in B \) if \( s \subseteq \bigcup G \), then \( s \in G \).

proof:

Suppose (1) and (2) hold but for some \( s \subseteq \bigcup G \) we have \( s \notin G \).

Note that there cannot be a sequence \((s_n : n \in \omega)\) starting with \( s_0 = s \), and with \( s_{n+1} < s_n \) and \( s_n \notin G \) for each \( n \). This is because if \( \{x\} = \bigcap_{n \in \omega} s_n \), then \( x \in s \subseteq \bigcup G \) and so for some \( t \in G \) we have \( x \in t \). But then for some sufficiently large \( n \) we have that \( s_n \subseteq t \) putting \( s_n \in G \) by (1).

Hence there must be some \( t \subseteq s \) with \( t \notin G \) but for all \( r < t \) we have \( r \in G \). This is a contradiction to (2).

QED

Let \( G \subseteq \mathcal{P}(B) \) be the set of all \( G \) which satisfy the hypothesis of Lemma 15. Then we have a \( G_\delta \) subset \( \mathcal{G} \) of \( 2^B \), which uniquely parameterizes the open sets. Hence the set \( U \) witnesses the unique universal property:

\[ U = \{(G, x) \in \mathcal{G} \times Y : x \in \bigcup G\}. \]

To finish the proof it is enough to see that no point in \( \mathcal{G} \) is isolated. If \( \mathcal{G} \) has an isolated point, there must be \( s_1, \ldots, s_n \) and \( t_1, \ldots, t_m \) from \( B \) such that

\[ N = \{G \in \mathcal{G} : s_1 \in G, \ldots, s_n \in G, t_1 \notin G, \ldots, t_m \notin G\} \]

is a singleton. Let \( W = s_1 \cup \cdots \cup s_n \). Since \( N \) contains the point \( G = \{s \in B : s \subseteq W\} \) it must be that \( N = \{G\} \). For each \( j \) choose \( x_j \in t_j \setminus W \). Since the union of the \( s_i \) and \( t_j \) does not cover \( Y \), we can choose

\[ z \in Y \setminus (s_1 \cup \cdots \cup s_n \cup t_1 \cup \cdots \cup t_m). \]

Take \( r \in B \) with \( z \in r \) but \( x_j \notin r \) for each \( j = 1, \ldots, m \). Let

\[ G' = \{s \in B : s \subseteq r \cup W\}. \]
Then \( G' \in N \) but \( G' \neq G \) which shows that \( N \) is not a singleton.

It follows from Lemma 8 and 12 that \( \omega^\omega \times Y \) has the UU.

QED

Proposition 2 and 14 show that:

**Corollary 16** For \( Y \) Polish:

\( \omega^\omega \times Y \) has UU iff \( Y \) is not compact.

**Question 17** Does \( 2^{\omega} \times \omega^\omega \) have UU?

**Question 18** Does \( 2^{\omega} \times \mathbb{R} \) have UU?

Our next result follows from Proposition 22 but has a simpler proof so we give it first.

**Proposition 19** If \( X \) is a countable metric space which is not compact, then \( \omega^\omega \times X \) has UU. So, for example, \( \omega^\omega \times \mathbb{Q} \) has UU.

**proof:**

We produce a uniquely universal set for the open subsets of \( X \).

First note that there exists a countable basis \( \mathcal{B} \) for \( X \) with the property that it is closed under finite unions and \( X \setminus B \) is infinite for every \( B \in \mathcal{B} \). To see this fix \( \{ x_n : n < \omega \} \subseteq X \) an infinite set without a limit point, i.e., an infinite closed discrete set. Given a countable basis \( \mathcal{B} \) replace it with finite unions of sets from

\[ \{ B \setminus \{ x_m : m > n \} : n \in \omega \text{ and } B \in \mathcal{B} \}. \]

We may assume also that \( \mathcal{B} \) includes the empty set.

Next let

\[ \mathcal{P} = \{ (B, F) : B \in \mathcal{B}, \ F \in [X]^{<\omega}, \text{ and } B \cap F = \emptyset \}. \]

Then \( \mathcal{P} \) is a partial order determined by \( p \leq q \) iff \( B_q \subseteq B_p \) and \( F_q \subseteq F_p \). For \( p \in \mathcal{P} \) we write \( p = (B_p, F_p) \). For \( p, q \in \mathcal{P} \) we write \( p \perp q \) to stand for \( p \) and \( q \) are incompatible, i.e., there does not exist \( r \in \mathcal{P} \) with \( r \leq p \) and \( r \leq q \).

We will code open subsets of \( X \) by good filters on \( \mathcal{P} \). Define the family \( \mathcal{G} \) of good filters on \( \mathcal{P} \) to be the set of all \( G \subseteq \mathcal{P} \) such that
1. If $p \leq q$ and $p \in G$, then $q \in G$.
2. For all $p, q \in G$, there exist $r \in G$ such that $r \leq p$ and $r \leq q$.
3. For all $x \in X$, there exists $p \in G$ such that $x \in B_p \cup F_p$.
4. For all $p \in P$, either $p \in G$ or there exists $q \in G$ such that $p \not\leq q$.

Since the poset $P$ is countable, we can identify $P$ with $P(\omega)$ and hence $2^\omega$.
We give $G \subseteq P(\omega)$ the subspace topology. Note that $G$ is $G_\delta$ in this topology.

Note also that the sets

$$[p] = \{G \in G : p \in G\}$$

form a basis for $G$ (use conditions (2) and (4) to deal with finitely many $p_i$ in $G$ and finitely many $q_j$ not in $G$).

Note that since $X \setminus B_p$ is always infinite, for any $p \in P$ there exists $r, q \leq p$ with $r \perp q$. Namely, for some $x \in X \setminus (B_p \cup F_p)$ put $x$ into $B_r \cap F_q$. It follows that no element of $G$ is isolated. So $G$ is a zero-dimensional Polish space without isolated points. Hence by Lemma 12 there is a continuous bijection $f : \omega^\omega \to G$.

For $G \in G$, let

$$U_G = \bigcup \{B_p : p \in G\}.$$ 

For any $U \subseteq X$ open, define

$$G_U = \{p \in P : B_p \subseteq U \text{ and } F_p \cap U = \emptyset\}.$$ 

The maps $G \to U_G$ and $U \to G_U$ show that there is a one-to-one correspondence between $G$ and the open subsets of $X$.

Finally define $U \subseteq G \times X$ by

$$(G, x) \in U \text{ iff } \exists p \in G \text{ } x \in B_p.$$ 

This witnesses the UU property for $G \times X$ and so by Lemma 8, we have UU for $\omega^\omega \times X$.

QED

**Question 20** Does $2^\omega \times \mathbb{Q}$ have UU?
Our next result Proposition 22 implies Proposition 19 but needs the following Lemma:

**Lemma 21 (Sierpinski)** Suppose $B$ is a Borel set in a Polish space for which every point is a condensation point. Then there exists a continuous bijection from $\omega^\omega$ to $B$.

**proof:**

We use that every Borel set is the bijective image of a closed subset of $\omega^\omega$. This is due to Lusin-Souslin see Kechris [5] p.83 or Kuratowski-Mostowski [7] p.426.

Using the fact that every uncountable Borel set contains a perfect subset it is easy to construct $K_n$ for $n < \omega$ satisfying:

1. $K_n \subseteq B$ are pairwise disjoint,
2. $K_n$ are homeomorphic copies of $2^\omega$ which are nowhere dense in $B$, and
3. every nonempty open subset of $B$ contains infinitely many $K_n$.

Let $B_0 = B \setminus \bigcup_{n<\omega} K_n$. We may assume $B_0$ is nonempty, otherwise just split $K_0$ into two pieces. Since it is a Borel set, there exists $C \subseteq \omega^\omega$ closed and a continuous bijection $f : C \to B_0$. Define

$$\Gamma = \{s \in \omega^{<\omega} : [s] \cap C = \emptyset \text{ and } [s^*] \cap C \neq \emptyset\}$$

where $s^*$ is the unique $t \subseteq s$ with $|t| = |s| - 1$. Without loss we may assume that $C$ is nowhere dense and hence $\Gamma$ infinite. Let $\Gamma = \{s_n : n < \omega\}$ be a one-one enumeration. Note that $\{C\} \cup \{[s_n] : n < \omega\}$ is a partition of $\omega^\omega$.

Inductively choose $t_n > t_{n-1}$ with $K_{t_n}$ a subset of the ball of radius $\frac{1}{n+1}$ around $f(x_n)$ for some $x_n \in C \cap [s_n^*]$. For each $n < \omega$ let $f_n : [s_n] \to K_{t_n}$ be a continuous bijection.

Then $g = f \cup \bigcup_{n<\omega} f_n$ is a continuous bijection from $\omega^\omega$ to $B_0 \cup \bigcup_{n<\omega} K_{t_n}$.

To see that it is continuous suppose for contradiction that $u_n \to u$ as $n \to \infty$ and $|g(u_n) - g(u)| > \epsilon > 0$ all $n$. Since $C$ is closed if infinitely many $u_n$ are in $C$, so is $u$ and we contradict continuity of $f$. If $u \in [s_n]$, then we contradict the continuity of $f_n$, So, we may assume that all $u_n$ are not in $C$ but $u$ is in
C. By the continuity of $f$ we may find $s \subseteq u$ with $f([s] \cap C)$ inside a ball of radius $\frac{\epsilon}{3}$ around $f(u)$. Find $n$ with $\frac{1}{n+1} < \frac{\epsilon}{3}$ for which there is $m$ such that $u_m \in [s_n]$ and $s_n \supseteq s$. But then $g(u_m) = f_n(u_m) \in K_{l_n}$ and $K_{l_n}$ is in a ball of radius $\frac{1}{n+1}$ around some $f(x_n)$ with $x_n \in [s_n] \cap C$. This is a contradiction:

$$d(g(u), g(u_m)) \leq d(f(u), f(x_n)) + d(f(x_n), f_n(u_m)) \leq \frac{2}{3}\epsilon.$$

Next let $I = \omega \setminus \{l_n : n < \omega\}$. Then there exists continuous bijection

$$h : I \times \omega^\omega \to \bigcup_{i \in I} K_i.$$

Finally $g \cup h$ is a continuous bijection from $\omega^\omega \oplus (I \times \omega^\omega)$ to $B_0 \cup \bigcup_{n < \omega} K_n = B$. Since $\omega^\omega \oplus (I \times \omega^\omega)$ is a homeomorphic copy of $\omega^\omega$ we are done.

QED

**Proposition 22** $\omega^\omega \times Y$ has UU for any $\sigma$-compact but not compact subspace $Y$ of a Polish space. So for example, $\omega^\omega \times (\mathbb{Q} \times 2^\omega)$ has UU.

**proof:**

Let $Y = \bigcup_{n < \omega} K_n$ where each $K_n$ is compact. Since $Y$ is not compact it contains an infinite closed discrete set $D$. Choose a countable basis $\mathcal{B}$ for $Y$ such that for any $b \in \mathcal{B}$ the closure of $b$ contains at most finitely many points of $D$. Define $G \subseteq \mathcal{B}$ to be good iff for every $b \in \mathcal{B}$ if $\text{cl}(b) \subseteq \bigcup G$ then $b \in G$. Let $\mathcal{G} \subseteq \mathcal{P}(\mathcal{B})$ be the family of good sets.

Note that $\mathcal{G}$ is a $\Pi^0_3$ set:

$$G \in \mathcal{G} \text{ iff } \forall b \in \mathcal{B} \ (\forall n \ \text{cl}(b) \cap K_n \subseteq \bigcup G) \rightarrow b \in G$$

Note that $\text{cl}(b) \cap K_n \subseteq \bigcup G$ iff there is a finite $F \subseteq G$ with $\text{cl}(b) \cap K_n \subseteq \bigcup F$.

To finish the proof it is necessary to see that basic open sets in $\mathcal{G}$ are uncountable. Given $b_i, c_j \in \mathcal{B}$ for $i < n$ and $j < m$ suppose that

$$N = \{G \in \mathcal{G} : b_0 \in G, \ldots, b_{n-1} \in G, \ c_0 \notin G, \ldots, c_{m-1} \notin G\}$$

is nonempty. Since it is nonempty we can choose points

$$u_j \in \text{cl}(c_j) \setminus \bigcup_{i < n} \text{cl}(b_i)$$
for $j < m$. Note that the set

$$Z = D \setminus \text{cl}(\bigcup_{i<n,j<m} b_i \cup c_j)$$

is an infinite discrete closed set. But then given any $Q \subseteq Z$ we can find an open set $U_Q$ with $\bigcup_{i<n} \text{cl}(b_i) \subseteq U_Q$, $u_j \notin U_Q$ for $j < m$, and $U_Q \cap Z = Q$. Let

$$G_Q = \{ b \in B : \text{cl}(b) \subseteq U_Q \}$$

Since each $G_Q \in N$ we have that $N$ is uncountable. By Lemma 21 and 8, we are done.

QED

**Question 23** For what Borel spaces $Y$ does $\omega^\omega \times Y$ have $UU$?

For example, does $\omega^\omega \times \mathbb{Q}^\omega$ have $UU$?

**Proposition 24** For every $Y$ a $\Sigma^1_1$ set there exists a $\Sigma^1_1$ set $X$ such that $X \times Y$ has $UU$.

**proof:**

Suppose $Y \subseteq Z$ where $Z$ is Polish and $B$ is a countable base for $Z$. Define $G \subseteq \mathcal{P}(B)$ by

$$G \in G \iff \forall b \in B \ [\ (b \cap Y \subseteq \bigcup G) \rightarrow b \in G]$$

Note that $b \cap Y \subseteq \bigcup G$ is $\Pi^1_1$ and so $G$ is $\Sigma^1_1$.

QED

We use the next Lemma for Example 26.

**Lemma 25** For any space $Y$

$$(\omega \times 2^\omega) \times Y \text{ has } UU \iff 2^\omega \times Y \text{ has } UU.$$ 

**proof:**

Suppose $C \subseteq (\omega \times 2^\omega) \times Y$ is a closed set uniquely universal for the closed subsets of $Y$. 


Since the whole space $Y$ occurs as a cross section of $C$ without loss we may assume that $Y = C_{(0,0)}$ where $\vec{0}$ is the constant zero function.

For each $n > 0$ let

$$K_n = \{(0^n\langle \star, x_0, x_1, \ldots \rangle, y) \in 2^\omega \times Y : ((n, x), y) \in C\}$$

By $0^n\langle \star, x_0, x_1, \ldots \rangle$ we mean a sequence of $n$ zeros followed by the special symbol $\star$ and then the (binary) digits of $x$. Note that the $K_n$ converge to $\vec{0}$.

Let $K_0 = \{(x, y) : ((0, x), y) \in C\}$

Let $B = \bigcup_{n<\omega} K_n \subseteq S \times Y$ where

$$S = 2^\omega \cup \bigcup_{n>0} \{0^n\langle \star, x_0, x_1, \ldots \rangle : x \in 2^\omega\}.$$ 

Note that $S$ is homeomorphic to $2^\omega$ and there is a one-to-one correspondence between the cross sections of $B$ and cross sections of $C$. Note that $B$ is closed in $S \times Y$: If $(x_n, y_n) \in B$ for $n < \omega$ is a sequence converging to $(x, y) \in S \times Y$ and $x$ is not the zero vector it is easy to see that $(x, y) \in B$. On the other hand if $x$ is the zero vector, then since $B_{\vec{0}} = Y$, it is automatically true that $(x, y) \in B$.

Hence UU holds for $2^\omega \times Y$.

For the opposite direction just note that $(\omega + 1) \times 2^\omega$ is homeomorphic to $2^\omega$ and there is a continuous bijection from $\omega \times 2^\omega$ onto $(\omega + 1) \times 2^\omega$.

QED

Next we describe our counterexample to a converse of Proposition 7. Let $Z = (\omega \times \omega) \cup \{\infty\}$. Let each $D_n = \{n\} \times \omega$ be an infinite closed discrete set and let the sequence of $D_n$ “converge” to $\infty$, i.e., each neighborhood of $\infty$ contains all but finitely many $D_n$. Equivalently $Z$ is homeomorphic to:

$$Z' = \{x \in \omega^\omega : |\{n : x(n) \neq 0\}| \leq 1\}.$$ 

The point $\infty$ is the constant zero map, while $D_n$ are the points in $Z'$ with $x(n) \neq 0$. Note that $Z'$ is a closed subset of $\omega^\omega$, hence it is Polish. This seems to be the simplest nonlocally compact Polish space.

**Example 26** $Z$ is a countable nonlocally compact Polish space such that $2^\omega \times Z$ has the UU.
Uniquely Universal Sets

proof:
\[ Z = \bigcup_{n<\omega} D_n \cup \{\infty\}. \]
Note that \( X \subseteq Z \) is closed iff \( \infty \in X \) or \( X \subseteq \bigcup_{i\leq k} D_i \) for some \( k < \omega \). By Lemma 25 it is enough to see that \( (\omega \times 2^\omega) \times Z \) has the UU.

Let \( P_n = \{n\} \times 2^\omega \).

Use \( P_0 \) to uniquely parameterize all subsets of \( Z \) which contain the point at infinity, see Remark 5.

Use \( P_1 \) to uniquely parameterize all \( X \subseteq D_0 \), including the empty set.

For \( n = 1 + 2^{k-1}(2l-1) \) with \( k, l > 0 \) use \( P_n \) to uniquely parameterize all \( X \subseteq \bigcup_{i\leq k} D_i \) such that \( D_k \) meets \( X \) and the minimal element of \( D_k \cap X \) is the \( l \)-th element of \( D_k \).

QED

Our next two results have to do with Question 17.

**Proposition 27** Existence of UU for \( 2^\omega \times \omega^\omega \) is equivalent to:

There exist a \( C \subseteq \mathcal{P}(\omega^{<\omega}) \) homeomorphic to \( 2^\omega \) such that every \( T \in C \) is a subtree of \( \omega^{<\omega} \) (possibly with terminal nodes) and such that for every closed \( C \subseteq \omega^\omega \) there exists a unique \( T \in C \) with \( C = [T] \).

proof:

Given \( C \subseteq 2^\omega \times \omega^\omega \) witnessing UU for closed subsets of \( \omega^\omega \). Let

\[ [T] = \{(s,t) : ([s] \times [t]) \cap C \neq \emptyset\}. \]

Define \( f : 2^\omega \rightarrow \mathcal{P}(\omega^{<\omega}) = 2^{\omega^{<\omega}} \) by \( f(x)(s) = 1 \) iff \( (x|n, s) \in T \) where \( n = |s| \).

Then \( f \) is continuous, since \( f(x)(s) \) depends only on \( x|n \) where \( n = |s| \).

If \( T_x = f(x) \), then \([T_x] = C_x \). Hence \( f \) is one-to-one, so its image \( \mathcal{C} \) is as described.

QED

**Proposition 28** Suppose \( \omega^{\omega} \times Y \) has UU where \( Y \) is any topological space in which open sets are \( F_\sigma \). Then there exists \( U \subseteq 2^\omega \times Y \) an \( F_\sigma \) set such that all cross sections \( U_x \) are open and for every open \( W \subseteq Y \) there is a unique \( x \in 2^\omega \) with \( U_x = W \).
proof:

Let $\omega \oplus 1$ denote the discrete space with one isolated point adjoined and let $\omega + 1$ denote the compact space consisting of a single convergent sequence. Then $(\omega \oplus 1)^\omega$ is homeomorphic to $\omega^\omega$ and $(\omega + 1)^\omega$ is homeomorphic to $2^\omega$.

Assume that $U \subseteq (\omega \oplus 1)^\omega \times Y$ is an open set witnessing $UU$. Then $U$ is an $F_\sigma$ set in $(\omega + 1)^\omega \times Y$.

To see this note that a basic clopen set in $(\omega \oplus 1)^\omega$ could be defined by some $s \in (\omega \oplus 1)^{<\omega}$ by

$$[s] = \{x \in (\omega \oplus 1)^\omega : s \subseteq x\}.$$ 

But it is easy to check that $[s] \subseteq (\omega + 1)^\omega$ is closed in the topology of $(\omega + 1)^\omega$. Since $U$ is open in $(\omega \oplus 1)^\omega \times Y$ there exists $s_n$ and open sets $W_n \subseteq Y$ such that

$$U = \bigcup_{n<\omega} [s_n] \times W_n.$$ 

Hence $U$ is the countable union of $F_\sigma$ sets and so it is $F_\sigma$ in $(\omega + 1)^{<\omega}$.

QED

Here $\oplus$ refers to the topological sum of disjoint copies or equivalently a clopen separated union.

**Proposition 29** Suppose $X_i \times Y_i$ has $UU$ for $i \in I$. Then

$$\left( \prod_{i \in I} X_i \right) \times \left( \bigoplus_{i \in I} Y_i \right) \text{ has } UU.$$ 

So, for example, if $2^\omega \times Y$ has $UU$, then $2^\omega \times (\omega \times Y)$ has $UU$.

proof:

Define

$$((x_i)_{i \in I}, y) \in U \text{ iff } \exists i \in I \ (x_i, y) \in U_i$$

where the $U_i \subseteq X_i \times Y_i$ witness $UU$.

QED

Except for Proposition 2 we have given no negative results. The following two propositions are the best we could do in that direction.
Proposition 30 Suppose $U \subseteq X \times Y$ is an open set universal for the open subsets of $Y$. If $X$ is second countable, then so is $Y$.

proof:

$U$ is the union of open rectangles of the form $B \times C$ with $B$ open in $X$ and $C$ open in $Y$. Clearly we may assume that $B$ is from a fixed countable basis for $X$. Since $\bigcup_i B \times C_i = B \times \bigcup_i C_i$ we may write $U$ as a countable union:

$$U = \bigcup_{n<\omega} B_n \times C_n$$

where the $B_n$ are basic open sets in $X$ and the $C_n$ are open subsets of $Y$. But this implies that $\{C_n : n < \omega\}$ is a basis for $Y$ since for each $x \in X$

$$U_x = \bigcup\{C_n : x \in B_n\}$$

QED

Proposition 31 There exists a partition $X \cup Y = 2^\omega$ into Bernstein sets $X$ and $Y$ such that for every Polish space $Z$ neither $Z \times X$ nor $Z \times Y$ has UU.

proof:

Note that up to homeomorphism there are only continuum many Polish spaces. If there is a UU set for $Z \times X$, then there is an open $U \subseteq Z \times 2^\omega$ such that $U \cap (Z \times X)$ is UU. Note that the cross sections of $U$ must be distinct open subsets of $2^\omega$. Hence it suffices to prove the following:

Given $U_\alpha$ for $\alpha < \mathfrak{c}$ such that each $U_\alpha$ is a family of open subsets of $2^\omega$ either

(a) there exists distinct $U, V \in U_\alpha$ with $U \cap X = V \cap X$ or

(b) there exists $U \subseteq 2^\omega$ open such that $U \cap X \neq V \cap X$ for all $V \in U_\alpha$.

And the same for $Y$ in place of $X$.

Let $P_\alpha$ for $\alpha < \mathfrak{c}$ list all perfect subsets of $2^\omega$ and let $\{z_\alpha : \alpha < \mathfrak{c}\} = 2^\omega$. Construct $X_\alpha, Y_\alpha \subseteq 2^\omega$ with

1. $X_\alpha \cap Y_\alpha = \emptyset$
2. $\alpha < \beta$ implies $X_\alpha \subseteq X_\beta$ and $Y_\alpha \subseteq Y_\beta$

3. $|X_\alpha \cup Y_\alpha| = |\alpha| + \omega$

4. If there exists distinct $U, V \in \mathcal{U}_\alpha$ such that $U \Delta V$ is a countable set disjoint from $X_\alpha$, then there exists such a pair with $U \Delta V \subseteq Y_{\alpha+1}$

5. If there exists distinct $U, V \in \mathcal{U}_\alpha$ such that $U \Delta V$ is a countable set disjoint from $Y_{\alpha+1}$, then there exists such a pair with $U \Delta V \subseteq X_{\alpha+1}$

6. $P_\alpha$ meets both $X_{\alpha+1}$ and $Y_{\alpha+1}$

7. $z_\alpha \in (X_{\alpha+1} \cup Y_{\alpha+1})$

First we do (4) then (5) and then take care of (6) and (7).

Let $X = \bigcup_{\alpha < \xi} X_\alpha$ and $Y = \bigcup_{\alpha < \xi} Y_\alpha$.

Fix $\alpha$ and let us verify (a) or (b) holds. Take any point $p \in X \setminus X_{\alpha+1}$. If (b) fails there must be $U, V \in \mathcal{U}_\alpha$ with $X = X \cap U$ and $X \setminus \{p\} = X \cap V$. Then $(U \Delta V) \cap X_{\alpha+1} = \emptyset$. Since $X$ is Bernstein and $(U \Delta V) \cap X$ has only one point in it, it must be that $U \Delta V$ is countable. Then by our construction we have chosen distinct $U, V \in \mathcal{U}_\alpha$ with $U \Delta V \subseteq Y$ therefore $U \cap X = V \cap X$, so (a) holds.

A similar argument goes through for $Y$.

QED

Finally, and conveniently close to the bibliography, we note some papers in the literature which are related to the property UU. Friedberg [3] proved that there is one-to-one recursively enumerable listing of the recursively enumerable sets. This is the same as saying that there is a (light-face) $\Sigma^0_1$ subset $U \subseteq \omega \times \omega$ which is uniquely universal for the $\Sigma^0_1$ subsets of $\omega$. Brodhead and Cenzer [1] prove the analogous result for (light-face) $\Sigma^0_1$ subsets of $2^\omega$.


References


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