There are no $Q$-Points in Laver's Model for the Borel Conjecture

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Published by: American Mathematical Society

Stable URL: http://www.jstor.org/stable/2043048

Accessed: 16/10/2009 11:30
THERE ARE NO Q-POINTS IN LAVER'S MODEL
FOR THE BOREL CONJECTURE

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ABSTRACT. It is shown that it is consistent with ZFC that no nonprincipal ultrafilter on \( \omega \) is a Q-point (also called a rare ultrafilter).

All ultrafilters are assumed to be nonprincipal and on \( \omega \).

DEFINITIONS. (1) \( U \) is a \( Q \)-point (also called rare \([C]\)) iff \( \forall f \in \omega^\omega \) if \( f \) is finite-to-one then \( \exists X \subseteq U, f \upharpoonright X \) is one-to-one.

(2) \( U \) is a \( P \)-point iff \( \forall f \in \omega^\omega, \exists X \subseteq U, f \upharpoonright X \) is constant or finite-to-one.

(3) \( U \) is a semi-\( Q \)-point (also called rapid \([C]\)), iff \( \forall f \in \omega^\omega, \exists g \in \omega^\omega, \forall n f(n) < g(n) \) and \( g^\omega \in U \).

(4) \( U \) is semiselective iff it is a \( P \)-point and a semi-\( Q \)-point.

(5) For \( f, g \in \omega^\omega, [f < g \text{ iff } \exists n (f(m) < g(m))]. \)

(6) For \( \mathbb{F} \subseteq \omega^\omega, [\mathbb{F} \text{ is dominant iff } \forall f \in \omega^\omega \exists g \in \mathbb{F}(f < g)] \).

THEOREM 1 (KETONEN [Ke]). If every dominant family has cardinality \( 2^\omega \), then there exists a \( P \)-point.

THEOREM 2 (MATHIAS, TAYLOR [M3]). If there exists a dominant family of cardinality \( \aleph_1 \), then there exists a \( Q \)-point.

Kunen [Ku1] showed that adding \( \aleph_2 \) random reals to a model of ZFC + GCH gives a model with no semiselective ultrafilters. More recently he showed [Ku2] that if one first adds \( \aleph_1 \) Cohen reals (then the random reals) then the resulting model has a \( P \)-point. In either case one has a dominant family of size \( \aleph_1 \) so there is a \( Q \)-point.

THEOREM 3. The following are equivalent:

(1) \( U \) is a semi-\( Q \)-point.

(2) Given \( P_n \subseteq \omega \text{ finite for } n < \omega \) there exists \( X \subseteq U \) such that \( \forall n, |X \cap P_n| < n \).

(3) \( \exists h \in \omega^\omega \text{ such that given } P_n \subseteq \omega \text{ finite for } n < \omega \) there exists \( X \subseteq U \) such that \( \forall n, |X \cap P_n| < h(n) \).

PROOF. (1) \( \Rightarrow \) (2). Let \( f(n) = \text{sup}(\bigcup_{m < n} P_m) + 1 \). Suppose that for all \( n, g(n) > f(n) \); then \( P_n \cap g^\omega \subseteq \{ g(0), \ldots, g(n - 1) \} \).

(3) \( \Rightarrow \) (1). Assume \( f \) increasing. Choose \( n_0 < n_1 < n_2 < \cdots \), so that \( h(k + 1) < n_k \). Let \( P_k = f(n_k) \) and let \( Y \subseteq U \) so that \( |Y \cap P_k| < h(k) \). Then, for each \( m > n_0, |Y \cap f(m)| < m \), since if \( n_k < m < n_{k+1} \) then

0002-9939/80/0000-0023/$02.00

Received by the editors April 3, 1978.

AMS (MOS) subject classifications (1970). Primary 02K05.
\[ |Y \cap f(n_{k+1})| < h(k + 1) < n_k < m. \]

Hence if \( g \in \omega^\omega \) enumerates \( Y - f(n_0 + 1) \) in increasing order then \( \forall n, f(n) < g(n). \) \( \square \)

Define \( U \times V = \{ A \subseteq \omega \times \omega: \{ n: \{ m: (n, m) \in A \} \in V \} \in U \}. \) Whilst \( U \times V \) is never a \( P \)-point or a \( Q \)-point, nevertheless:

**Theorem 4.** \( U \times V \) is a semi-\( Q \)-point iff \( V \) is a semi-\( Q \)-point.

**Proof.** \((\Rightarrow)\) Given \( P_k \subseteq \omega \) finite let \( P_k^* = \{ \langle n, m \rangle: m \in P_k \text{ and } n < m \}. \) Choose \( Z \subseteq U \times V \) so that \( \forall k, |Z \cap P_k^*| < k. \) Let \( n \in \omega \) so that \( Y = \{ m > n: (n, m) \in Z \} \in V \) then \( \forall k, |Y \cap P_k| \leq k. \) (More generally if \( f \cdot U = V \) and \( U \) is a semi-\( Q \)-point and \( f \) is finite-to-one then \( V \) is a semi-\( Q \)-point.)

\((\Leftarrow)\) Given \( P_k \subseteq \omega^2 \) finite, choose \( n_k \) increasing so that \( P_k \subseteq n_k^2. \) Let \( Y \in V \) so that \( \forall k, |n_k \cap Y| < k. \) Let \( Z = \bigcup_{k < \omega} \langle k \rangle \times \{ m: m \in Y \text{ and } m > n_k \} \)

\[ Z \cap P_k \subseteq Z \cap n_k^2 \subseteq k \times (n_k \cap Y) \]

which has cardinality \( \leq (k + 1)^2. \) \( \square \)

**Theorem 5.** In Laver's model \( N \) for the Borel conjecture \([L] \) there are no semi-\( Q \)-points.

**Proof.** Some definitions from [L]:

1. \( T \in \mathfrak{S} \) iff \( T \) is a subtree of \( \omega^{<\omega} \) with the property that there exist \( s \in T \) (called stem \( T \)) so that \( \forall t \in T, t \subseteq s \) or \( s \subseteq t, \) and if \( t \supseteq s \) and \( t \in T \) then there are infinitely many \( n \in \omega \) such that \( t^{<\langle n \rangle} \in T. \)

2. \( \hat{T} > T \) iff \( \hat{T} \subseteq T. \)

3. \( T_s = \{ t \in T: s \subseteq t \text{ or } t \subseteq s \}. \)

4. \( T^0 > \hat{T} \) iff \( T > \hat{T} \) and they have the same stem.

5. For \( x < y < \omega \) let \( \langle x, y \rangle = \{ n < \omega: x < n < y \}. \)

**Lemma 1.** Suppose we are given \( T \in \mathfrak{S} \) and finite sets \( F_s \) for each \( s \in T - \{ \emptyset \} \) such that for each \( s \in T - \{ \emptyset \}: \)

- (a) if \( s = (k_0, \ldots, k_n, k_{n+1}) \), then \( F_s \subseteq [k_n, k_{n+1}] \);
- (b) if \( s = \langle n \rangle \), then \( F_s \subseteq \{0, n\} \);
- (c) \( \exists N < \omega \text{ such that } \forall t \text{ immediately below } s \text{ in } T[F_s] < N. \) For any \( \hat{T} > T \) let \( H_{\hat{T}} = \bigcup \{ F_s: s \in \hat{T} \}. \) Then \( \exists T^1, T^0 \supseteq T \) such that \( H_{T^0} \cap H_{T^1} \) is finite.

**Proof.** We may as well assume that the stem of \( T \) is \( \emptyset. \) Given \( Q \) any infinite family of sets of cardinality \( < N < \omega \) there exists \( G, |G| < N, \exists \hat{Q} \subseteq Q \) infinite so that \( \forall F, \hat{F} \in \hat{Q}, F \cap \hat{F} \subseteq G \) (i.e., a \( \Delta \)-system). Now trim \( T \) to obtain \( \hat{T} > T \) so that \( \forall s \in T, \exists G_s \subseteq [k_n, \omega] \) finite \( (s = (k_0, \ldots, k_n)) \) and for all \( t, \hat{t} \) immediately below \( s \) in \( \hat{T}, (F_t \cap F_{\hat{t}}) \subseteq G_s. \) Build two sequences of finite subtrees of \( \hat{T}: \)

\[ T^0_n \subseteq T^0_{n+1} \cdots, \quad T^1_n \subseteq T^1_{n+1} \cdots \]
so that
\[
\left[ \bigcup_{s \in T_0^i} (F_s \cup G_s) \right] \cap \left[ \bigcup_{s \in T_1^j} (F_s \cup G_s) \right] \subseteq G_{\emptyset}
\]
and \( \bigcup_{n<\omega} T_n^i = T_i \triangleright \hat{T} \) for \( i = 0, 1 \).

This is done as follows: Suppose we have \( T_n^0, T_n^1 \) and we are presented with \( s \in T_n^0 \) and asked to add an immediate extension of \( s \) to \( T_n^0 \). Then since \( \{ F_i - G_i : t \text{ immediately below } s \text{ in } \hat{T} \} \) is a family of disjoint sets and \( G_i \subseteq [k_n, \omega) \) where \( t = (k_0, \ldots, k_n) \) we can find infinitely many \( t \) immediately below \( s \) in \( \hat{T} \) so that
\[
\left[ (F_i - G_i) \cup G_i \right] \cap \left[ \bigcup_{s \in T_i^j} (F_s \cup G_s) \right] = \emptyset. \quad \square
\]

The above is a double fusion argument.

Some more definitions from [L]:
(1) Fix a natural \( \omega \)-ordering of \( \omega^{<\omega} \) and for any \( T \in \mathcal{F} \) transfer it to \( \{ t \in T : \text{stem } T \subseteq t \} \) in a canonical fashion. \( T\langle n \rangle \) denotes the \( n \)th element of \( \{ t \in T : \text{stem } T \subseteq t \} \).
(2) \( \hat{T}^n \triangleright T \) iff \( \hat{T} \triangleright T \) and \( \forall i \geq n, \hat{T}\langle i \rangle = T\langle i \rangle \).
(3) The p.o. \( P_w \) is the \( \omega_2 \) iteration of \( \mathcal{F} \) with countable support \( (p \upharpoonright_a \vdash \text{"}p(a) \in \mathcal{P}^m(G_a)\text{ for all } a \text{ and } \text{supp}(p) = \{ a : p(a) \neq \omega^{<\omega} \} \) is countable).\)
(4) For \( K \) finite and \( n < \omega, p^K_n \triangleright q \) iff \( p \triangleright q \) and \( \forall \alpha \in K, p \upharpoonright_a \vdash \text{"}p(a) \triangleright q(a)\text{"} \).

**Lemma 2.** Let \( f \) be a term denoting the first Laver real and \( \tau \) any term. If \( p \in P_{\omega_2} \) and \( p \vdash \text{"}\tau \in \omega^{<\omega}, \forall n (f(n) < \tau(n)) \text{ and } \tau \text{ increasing} \" \) then \( \exists Z_0, Z_1 \) such that \( Z_0 \cap Z_1 \) is finite and \( \exists p_0, p_1 \) such that \( p_i \vdash \text{"}\tau \omega \subseteq Z_i \text{"} \) for \( i = 0, 1 \).

**Proof.** Construct a sequence \( p \triangleleft_0 p_0 \triangleleft_0 p_1 \ldots \) so that \( \bigcup_{n<\omega} K_n = \bigcup_{s<\omega} \text{supp}(p_n) \) and \( 0 \in K_0 \). Having gotten \( p_n \), let \( s = (k_0, \ldots, k_m) \) be \( p_n(0)\langle n \rangle \). Fix \( t = (k_0, \ldots, k_m, k_{m+1}) \) in \( p_n(0) \). Then for each \( i < m + 1 \),
\[
p_t = \langle p_n(0), t \rangle \cup p_n \upharpoonright [1, \omega_2] \vdash \text{"}\tau(i) > k_{m+1} \text{ or } \forall \omega, \xi \leq k_{m+1}, \tau(i) = 1\text{"}.
\]
Hence by applying Lemma 6 of [L] \( m + 2 \) many times we can find \( q_s \triangleleft p_t \) and \( F_s \subseteq [k_m, k_{m+1}] \) such that \( |F_s| < (m + 2)(n + 1)|K_n| \) and \( q_s \vdash \text{"}\tau \omega \cap \{k_m, k_{m+1}\} \subseteq F_t\text{"} \). (Note \( p_t \vdash \text{"}\forall i \geq m + 1, \tau(i) > k_{m+1}\text{"} \). Let \( p_{n+1}(0) = (p_n(0) - p_n(0), t \text{ is immediately below } s \text{ in } p_n(0)) \). Let \( p_{n+1}(1, \omega_2) \) be a term denoting \( q_t \upharpoonright [1, \omega_2] \) if \( q_t(0) \) or \( q_t(1, \omega_2) \) if \( p_n(0) - \{ t : s \subseteq t \} \). Hence \( p_{n+1} \triangleleft_0 p_n \). Now let \( \tilde{p} \) be the fusion of the sequence of \( p_n \) (see [L, Lemma 5]). Then for each \( t \in \tilde{p}(0) \) if \( t = (k_0, \ldots, k_m, k_{m+1}) \) and \( t \supseteq \text{stem } \tilde{p}(0), \) then \( \langle \tilde{p}(0) \cup \tilde{p} \upharpoonright [1, \omega_2] \rangle \vdash \text{"}\tau \omega \cap \{k_n, k_{n+1}\} \subseteq F_t\text{"} \). For \( t \in \tilde{p}(0) \) and \( t \nsubseteq \text{stem } \tilde{p}(0) \) let \( F_t = k_{m+1} \). Applying Lemma 1 obtain \( T_0, T_1 \triangleright \tilde{p}(0), Z_0 \) and \( Z_1 \) such that \( Z_0 \cap Z_1 \) is finite, and \( \langle T_t \cup p \upharpoonright [1, \omega_2] \rangle \vdash \text{"}\tau \omega \subseteq Z_i\text{"} \) for \( i = 0, 1 \). \( \square \)
Proof of Theorem 5. Suppose $M[G_{\omega_2}] \models "U is a semi-$Q$-point". Applying an argument of Kunen's we get $\alpha < \omega_2$ such that $U \cap M[G_\alpha] \in M[G_\alpha]$. $(M[G_\beta] \models "\text{CH}"$ for all $\beta < \omega_2$ so construct using $\omega_2$-c.c., $\alpha_\lambda < \omega_2$ for $\lambda < \omega_1$ so that $\forall x \in M[G_{\alpha_\lambda}] \cap 2^\omega, \text{P}_{\alpha_\lambda+1}$ decides "$x \in U"$. Let $\alpha = \text{sup} \alpha_\lambda$. Note $M[G_\alpha] \cap 2^\omega = \bigcup_{\beta < \alpha} M[G_\beta] \cap 2^\omega$ since $\aleph_1$ is not collapsed.) By [L, Lemma 11] we may assume $U \cap M \in M$. But Lemma 2 clearly implies that for any $V \text{ult. in } M, M[G_{\omega_2}] \models "\text{no extension of } V \text{ is a semi-} Q\text{-point."} \tag*{□}

Remarks. (1) A similar argument shows that in the model gotten by $\omega_2$ iteration of Mathias forcing with countable support there are no semi-$Q$-points. In fact, as Mathias later pointed out to me, the appropriate argument needed is an easy generalization of Theorem 6.9 of [M2].

(2) In [M1] Mathias shows $[\omega \rightarrow (\omega)_{\omega_1}] \Rightarrow [\text{There are no rare filters or nonprincipal ultrafilters].}$

(3) In neither the Laver or Mathias models are there small dominant families so by Ketenen [Ke] there is a $P$-point. Also it is easily shown no ultrafilter is generated by fewer then $\aleph_2$ sets.

(4) Not long after the results of this paper were obtained, Shelah showed that it is consistent that no $P$-points exist [W]. In his model there is a dominant family of size $\aleph_1$, so there are $Q$-points. It remains open whether or not it is consistent that there are no $P$-points or $Q$-points.

Conjecture. Borel conjecture $\Leftrightarrow$ there does not exist a semi-$Q$-point.

References


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