A MINIMAL DEGREE WHICH COLLAPSES $\omega_1$

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Abstract. We consider a well-known partial order of Prikry for producing a collapsing function of minimal degree. Assuming $\text{MA} + \neg \text{CH}$, every new real constructs the collapsing map.

Let $\omega^{<\omega}_1$ be the tree of finite sequences from $\omega_1$. Define the partial order $P$ to be the set of all nonempty subtrees $T$ of $\omega^{<\omega}_1$ which satisfy: for all $s \in T$ there exists $t \geq s$ such that $\{x : t^x x \in T\}$ is uncountable. The ordering on $P$ is inclusion. This partial order was first considered by Prikry, who also showed that it gives a minimal collapsing function (see Abraham (1987)).

Theorem. Suppose $M \models \text{"ZFC + MA + \neg CH"}$. Then for any $G$ $P$-generic over $M$,

1. $M[G] \models \text{"$\omega_1$ is countable"};
2. for every real $x \in M[G]$, $x \in M$ or $G \in M[x]$.

Note that (1) and (2) are impossible if $M \models \text{"CH"}$. This is because collapsing the continuum to $\omega$ always introduces Cohen reals, random reals, etc.

Let us give some definitions. For $p \in P$ we say that $s \in p$ is a splitting node of $p$ iff $\{\alpha : s^\alpha x \in p\}$ is uncountable. We say that $s \in p$ is a level $n$ node iff $\{t : t \subseteq s$ and $t$ is a splitting node of $p\}$ has size $n$. We say that $p \leq n q$ iff $p \leq q$ and all level $n$ nodes of $q$ are still in $p$. The standard fusion argument shows that if $P_n \leq n P_n$ for each $n < \omega$, then the fusion $(\bigcap_{n<\omega} P_n)$ is an element of $P$. For any $p \in P$ and $s \in p$ define $p_s = \{t \in p | t \subseteq s$ or $s \subseteq t\}$.

Now suppose $q \in P$ and $\tau$ is a term such that $q \models \neg \text{"$\tau \in 2^{<\omega}$"}$.

Lemma 1. There exist $p \leq q$ and $F : p \rightarrow 2^{<\omega}$ such that

(a) for all $n < \omega$, $F^P(p \cap \omega_1^n) \subseteq 2^n$, and
(b) for all $s \in p$, $p_s \models \text{"$F(s) \subseteq \tau$"}$.

Proof. This is an easy fusion argument. Given $q \in P$ and $s \in q \cap \omega_1^n$ which we want to retain as a splitting node, simply extend each $q_{s^{-1}}$ to decide $\tau \upharpoonright n$, then $\omega_1$ of $q_{s^{-2}}$ decide $\tau \upharpoonright n$ the same way. So build a sequence $q_{n+1} \leq n q_n \leq q$ such that for every level $n$ node $s$ of $q_{n+1}$, $(q_{n+1})_s$ decides $\tau \upharpoonright \text{length}(s)$. The fusion of the $q_n$'s is $p$. \hfill $\Box$

From now on assume that $p \models \neg \text{"$\tau \notin M$"}$ and $p$ and $F : p \rightarrow 2^{<\omega}$ are from Lemma 1.

Lemma 2. Suppose $p_\alpha \leq p$ for $\alpha < \omega_1$. Then there exist $q_\alpha \leq p_\alpha$ and $C_\alpha \subseteq 2^\omega$ closed

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for each \( \alpha < \omega_1 \) such that the \( \{ C_\alpha : \alpha < \omega_1 \} \) are disjoint and for each \( \alpha < \omega_1 \):

\[
q_\alpha \models \"\tau \in C_\alpha \".
\]

**Proof.** Define the partial order \( Q_\alpha \) by \((T_1, T_2, n) \in Q_\alpha \) iff

1. \( T_1 \) is a finite subtree of \( p_\alpha \cap \omega_1^{\leq n} \) with every branch of length \( n \);
2. \( T_2 \) is a finite subtree of \( 2^{\leq n} \) with every branch of length \( n \); and
3. for all \( s \in T_1, F(s) \in T_2 \).

We define \((\hat{T}_1, \hat{T}_2, \hat{n}) \leq (T_1, T_2, n) \) iff

1. \( \hat{n} \geq n \);
2. \( \hat{T}_1 \supseteq T_1 \); and
3. \( \hat{T}_2 \) is an end extension of \( T_2 \) (i.e. \( \hat{T}_2 \cap 2^{\leq n} = T_2 \)).

It is easy to see that \( Q_\alpha \) has the countable chain condition, since if \( T_1^p = T_1^q \), then \( p \) and \( q \) are compatible. Now let \( Q \) be the direct sum of \( \{ Q_\alpha : \alpha < \omega_1 \} \).

Since each \( Q_\alpha \) has property \( K \) (in fact is \( \sigma \)-centered), \( Q \) has the c.c.c. A partial order has property \( K \) if every subset of cardinality \( \omega_1 \) contains a subset of \( \omega_1 \) pairwise compatible elements.

It is not hard to see that the product of two partial orders with property \( K \) has property \( K \), and the direct sum of such orders has the property. Also MA + \( \neg \)CH implies that every c.c.c. order has property \( K \).

**Claim 1.** Given \( q \in Q_\alpha \) and \( r \in Q_\beta \) there exist \( \hat{q} \leq q \) and \( \hat{r} \leq r \) with the same \( n \) and \( T_1^q \cap T_2^r \cap 2^n = \emptyset \).

**Proof.** This is where \( \neg \)CH is used. For each \( s \in T_1^q \) let \( x_s : \omega \to \omega_1 \) be a branch of \( p \) extending \( s \) and let \( y_s : \omega \to 2 \) be \( \{ F(x_s \upharpoonright n) : n < \omega \} \). (I.e. so \( p_{x_s \upharpoonright n} \models \neg \tau \upharpoonright n = y_s \upharpoonright n \)). Since \( p \models \neg \tau \in M \), there exists for each \( s \in T_1^q \) some \( \hat{s} \supseteq s \) such that \( F(\hat{s}) \) is incompatible with all of the \( y_s \)'s. Now it is easy to prove Claim 1. \( \square \)

For any \( G \) a \( Q_\alpha \) filter let \( q_\alpha = \bigcup \{ T_1^p : p \in G \} \) and let \( \dot{C}_\alpha = \bigcup \{ T_2^p : p \in G \} \).

**Claim 2.** There are \( \omega_1 \) dense subsets of \( Q_\alpha \) such that if \( G \) is any \( Q_\alpha \) filter meeting them all, then \( q_\alpha \in P \).

**Proof.** For any \( s \in p \) and \( \beta < \omega_1 \) let \( D_\beta^s = \{ q \in Q_\alpha : s \in T_1^q \) and there exists \( t \in T_1^q, t \supseteq s, \) and range\( (t) \) contains some \( \gamma > \beta \} \). It is easy to see that \( D_\beta^s \) is dense beneath the set of \( q \) such that \( s \in T_1^q \). Consequently if we let

\[
E_\beta^s = D_\beta^s \cup \{ q : q \models \neg \tau \in T_1^q \}) \},
\]

then \( E_\beta^s \) is dense in \( Q_\alpha \). If \( G \) meets each \( E_\beta^s \) for \( s \in p \) and \( \beta < \omega_1 \), then \( q_\alpha \in P \). \( \square \)

Note that \( q_\alpha \models \forall n \tau \upharpoonright n \in \dot{C}_\alpha \). The lemma follows easily from the claims and MA + \( \neg \)CH. \( \square \)

Using Lemma 2 and a fusion argument, find \( q \leq p \) such that for all \( s \in q \) there exists \( \langle C_\alpha^s : s \upharpoonright \alpha < q \rangle \), a family of disjoint closed sets, such that \( q_\alpha \upharpoonright \tau \models \tau \in C_\alpha \)”. Thus \( q \models \neg \tau \in M[\tau^G] \) and the theorem is proved. \( \square \)

**Remarks.** Assume that \( M \models \text{"MA + \( \neg \)CH"} \) and \( G \) is Prikry collapsing generic over \( M \). Then for \( f \in M[G] \cap \omega_\omega \) there exists \( g \in M \cap \omega_\omega \) such that for every \( n < \omega, f(n) < g(n) \). Also for every \( X \in M[G] \cap [\omega]^\omega \) there exists \( Y \in M \cap [\omega]^\omega \) such that \( Y \subseteq X \) or \( X \cap Y = \emptyset \). And every meager set (measure zero set) coded in \( M[G] \) is covered by one coded in \( M \). All of these properties are true when \( G \) is Sacks generic over \( M \). The proofs are similar here with the addition of a suitable forcing notion to apply Martin’s axiom.
The fact that \( \omega_1 \) is collapsed but every element of \( \omega^\omega \) is dominated by a ground model element of \( \omega^\omega \) implies that in the ground model the Boolean algebra associated with the Prikry collapse is \((\omega, \omega)\)-weakly distributive but not \((\omega, \omega_1)\)-weakly distributive. This is also true of Namba forcing (see Namba (1972)).

Of course, in our theorem we only needed that \( M \models "\text{MA}(K)" \), since we only did property \( K \) forcing. If, in addition, \( M \models "\text{there are no Souslin trees}" \), then for every set of ordinals \( X \in M[G], X \in M \) or \( G \in M[X] \). Since a branch through a Souslin tree cannot be minimal, this assumption is necessary. The proof is left as an exercise for the reader.

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