Math 421: Theory of Calculus

Preliminaries: Logic and Proofs

- A **proposition** is a sentence that has exactly one truth value: T = true or F = false.

$7^2 = 60$

Euclid was left handed.

This sentence is false.

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- A **proposition** is a sentence that has exactly one truth value: T = true or F = false.

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This sentence is false.

(self referential, logical paradox, see Bertrand Russell)

- The **negation** of a proposition P is “not P”: ~P

Truth Table for negation:

True = 1

False = 0

P	**~P**
0	1
1	0

Note: ~(~P) = P.

Conjunction

- The **conjunction** of propositions P and Q:

P **and** Q

is true exactly when both P and Q are true.

P	Q	P and Q
1	1	1
1	0	0
0	1	0
0	0	0

Disjunction

- The **disjunction** of propositions P and Q:

P **or** Q

is true when one of P or Q is true.

P	Q	P and Q
1	1	1
1	0	1
0	1	1
0	0	0

Exercise

- Make a truth table for

  (P or ~Q) and R

Tautology

- A **tautology** is a propositional form that is true for every assignment of truth values to its components.

Contradiction

- A **contradiction** is a propositional from that is false for every assignment of truth values to its components.

Law of the Excluded Middle

Theorem: P or ~P is a tautology.

Check it!

Can you find a contradiction?

Equivalence

- *P* is **equivalent** to *Q*: $P \Leftrightarrow Q$

If *P* and *Q* have the same truth tables.

Exercise

P xor *Q*, means either *P* or *Q* is true, but not both (exclusive or).

Show that:

P xor *Q* $\Leftrightarrow$ (*P* or *Q*) and ~(*P* and *Q*)

Conditionals

- *P* **implies** *Q*, “if *P* then *Q*”: $P \Rightarrow Q$

Has the truth table:

P	Q	P ⇒ Q
1	1	1
1	0	0
0	1	1
0	0	1

From falsehood anything can follow!

Exercise:

Show that

$$(P \Rightarrow Q) \Leftrightarrow (\sim P \text{ or } Q)$$

- The **converse** of $P \Rightarrow Q$ is $Q \Rightarrow P$.

Show that $P \Rightarrow Q$ is **not** equivalent to $Q \Rightarrow P$.

- The **contrapositive** of $P \Rightarrow Q$ is $\sim Q \Rightarrow \sim P$.

Show that $P \Rightarrow Q$ is equivalent to $\sim Q \Rightarrow \sim P$.

Proof by Contradiction

If you want to prove that $P \Longrightarrow Q$ **by contradiction**, assume (P and ~Q) and come to a contradiction.

Show: ((P and ~Q) = F)$\Leftrightarrow$ ($P \Longrightarrow Q$)

“There is no royal road to mathematics.”

after ***Euclid***

Quantifiers

- Let P(x) denote a property of the object x.
- **There exists**: $\exists$

$\exists\, x \in X:\ P(x)$

Means: there exists at least one object x in the set X which has the property P.

- **For all**: $\forall$

$\forall\, x \in X:\ P(x)$

Means that for each object x in the set X, x has property P.

Note that for the **negation of a statement** we have:

De Morgan's Law:

- (i) ~(P and Q) = (~P) or (~Q)
- (ii) ~(P or Q) = (~P) and (~Q)

and

- (iii) $\sim(\forall x \in X : P(x)) = (\exists x \in X : \sim P(x))$
- (iv) $\sim(\exists x \in X : P(x)) = (\forall x \in X : \sim P(x))$.

Negation with quantifiers

Negate: $\forall x \in X : P(x)$

- Replace ∃ with ∀
- Replace ∀ with ∃
- Replace *P(x)* with ~ *P(x)*

$\exists x \in X : \sim P(x)$

Show:

(iii) $\sim(\forall x \in X : P(x)) \Leftrightarrow (\exists x \in X : \sim P(x))$

Exercise:

Consider the statements

(a) $\exists x \in \mathbb{R}\ \forall y \in \mathbb{R}\ x + y > 0$;
(b) $\forall x \in \mathbb{R}\ \exists y \in \mathbb{R}\ x + y > 0$;

1. Are the statements true or false?
2. Find their negations.

Exercise

Let f, g be two functions defined from $\mathbb{R}$ into $\mathbb{R}$. Translate using quantifiers the following statements:

1. f is bounded above;

2. f is even;

3. f is never equal to 0;

4. f is increasing;

5. f is less than g.

Exercise:

Let $f : \mathbb{R} \to \mathbb{R}$. Find the negations of the following statements:

a. For any $x \in \mathbb{R}$: $f(x) \leq 1$.

b. The function f is increasing.

Elementary Set Theory

- If ***x* is an element of the set *A***: $x \in A$, otherwise x is not in A: $x \notin A$.
- If A and B are sets,

then $A \subseteq B$, ***A* contained in *B***, if each element of A is also an element of B.

Equivalently: $B \supseteq A$, B contains A.

Sets A and B are **equal**: $A = B$, if $A \subseteq B$ and $B \subseteq A$

Union and Intersection

Given sets *A* and **B**, we define:

- "*A* union *B*" :

  $A \cup B = \{x\text{: } x \in A \text{ or } x \in B \text{ or both}\}$

- "*A* intersect *B*" :

  $A \cap B = \{x\text{: } x \in A \text{ and } x \in B\}$

The Complement

Let A and B be subsets of X. Then

$A \setminus B = A$ *minus* $B = \{x \in X : x \in A$ and $x \notin B\}$

is the ***relative complement*** of B in A. When the set X is clear from the context we write also

$A^c = A$ complement $= X \setminus A$

and call A^c the ***complement*** of A.

Exercise

For two sets A and B show that the following statements are equivalent:

a) $A \subseteq B$

b) $A \cup B = B$

c) $A \cap B = A$