Differentiation

Definition of the Derivative

Question

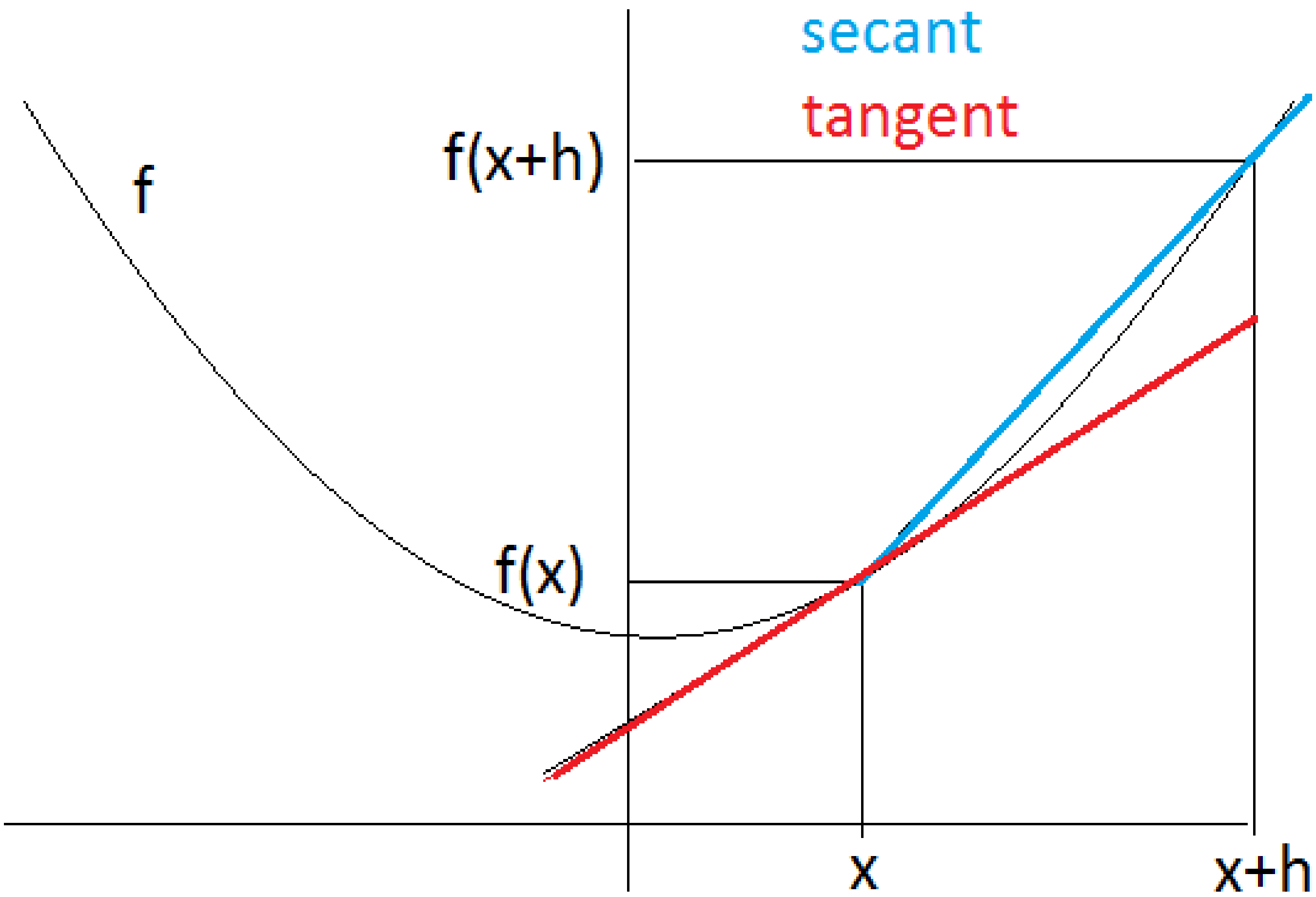
Given a "well-behaved" function graph, can we draw a tangent line at *x*? Which *x*?

for $f(x) = |x|$?

for $f(x) = \begin{cases} x^2 & x \quad 0 \\ x & x > 0 \end{cases}$?

for $f(x) = \sqrt{|x|}$?

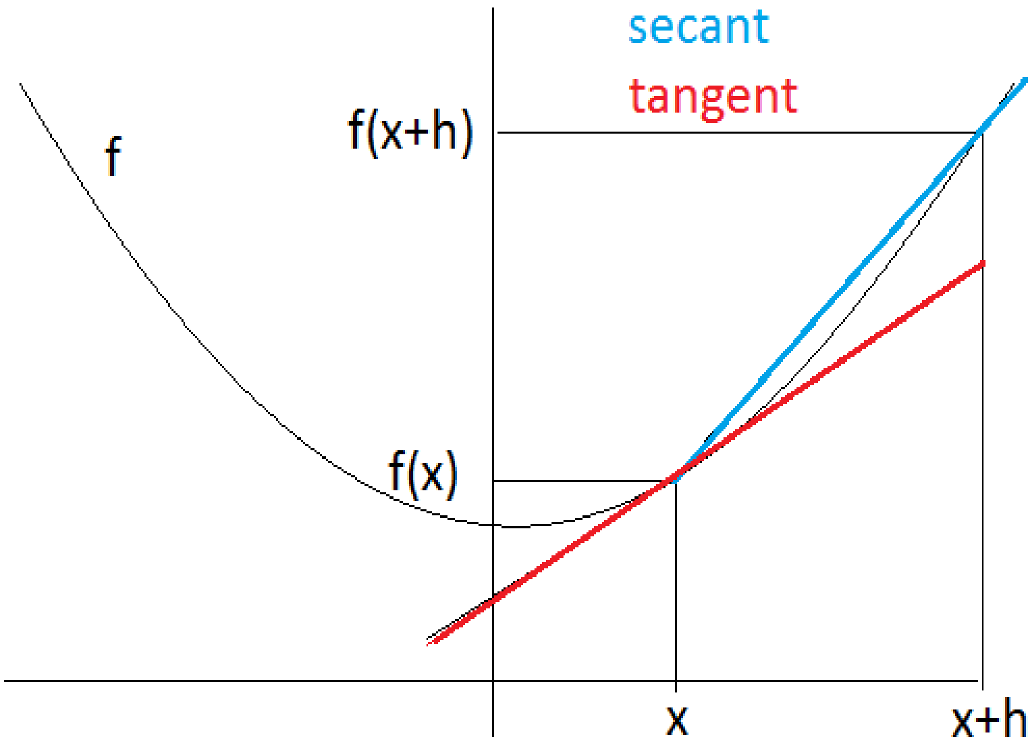
Conceptually the tangent is the limit of the secant lines:



the secant line through (a,f(a)) and (a+h,f(a+h)) has slope $\frac{f(a+h)-f(a)}{h}$

so the tangent through (a,f(a)) has slope

$$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$



Definition

f is differentiable at a if

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \text{ exists.}$$

$f'(a)$ is the derivative of f at a.

f is differentiable at a.

Exercises

Show that the derivative of

1. $f(x) = x^3 - x$ at x = 2 is 11
2. $f(x) = \frac{x}{x+1}$ at x = 2 is $\frac{-1}{9}$
3. $f(x) = \sqrt{x+1}$ at x = 3 is $\frac{1}{4}$

Exercise

Find the onesided derivatives of

1. $f(x) = |x|$ at x = 0

2. f(x) = $\begin{cases} x^2 & x \quad 0 \\ x & x > 0 \end{cases}$ at x = 0

3. $f(x) = \sqrt{|x|}$ at x = 0

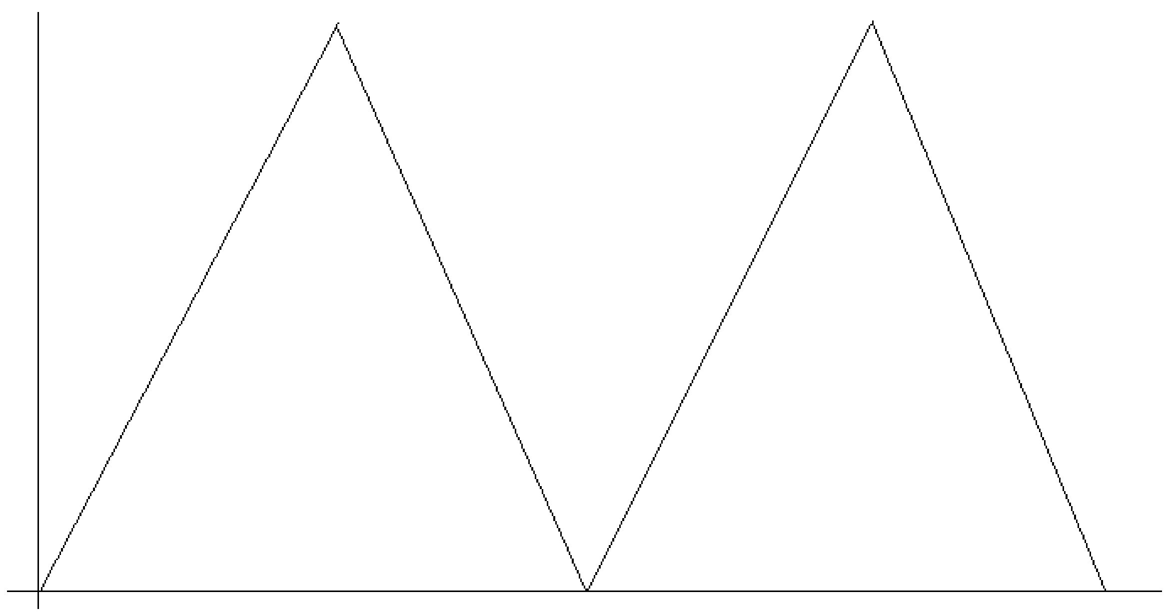
Theorem 9.1

If f is differentiable at a, then f is continuous at a.

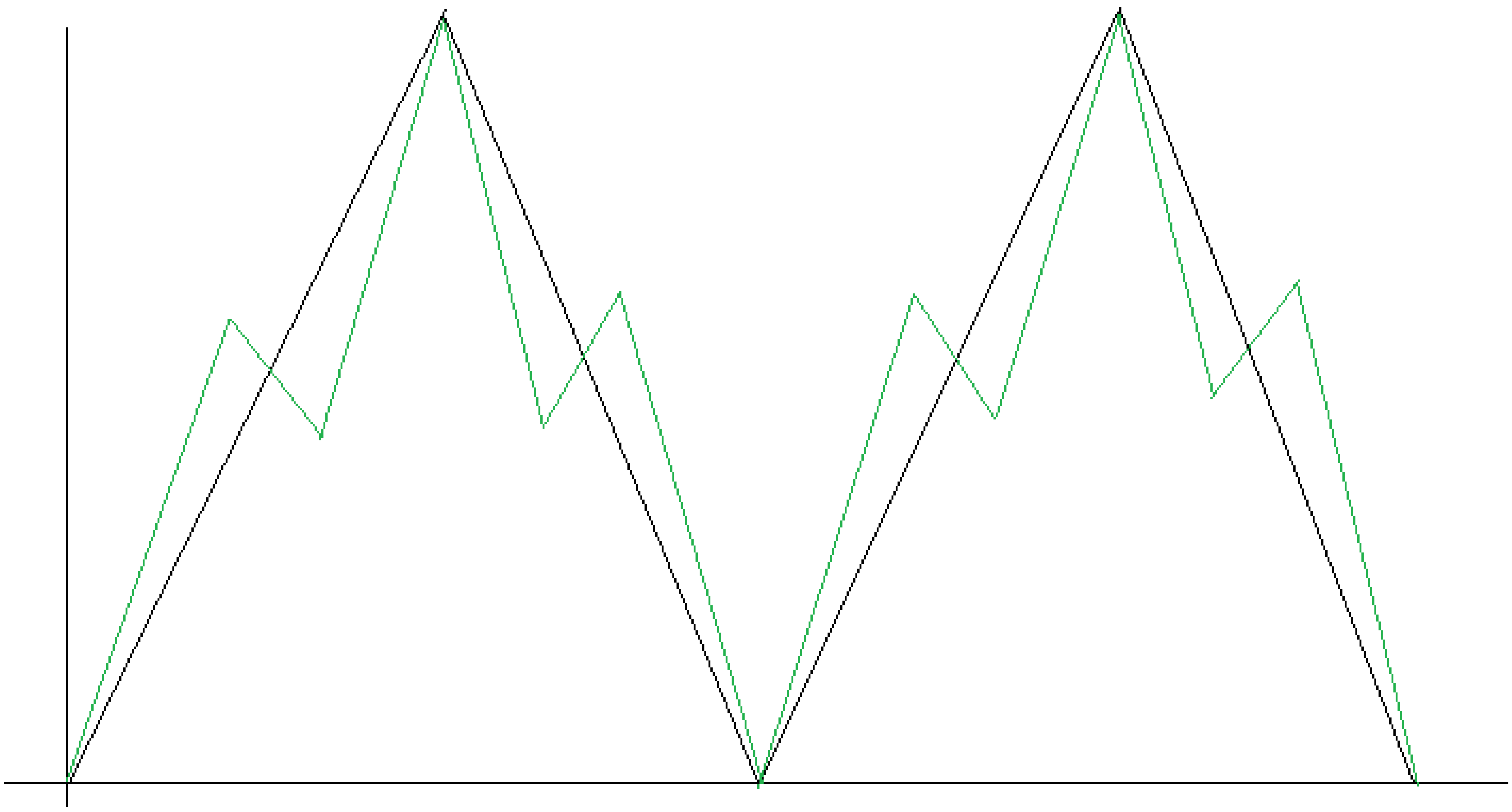
Try to prove it.

But: f continuous **does not imply** f differentiable

First iteration of the construction of a continuous but not differentiable function



second iteration in the construction of a continuous but not differentiable function, etc.



Exercise

Suppose f is differentiable and $g(x) = f(x+c)$.
Prove $g'(x) = f\,'(x+c)$.

Exercise

Let f be a function such that $|f(x)| \leq x^2$ for all x.

Then f is differentiable at 0.

Exercise

Determine if these functions are differentiable at 0

(i) $f(x) = \begin{cases} 0 & x = 0 \\ \sin(\frac{1}{x}) & x \neq 0 \end{cases}$

(ii) $f(x) = \begin{cases} 0 & x = 0 \\ x\sin(\frac{1}{x}) & x \neq 0 \end{cases}$

(iii) $f(x) = \begin{cases} 0 & x = 0 \\ x^2\sin(\frac{1}{x}) & x \neq 0 \end{cases}$

Squeeze Theorem

Suppose $f(a) = g(a) = h(a)$ and for all x

$f(x) \leq g(x) \leq h(x)$. Also $f'(a) = h'(a)$.

Then:

g is differentiable at a and

$$f'(a) = g'(a) = h'(a).$$

Try to prove it.

Higher Order Differentiation

We can compute higher order derivatives:

$$f'(x), f''(x) = f^{(2)}(x), f^{(3)}(x), \ldots f^{(n)}(x), \ldots$$

in Leibniz notation:

$$\frac{\partial f}{\partial x}, \frac{\partial^2 f}{\partial x^2}, \frac{\partial^3 f}{\partial x^3}, \ldots, \frac{\partial^n f}{\partial x^n}, \ldots$$

Exercise

A function *f* has a ***symmetric derivative*** at a point if

$$f'_s(x) = \lim_{h \to 0} \frac{f(x+h) - f(x-h)}{2h}$$

exists. Show that $f'_s(x)$ = *f'*(*x*) at any point at which the latter exists but

that $f'_s(x)$ may exist even when *f* is not differentiable at *x*.

Exercise

Find all points where

$$f(x) = \sqrt{1 - \cos x}$$

is not differentiable and at those points find the one-sided derivatives.

Exercise

Let a function $f : \mathbb{R} \rightarrow \mathbb{R}$ be defined by setting $f(1/n) = c_n$ for $n = 1, 2, 3,$

. . . where $\{c_n\}$ is a given sequence and elsewhere $f(x) = 0$.Find a condition

on that sequence so that $f'(0)$ exists.

Exercise

Show that a function f that satisfies an inequality of the form

$$|f(x) - f(y)| \leq M|x - y|$$

for some constant M and all x, y must be everywhere continuous but need

not be everywhere differentiable

Exercise

Find an example of an everywhere differentiable function f so that f' is not everywhere continuous.

Exercise

The Dirichlet function is discontinuous at each rational number. By Theorem 9.1 this function has no derivative at any rational number. Does it have a derivative at any irrational number?

Exercise

Let $f : (-a, a) \to \mathbb{R}$, with $a > 0$. Assume $f(x)$ is continuous at 0 and such that the limit

$$\lim_{x \to 0} \frac{f(x) - f(\kappa x)}{x} = l$$

exists, where $0 < \kappa < 1$. Show that $f'(0)$ exists. What happens to this conclusion when $\kappa > 1$?

Exercise

Let $f : [a, b] \to \mathbb{R}$ be continuous and differentiable everywhere in (a, b) except maybe at $c \in (a, b)$. Assume that

$$\lim_{x \to c} f(x) = l$$

Show that $f(x)$ is differentiable at c and $f(c) = l$.