Differentiation

Differentiation Rules

Theorem 1

If f is a constant function $f(x) = c$, then $f'(x) = 0$ for all x.

Theorem 2

If $f(x) = x$ then $f'(x) = 1$.

Try to prove it.

Theorem 3

If f, g are differentiable at a then

(i) $(f+g)'(a) = f'(a) + g'(a)$

(ii) $(c \cdot f)'(a) = c \cdot f'(a)$, for any constant g

(iii) $(f \cdot g)(a) = f'(a) \cdot g(a) + f(a) \cdot g'(a)$

(iv) and if $g(a) \neq 0$:

$$\left(\frac{1}{g}\right)'(a) = \frac{-g'(a)}{\left(g(a)\right)^2}$$

(v) $\left(\frac{f}{g}\right)'(a) = \frac{f'(a)g(a) - f(a)g'(a)}{\left(g(a)\right)^2}$

Exercise

Prove by induction:

If $n \in \mathbb{N}$, and $f(x) = xn$,

then $f'(x) = n\ x^{n-1}$.

Note:

- Using (i), (ii), (iii) and the previous exercise, we can differentiate polynomials
- using (iv), (v), we can differentiate rational functions

Exercise (higher order derivatives)

Suppose $f(x) = x^{-n},\ n \in \mathbb{N}$

Prove by induction that

$$f^{(k)}(x) = (-1)^k\, k! \binom{n+k-1}{n-1} x^{-n-k}$$

Given without proof

$$\frac{d}{dx}\sin x = \cos x$$

$$\frac{d}{dx}\cos x = \sin x$$

Exercise

Prove by induction:

$$(f_1 f_2 \ldots fn)'(x) = \sum_{i=1}^{n} f_1(x) \ldots fi_{-1}(x) f_i'(x) f_{i+1}(x) \ldots fn(x)$$

Theorem (Chain Rule)

If g is differentiable at a and f is differentiable at g(a), then $f \circ g$ is differentiable at a and

$$(f \circ g)'(a) = f'(g(a)) \cdot g'(a)$$

go over proof.

Exercise

1. For $f(x) = \frac{\sin(\cos x)}{x}$, find $f'(x)$.
2. For $f(x) = \frac{1}{1+x}$, find $f'(f(x))$.

Exercise

1. If $f + g$ is differentiable at a, are f and g necessarily differentiable at a?
2. If $f \cdot g$ and f are differentiable at a, what conditions on f imply that g is differentiable at a?

Exercise

Show that

$$(fg)''(x) = f''(x)g(x) + 2f'(x)g'(x) + f(x)g''(x)$$

under appropriate hypotheses.

Exercise

Suppose f is differentiable at 0 and that f(0) = 0.

Prove that f(x) = xg(x) for some function g, which is continuous at 0.

What happens if we try to write $g(x) = \frac{f(x)}{x}$?

Exercise (Chain Rule related)

Give an explicit example of functions *f* and *g* such that the "proof" of the

chain rule based on the equation

$$\frac{f\big(g(x)\big) - f(g(x_0))}{x - x_0} = \frac{f\big(g(x)\big) - f(g(x_0))}{g(x) - g(x_0)} \cdot \frac{g(x) - g(x_0)}{x - x_0}$$

fails.