Significance of the Derivative

Maxima, Minima and the Mean Value Theorem

Definition

Suppose f is a function, $A \subset \text{domain of } f$,

$x \in A$ is a ***maximum value for f over A*** if

$\forall\, x, y \in A$: $f(x) \geq f(y)$

Analogously,

$x \in A$ is a ***minimum value for f over A*** if

$\forall x, y \in A$: $f(x) \leq f(y)$

Theorem 11.1

f a function on (a,b). If x is a max (min) for f on (a,b) and f differentiable at x, then

$$f'(x) = 0.$$

Definition

Suppose f is a function, $A \subset \text{domain of } f$,

$x \in A$ is a ***local maximum for f*** if

$$\exists \delta > 0, x \text{ is a max on } A \cap (x - \quad, x + \delta)$$

Analogously,

$x \in A$ is a ***local minimum for f*** if

$$\exists \delta > 0, x \text{ is a min on } A \cap (x - \quad, x + \delta)$$

Theorem 11.2

If f is defined on (a,b) and has a local max/min at x in (a,b) and f differentiable at x, then

$$f'(x) = 0.$$

Question

Does $f'(x) = 0$ imply f has a local max/min at x?

Definition

A ***critical point*** of f is a number x such that $f'(x) = 0$.

Exercise

Let (a,b) be a point in the plane, L a line $y = mx + b$.

Find the point P on the line such that P minimizes the distance of the line to (a,b).

Work up to the Mean Value Theorem

Theorem 3: Rolle's Theorem

If f is continuous on [a,b] and differentiable on (a,b) and $f(a) = f(b)$, then there is a number $x \in (a, b)$ such that $f'(x) = 0$.

Exercise

Prove that if

$$\frac{a_0}{1} + \frac{a_1}{2} + \cdots + \frac{a_n}{n+1} = 0$$

then

$$a_0 + a_1 x + \cdots + anxn = 0$$

Theorem 4: Mean Value Theorem

If f is continuous on [a,b] and differentiable on (a,b), then there is a number $x \in (a, b)$ such that $f'(x) = \frac{f(b)-f(a)}{b-a}$.

Corollary 11.1

If f is defined on the interval I and $f'(x) = 0$ for all x in I, then f is constant on I.

Can you prove it?

Corollary 11.2

If f is defined on the interval I and $f'(x) = g'(x)$ for all x in I,

then there exists a constant c such that $f(x) = g(x) + c$.

Definition

f is ***increasing*** on the interval I,

if for all $a, b \in I, a < b$: $f(a) < f(b)$

f is ***decreasing*** on the interval I,

if for all $a, b \in I, a < b$: $f(a) > f(b)$

Corollary 11.3

Given $f'(x) > 0$ for all *x* in the interval *I*, then f is increasing on *I*.
Given $f'(x) < 0$ for all *x* in the interval *I*, then f is decreasing on *I*.

Can you prove it?

Exercise

Prove that if f' is increasing, then every tangent line intersects the graph of f only once.

Theorem 12.5: Second Derivative Test

Suppose $f'(a) = 0$.

If $f''(a) > 0$, then f has a local min at a.

If $f''(a) < 0$, then f has a local max at a.

Exercise

Suppose that $f: [0,1] \to [0,1]$ is continuous and differentiable and $f(x) \neq 1$ for all $x \in [0,1]$.

Show: There exists exactly one number $x \in [0,1]$ such that $f(x) = x$.

Theorem 11.6

Suppose $f''(a)$ exists.

If f has a local min at a, then $f''(a) \geq 0$.

If f has a local max at a, then $f''(a) \leq 0$.

Theorem 11.7

Suppose f continuous at a,

f'' exists for all x in some interval containing a,

except perhaps for $x = a$.

Suppose $\lim_{x \to a} f'(x)$ exists.

Then $f'(a)$ exists and $f'(a) = \lim_{x \to a} f'(x)$.

Exercise

Find a function f such that $|f|$ is differentiable, but f is not.

Exercise

Show: If $|f|$ is differentiable at a, f is continuous at a,

then f is differentiable at a.

Theorem 11.8
Cauchy Mean Value Theorem

Suppose f, g continuous on [a,b] and differentiable on (a,b)

Then there is an $x \in (a, b)$ such that

$$[f(b) - f(a)]g'(x) = [g(b) - g(a)]f'(x)$$

The point is: it's not x_1 for g and x_2 for f, it's the same x for both!

Theorem 11.9
L'Hopital's Rule (follows from 11.8)

Given $\lim_{x \to a} f(x) = 0$, $\lim_{x \to a} g(x) = 0$ and suppose $\lim_{x \to a} \frac{f'(x)}{g'(x)}$ exists.

Then $\lim_{x \to a} \frac{f(x)}{g(x)}$ exists and is equal to $\lim_{x \to a} \frac{f'(x)}{g'(x)}$.

Exercise

Prove that if the function $f : I \to \mathbb{R}$ has a bounded derivative on I, then f is uniformly continuous on I. Is the converse true?

Exercise

Let *f* be a function on [*a, b*] that is differentiable at *c*. Let *L*(*x*) be the tangent

line to *f* at *c*. Prove that *l* is the unique linear function with the property that

$$\lim_{x \to c} \frac{f(x) - L(x)}{x - c} = 0.$$

Exercise

Problem 6.17 Let $f : \mathbb{R} \to \mathbb{R}$ differentiable everywhere. Let $x_0 \in \mathbb{R}$ and $h \in \mathbb{R}$. Show that there exists $\vartheta \in (0, 1)$ such that

$$f(x_0 + h) - f(x_0) = hf(x_0 + \vartheta h) .$$

Set $f(x) = \frac{1}{1+x}$

and $x_0 > 0$. Find the limit of ϑ when $h \to 0$.

Exercise

Show the following inequalities:

(a) $\ln(1 + x) \leq x$, for any $x \geq 0$;

(b) $x + x^3 \leq 3 \leq \tan(x)$, for any $x \in (0, \frac{\pi}{2})$;

Exercise

Let $f(x)$ be a continuous function on $[a, b]$, differentiable on (a, b), and $f(x) = 0$

for any $x \in (a, b)$.

a) Show that $f(x)$ is one-to-one.

b) Then show that $f(x) > 0$ for every $x \in (a, b)$,

or $f(x) < 0$ for every $x \in (a, b)$.

c) Deduce from this that $f(x)$ satisfies the Intermediate Value Theorem without use of any continuity of $f(x)$.

Exercise

Let $f : [0, 1] \rightarrow R$ be continuous and differentiable inside $(0, 1)$ such that

(i) $f(0) = 0$,

(ii) and there exists $M > 0$ such that

$$|f'(x)| \leq M|f(x)|, \text{ for } x \in (0, 1).$$

Show that $f(x) = 0$ for $x \in [0, 1]$.