Properties of Real Numbers

Associativity with respect to Addition

(P1) If a, b and c are any numbers, then

$$a + (b + c) = (a + b) + c$$

Additive Neutral Element

(P2) If a is any number, then

$$a + 0 = 0$$

$$0 + a = a.$$

Additive Inverse

(P3) For every number a, there is a number -a such that

$$a + (-a) = 0 = (-a) + a$$

Commutativity with respect to Addition

(P4) If a, b are any numbers, then

$$a + b = b + a$$

Associativity with respect to Multiplication

(P5) If a, b and c are any numbers, then

$$a \cdot (b \cdot c) = (a \cdot b) \cdot c$$

Multiplicative Neutral Element

(P6) If a is any number, then there exists a number $1 \neq 0$ such that

$$a \cdot 1 = 1 \cdot a = a$$

Multiplicative Inverse

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(P7) For every number $a \neq 0$, there is a number a^{-1} such that

$$a \cdot a^{-1} = a^{-1} \cdot a = 1$$

Since division is defined in terms of multiplication, division by 0 is undefined.

Consequences

- If $a \neq 0$ and $a \cdot b = a \cdot c$, then $b = c$.
  If $a \cdot b = 0$, then $a = 0$ or $b = 0$.

Commutativity with respect to Mutiplication

(P8) If a, b are any numbers, then

$$a \cdot b = b \cdot a$$

Distributive Law

If a, b and c are any numbers, then

$$a \,.\, (b + c) = a \,.\, b + a \,.\, c$$

Consequence

$(-a) . (-b) = a . b$

Positive Numbers

Suppose the numbers greater than 0 are called positive, i.e. they belong to the collection P of positive numbers.

- (P10) For every number a, one and only one of the following holds:
(i) $a = 0$,
(ii) a is in P,
(iii) $-a$ is in P.

(P11) If a, b are in P, then $a + b$ is in P.

(P12) If a, b are in P, then a b is in P:

Def: The absolute value $|a|$ of a:

$$|a| = a \text{ if } a \geq 0$$
$$|a| = -a \text{ if } a < 0$$

Triangle inequality

Theorem 1. For all numbers a, b, we have

$$|a + b| \le |a| + |b|.$$

Archimedian Property

The reals have one further property, the least upper bound property, introduced two chapters further on.

Exercise

Find all numbers *x* for which

$$|x+5| < 2.$$

Exercise

Find all numbers x, for which

$|x - 1| + |x - 2| > 1$

Exercise

- Find the points in (x,y) in x such that

$$|x + 1| - |y - 2| < 4$$

Draw a graph, shading the area in x which satisfies the inequality.

Exercise

Express without absolute value signs, treating various cases separately, when necessary:

$$|2 - |x||$$