Numbers of Various Sorts

Natural Numbers

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- Do not satisfy (P3): no additive inverse:

  $-a \notin \mathbb{N}$

Natural Numbers

$\mathbb{N} = \{1, 2, 3, \ldots\}$

- Have the property that every nonempty subset has a smallest element

  Well Ordering Principle,

  which leads to:

Finite Induction

Suppose P(x) means that property P is true for x.

If

1. P(1), i.e. P is true for 1 and
2. For k $\in \mathbb{N}$, P(k) $\Rightarrow$ P(k+1)

Then P(n), for all n $\in \mathbb{N}$.

We will show that

Finite Induction $\Leftrightarrow$ ***Well Ordering Principle***.

Complete Induction

Suppose P(x) means that property P is true for x.

If

1. P(1) and
2. For k $\in \mathbb{N}$,

   (P(1) and P(2)and … and P(k)) $\Rightarrow$ P(k+1)

Then P(n), for all n $\in \mathbb{N}$.

We will show that

Finite Induction $\Leftrightarrow$ ***Complete Induction***

$\Leftrightarrow$ ***Well Ordering Principle***.

Example: Finite Induction

- Prove that
- $$1+2+\ldots+n=n(n+1)/2$$

Factorization into Primes

$p \in \mathbb{N}$ is a **prime** if

$$(p = ab,\ a, b \in \mathbb{N}) \Rightarrow (a = 1 \text{ or } b = 1)$$

Thus:

$n \in \mathbb{N}$ not a prime

$\Rightarrow n$ is a ***composite number***, i.e.

$(n = ab$ and $1 < a, b < n$ and $a, b \in \mathbb{N})$

Example: Complete Induction

Prove, that if $n \in \mathbb{N}$, n not a prime, then n is a product of primes.

Theorem:

Finite Induction $\Leftrightarrow$ Well Ordering Principle

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Theorem:

Finite Induction $\Leftrightarrow$ Complete Induction

Proof components:

"$\Leftarrow$" is obvious

"$\Rightarrow$" needs work

Binomial Coefficients

For $n, k \in \mathbb{N}$, $n, k \geq 1$, if we select k elements at random from $\{1, 2, \ldots, n\}$ without replacement and without order. There are

"n choose k" $= C_n^k$

such choices.

$$C_n^k = n!/(k!(n-k)!) \text{ and } C_n^0 = 1$$

Exercise

Show that

$$C_{n+1}^{k} = C_{n}^{k} + C_{n}^{k-1}$$

You won't need induction for this.

Exercise

Use this to prove by induction that

$$C_n^0 + C_n^1 + \ldots + C_n^k + \ldots + C_n^n = 2^n .$$

Then prove it without induction (use the Binomial Theorem).

Binomial Theorem

Let a, b be any numbers, $n \in \mathbb{N}$, then

$(a+b)^n = a^n + C_n^1 a^{n-1} b + \ldots + C_n^k a^{n-k} b^{k+\ldots+} b^n$

Do proof

Geometric Series

Prove, that for $r \neq 1$,

$$1 + r + r^2 + r + \ldots + r^{n-1} = (1-r^n)/(1-r)$$

Do proof.

Natural Numbers

$\mathbb{N}=\{1, 2, 3, \ldots\}$

no additive neutral element,

no additive inverses.

Integers

$\mathbb{Z} = \{\ldots -3, -2, -1, 0, 1, 2, 3, \ldots\}$

additive neutral element,

additive inverses,

multiplicative neutral element,

but no multiplicative inverses.

Rational Numbers

$\mathbb{Q}=\{ p/q \mid p, q \ \mathbb{N} \}$

additive neutral element,

additive inverses,

multiplicative neutral element,

multiplicative inverses,

but only finitely many non zero decimal digits

Real Numbers

$\mathbb{R}$: any number with decimal digits 0,…,9

additive neutral element,

additive inverses,

multiplicative inverses,

also infinitely many decimal digits.

This includes e and π, both of which are *irrational*, i.e. they cannot be expressed as fractions.

Theorem: $\sqrt{2}$ is irrational.

Do proof.

Theorem: $\sqrt{2}$ is irrational.

We don't know yet whether $\sqrt{2}$ exists.

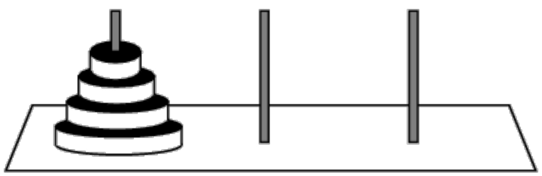
That will come out of the continuity of the $\sqrt{x}$ function.

In order to show this, we need the concept of the ***least upper bound property*** of real numbers.

Theorem: $\sqrt{2} + \sqrt{6}$ is irrational

Towers of Hanoi

Move the disks from pole 1 to pole 3, so if disk a lies on top of disk b, then disk a is smaller than disk b.



Prove that the entire stack of n disks can be moved in

$$2^n - 1$$

moves.