Functions

Provisional definition: a function is a rule which assigns to each of certain real numbers some other real number.

Polynomial

- $f(x) = x^2$ all x

Rational Function

- g(y) = y^3+3y+5/y^2+1 all y

Rational Function

- h(c) = $c^3+3c+5/(c^2-1)$ all c, c ≠ $\underline{+}1$

Polynomial with restricted domain

- r(x) = x^2 $-17 \leq x \leq \pi/3$

Function examples

- $s(x)$={0 if x is rational and 1 otherwise}

A *function*

is a **collection** of pairs of numbers such that if *(a,b), (a,c)* are both in the collection, then $b = c$.

Domain and Range

If f is a function, then the ***domain*** A of f is the set of all a such that there is a b such that (a,b) is in f. Thus b is unique.

We write $f(a) = b$.

Domain and Range

b is in the ***range*** B of the function *f*, if for all (*a*,*b*) in *f*, *b* is in *B*. (There could be more in B.)

If *A* is the domain of *f*, *B* the range of *f*,

we say

$$f: A \rightarrow B.$$

Theorem:

If f, g are functions

then

$f \pm g$, $f \cdot g$, f/g, $f \circ g$ are functions

under suitable restrictions of the domain.

Exercise

Suppose $f(x) = 1/x-1$,

and $g(x) = 1/x-2$,

find the domain of $f + g$.

Exercise

If $g(x) = x^2$

and $h(x) = \{0$ if x is rational and 1 otherwise$\}$

What is *g(h(z)) – h(z)?*

The function f: $A \to B$ is ***one to one (1-1)***,
if $f(a) = b = f(a')$ implies that a = a'.

The function f: $A \to B$ is ***onto***,
if for each $f(a) = b = f(a')$ implies that a = a'.

The function f: $A \to B$ is a ***bijection***,
if f is both one to one and onto.

Exercise

Suppose $f : A \to B$ and $g : B \to C$ are functions, show that

a) If both f and g are one-to-one, then $g \circ f$ is one-to-one.

b) If both f and g are onto, then $g \circ f$ is onto.

c) If both f and g are bijections, then $g \circ f$ is a bijection.

Let $f : X \to Y$ be a function, and $A \subset X$ and $B \subset Y$ are subsets.

The ***image of A under f***, $f(A)$ is defined as

$$f(A) = \{f(x) \in Y : x \in A\}.$$

The ***inverse image of B under f***

or pre-image of B, $f^{-1}(B)$ is defined as

$$f^{-1}(B) = \{x \in X : f(x) \in B\}.$$

Exercise:

For an arbitrary function $f : X \to Y$, prove that the following relations hold:

a)

$f(A_1 \cup A_2 \cup \ldots \cup A_n) = f(A_1) \cup f(A_2) \cup \ldots \cup f(A_n)$

b)

$f(A_1 \cap A_2 \cap \ldots \cap A_n) \subseteq f(A_1) \cap f(A_2) \cap \ldots \cap f(A_n)$

c) Give a counterexample to show that in b) equality is not always true.

Exercise:

For a function $f : X \to Y$, show that the following statements are equivalent:

a) f is one-to-one.

b) $f(A \cap B) = f(A) \cap f(B)$ holds for all $A, B \subseteq X$.

Exercise

For which numbers a, b, c, d will the function

$$f(x) = ax + (b/c)x + d$$

satisfy f(f(x)) = x for all x?

Exercise

f a function.

f is ***even*** if $f(x) = f(-x)$ for all x.

f is ***odd*** if $f(x) = -f(-x)$ for all x.

Fill in the table.

f + g	f even	f odd
g even		
g odd		

Similarly for $f \cdot g$ and $f \circ g$.

Exercise

If f satisfies $f(x+y) = f(x) + f(y)$ for all x, y,
Show that

a) $f(x_1+x_2+\ldots x_n) = f(x_1) + f(x_2) + \ldots f(x_n)$,

b) There exists a c such that $f(x) = cx$ for all rational x.

Exercise

Suppose that $f\colon A \to B$, and $g, h\colon B \to A$

$$h \circ f = I_A = \text{identity on } A,$$

and

$$f \circ g = I_B = \text{identity on } B,$$

then $g = h$.

Exercise

Find a function f, such that for some function g

$g \circ f =$ identity on the domain of f,

but there is no function h, such that

$f \circ h =$ identity on the domain of h.

In other words, f has a left inverse but no right inverse.

f *bounded*

A function f on an interval (a, b) is said to be ***bounded*** if there exists a number M such that

$$|f(x)| \leq M$$

for all x in the interval (a, b).

Exercise

Show that the function

$$f(x) = 1/x$$

is not bounded on the interval $(0,1)$.

Exercise

A function f which is defined on the interval [0,3] is known to be bounded on [0,1] and on (2,3].

Must f be bounded on the interval [0,3]?

Example…