Limits of Functions

The concept of a ***limit***

- is the basis for the definition of the **derivative** and the **integral**
- was developed about 100 years after **calculus** was discovered by **Newton** and **Leibniz**.
- was ultimately **defined** by **Cauchy** after many preliminary attempts that realized parts of the concept.

Definition of Limit

- The function f ***approaches the limit L near a***

means:
for every $\epsilon > 0$, there exists a $\delta > 0$
such that for all x, satisfying $0 < |x-a| < \delta$,
we have $|f(x) - L| < \epsilon$.

The limit in logical symbols

- $\forall\, \epsilon > 0\ \ \exists\, \delta > 0$, such that $\forall\, x$:
$0 < |x\text{-}a| < \delta \Rightarrow |f(x) - L| < \epsilon$.

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Negation:

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- $\forall \epsilon > 0 \;\; \exists \delta > 0$, such that $\forall x$:
  $0 < |x-a| < \delta \Rightarrow |f(x) - L| < \epsilon$.

Negation:

$\exists \epsilon > 0 \;\; \forall \delta > 0,\; x$: even though

$0 < |x-a| < \delta$ still we get $|f(x) - L| \geq \epsilon$.

The negation of the limit of f

$\exists \epsilon > 0 \ \ \forall \ \delta > 0,\ \ x$: even though
$0 < |x-a| < \delta$ still we get $|f(x) - L| \geq \ \epsilon$.

It means:
There exists an $\epsilon > 0$, such that for any $\delta > 0$, I can find an x, which will satisfy $0 < |x-a| < \delta$,
and yet $|f(x) - L| \geq \epsilon$.

Exercise

Show that $f(x) = x^2$ approaches 9 near x=3.

Theorem 1

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A function cannot approach two limits near a. In order words, if f approaches l near a and f approaches m near a, then l=m.

Lemma 1(a)

If $|x - x_0| < \epsilon/2$ and $|y - y_0| < \epsilon/2$ then $|(x+y) - (x_0+y_0)| < \epsilon$.

Lemma 1(b)

- If

$$|x - x_0| < \min\left(1, \frac{\varepsilon}{2(|y_0| + 1)}\right) \quad \text{and} \quad |y - y_0| < \frac{\varepsilon}{2(|x_0| + 1)},$$

then

$$|xy - x_0 y_0| < \varepsilon.$$

Lemma 1(c)

- If y_0 is different from 0 and

$$|y - y_0| < \min\left(\frac{|y_0|}{2}, \frac{\varepsilon |y_0|^2}{2}\right),$$

- Then y is different from 0 and

$$\left|\frac{1}{y} - \frac{1}{y_0}\right| < \varepsilon.$$

Theorem 2

- If $\lim_{x \to a} f(x) = l$ and $\lim_{x \to a} g(x) = m$, then

$$\text{(1)} \quad \lim_{x \to a} (f + g)(x) = l + m;$$

$$\text{(2)} \quad \lim_{x \to a} (f \cdot g)(x) = l \cdot m.$$

Moreover, if $m \neq 0$, then

$$\text{(3)} \quad \lim_{x \to a} \left(\frac{1}{g}\right)(x) = \frac{1}{m}.$$