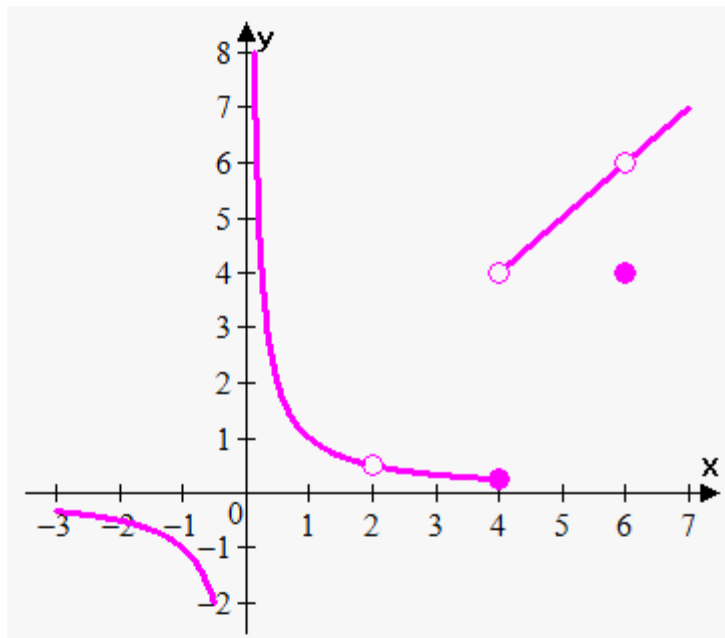
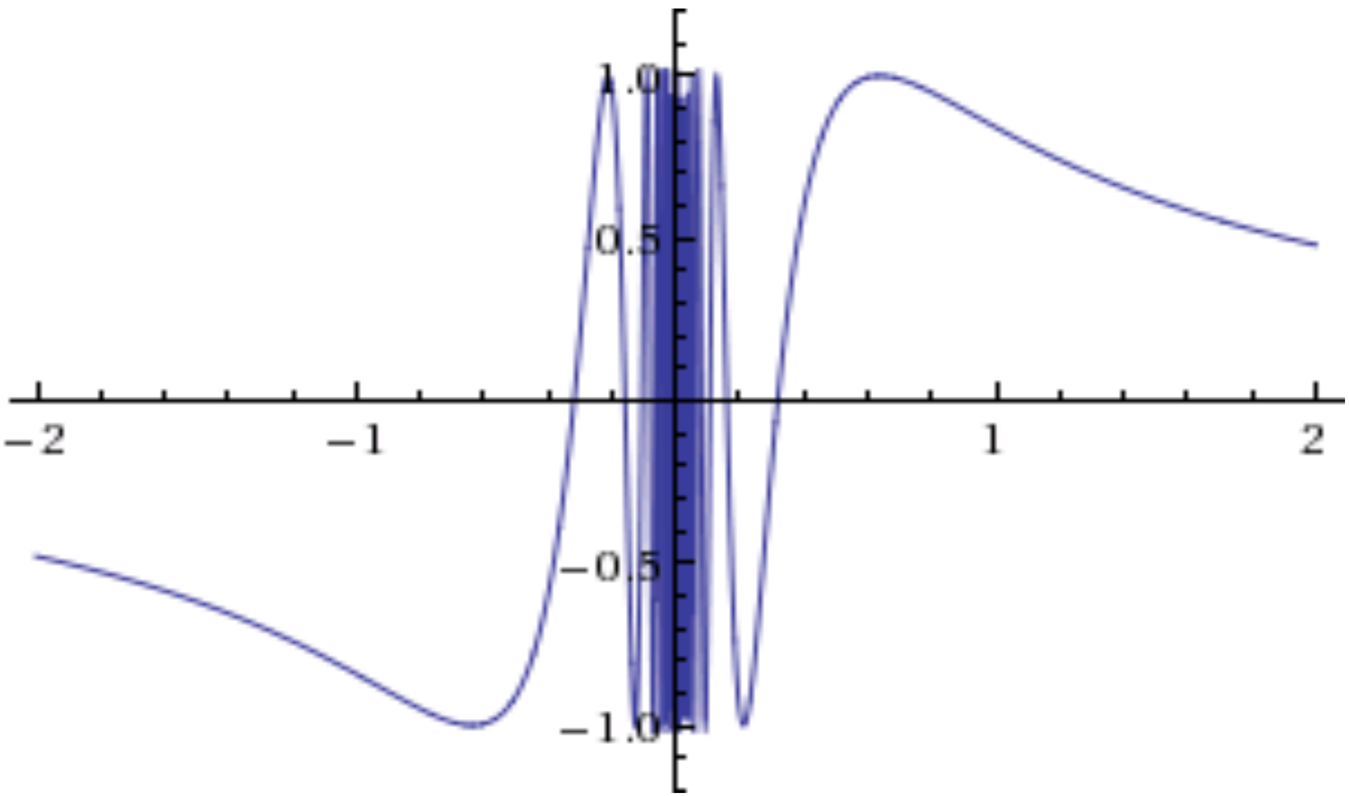
Continuous Functions

Types of Discontinuities



Types of Discontinuities



f continuous at a

iff (if and only if)

$\lim_{x \to a} f(x) = f(a)$

i.e.

for $\varepsilon > 0$, there is a $\delta > 0$ such that for all x satisfying $|x - a| < \delta$, we have

$$|f(x) - f(a)| < \varepsilon$$

Exercise 1

(i) Show that $f(x) = \begin{cases} 0 & x = 0 \\ \sin(1/x) & x \neq 0 \end{cases}$

is not continuous at 0.

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(i) Show that $f(x) = \begin{cases} 0 & x=0 \\ \sin(1/x) & x\neq 0 \end{cases}$

is not continuous at 0.

(ii) Show that $f(x) = \begin{cases} 0 & x=0 \\ x\sin(1/x) & x\neq 0 \end{cases}$

is continuous at 0.

Exercise 2

Show that f(x) = $\{$ 0 x irrational x x rational

is continuous at 0, but nowhere else.

Exercise 3

Give an example of a function f, such that f is nowhere continuous, but $|f|$ is continuous every where.

Exercise 4

Suppose that f is a function satisfying

$|f(x)| \leq |x|$ for all x.

Show that f is continuous at 0.

Exercise 5

Show that

$$f(x) = \begin{cases} 0 & x \text{ irrational} \\ 1/q & x = p/q \text{ in lowest terms} \end{cases}$$

is continuous for irrational x.

Theorem 1

If f and g are continuous at a then

(i) $f+g$ is continuous at a

(ii) $f \cdot g$ is continuous at a

(iii) $1/g$ is continuous at a, provided $g(a) \neq 0$.

Exercise 6

Suppose $f(x+y) = f(x) + f(y)$ for all x, y and that f is continuous at 0.

Show that f is continuous at a for all a.

Theorem

If g is continuous at a and f is continuous at g(a),

then $f \circ g$ is continuous at a.

Proof:

Exercise 7

Prove: Suppose that f is continuous at l and $\lim_{x \to a} g(x) = l$, then $\lim_{x \to a} f(g(x)) = f(l)$.

Note: g need not be continuous or even defined at a.

Continuity over an interval

f: $(a,b) \to \mathbb{R}$ is continuous iff f is continuous for all $x \in (a,b)$.

f: $[a,b] \to \mathbb{R}$ is continuous iff

- f is continuous for all $x \in (a,b)$, and
- $\lim_{x \to a+} f(x) = f(a)$, and
- $\lim_{x \to b-} f(x) = f(b)$.

Theorem

Suppose *f* is continuous at a and $f(a) > 0$.
Then there exists a $\delta > 0$ such that $f(x) > 0$ for all x satisfying $|x - a| < \delta$.

If *f* is continuous at a and its function value at *a* is positive, then *f* is positive in an open interval containing *a*.