Three Beautiful Theorems

properties of continuous functions defined on closed and bounded intervals on the real line. Officially, we don't know yet about the "***least upper bound property***" of the real numbers. This will be the topic of chapter 8. But this chapter will use it to show what can be done with it.

Theorem 1
when f has a zero

If f is continuous on $[a,b]$ and $f(a) < 0 < f(b)$, then there is an x in (a,b) such that $f(x) = 0$.

Proof: will have to wait until we have learnt about the ***least upper bound property*** (lub) of the real numbers in chapter 8.

Consequences:

Theorem 1 is a special case of the ***Intermediate Value Theorem***.

Theorems 4,5, and 9 follow from Theorem 1.

Are the conditions tight?

Do we need the interval $[a,b]$?

find an example where f is continuous on [a,b) and the conclusion of the theorem doesn't hold.

Do we need f to be continuous?

Are the conditions tight?

Do we need the interval $[a,b]$? yes

Consider $f(x) = \begin{cases} 0 & x \in [0,1) \\ 1 & x = 1 \end{cases}$

Do we need f to be continuous? yes

same example.

Theorem 2
f is bounded on $[a,b]$

Suppose f is continuous on $[a,b]$, then

(i) f is bounded above

$\exists M$ such that $\forall\ x \in [a,b]$: $f(x) \leq M$

(ii) f is bounded below

$\exists N$ such that $\forall\ x \in [a,b]$: $f(x) \geq N$

Theorem 2: f is bounded on $[a,b]$

Suppose f is continuous on $[a,b]$, then

(i) f is bounded above

$\exists M$ such that $\forall\ x \in [a,b]: f(x) \leq M$

Proof: will be done in Chapter 8.

Theorem 2: f is bounded on $[a,b]$

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(ii) f is bounded below

$\exists N$ such that $\forall x \in [a,b]$: $f(x) \geq N$

Proof of (ii): can you prove (ii) from (i)?

Are the conditions tight?

Do we need the interval $[a,b]$?

Do we need f to be continuous?

Are the conditions tight?

Do we need the interval [*a*,*b*]? yes

Consider $f(x) = \frac{1}{x}$, x $\in$(0,1].

Do we need f to be continuous? yes

Consider $f(x) = \begin{cases} \frac{1}{x}, & x \in (0,1] \\ 0 & x = 0 \end{cases}$

Theorem 3

f takes on its max/min on $[a,b]$

Suppose f is continuous on $[a,b]$, then

(i) f takes on its max on $[a,b]$

$$\exists y \in [a,b] \text{ such that } \forall\, x \in [a,b]: f(x) \leq f(y)$$

(ii) f takes on its min on $[a,b]$

$$\exists y \in [a,b] \text{ such that } \forall\, x \in [a,b]: f(x) \geq f(y)$$

Are the conditions tight?

Do we need the interval to be $[a,b]$?

Do we need f to be continuous?

Are the conditions tight?

Do we need the interval to be [a,b]? yes

Consider $f(x) = \frac{1}{x}$ on (0,1]?

Do we need f to be continuous? yes

Consider $f(x) = \begin{cases} x+1 & -1 \leq x < 0 \\ 0 & x = 0 \\ x-1 & 0 < x \leq 1 \end{cases}$

Theorem 4
Intermediate Value Theorem

If f is continuous on $[a,b]$ and $f(a) < c < f(b)$, then there is an x in (a,b) such that $f(x) = c$.

Proof: try to prove this from Theorem 1.

Theorem 5

Every positive number has a square root.

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Proof: try to prove this.

Exercise:

Show that $f(x) = x^3 - x + 2$ has a real root.

Theorem 9

If *n* is odd, then any equation

$$x^n + an\ _{-\ 1}x^{n-1} + \cdots + a_1 x + a_0 = 0$$

has a root.

Proof: look at the text book. A bit involved.

Theorem 10

If n is even and

$$f(x) = x^n + an\_{1}x^{n-1} + \cdots + a_1 x + a_0$$

then there is a number y such that

$$f(y) = f(x)$$

for all x.

Proof: see text, similar in style to proof of Theorem 9.

Theorem 11:

Consider the equation:

$$(*) \quad x^n + an \, \_ \, {}_1x^{n-1} + \cdots + a_1x + a_0 = c, n \text{ even}$$

then there exists an M such that (*) has a solution for $c \geq M$.

Exercise:

Define

$$f(x) = \begin{cases} \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Show that

(i) f is not continuous on [-1,1].

(ii) f satisfies the conclusion of the Intermediate Value Theorem.