Least Upper Bounds

where we prove Theorems 1 and 2
from the last chapter.

Definitions

A set A of real numbers is ***bounded above***, if there exists a number x, such that $x \geq a$, for all $a \in A$.

x is an ***upper bound*** for A.

x is a ***least upper bound, supremum, sup(A)*** for A, if

(i) x is an upper bound for A, and

(ii) if y is an upper bound for A, then $x \leq y$.

Similarly, $z = \inf A =$ infimum of A

$= $ greatest lower bound of A

Note: neither the supremum nor the infimum have to be in A.

Exercise

Find the least upper bound u of $A = [0,1) \cap \mathbb{Q}$ in $\mathbb{R}$.

Is u in A?

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Consider the sequence defined recursively by

$$x_1 = \sqrt{2}$$
$$x_{n+1} = \sqrt{2 + x_n}$$

Show, by induction that $x_n < 2$, for all n.

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$$x_1 = \sqrt{2}$$
$$x_{n+1} = \sqrt{2 + x_n}$$

Show, by induction that $x_n < x_{n+1}$, for all n.

Note:

Example of a set that is bounded above but doesn't have a least upper bound: $\emptyset$, the empty set.

Any real number is an upper bound.

The Least Upper bound Property of Real Numbers

If A is a set of real numbers, $A \neq \emptyset$, and A bounded above, then A has a least upper bound.

Note:

As we have seen, $\mathbb{Q}$ does not have the least upper bound property.

Exercise

Suppose A, B are nonempty sets of real numbers, such that $a \leq b$, for all $a \in A$, $b \in B$.

(i) Prove $\sup A \leq b$ for all $b \in B$.

(ii) Prove $\sup A \leq \inf B$.

Proof of Theorem 7.1

If f is continuous on $[a,b]$ and $f(a) < 0 < f(b)$, then there is an x in (a,b) such that $f(x) = 0$.

Proof idea:

find the smallest $x \in [a,b]$, such that $f(x) = 0$.

Question

In the proof of Theorem 7.1, we showed that there exists a smallest x in (a,b) with $f(x) = 0$.

Is there necessarily a second smallest x in (a,b) with $f(x) = 0$?

Exercise

Show that there is a largest x in (a,b) with

$$f(x) = 0.$$

Exercise

Given f continuous, nonnegative on $[a,b]$,

$f(a) = f(b) = 0$ and there exists an x_0, such that $f(x_0) > 0$.

Then there exist c, d, such that

$$a \leq c < x_0 < d \leq b \text{ and}$$

$f(c) = f(d) = 0$ and for all x in (c,d): $f(x) > 0$.

Theorem 8.1

If f is continuous at a, then there exists $\delta>0$ such that f is bounded above on $(a-\delta, a+\delta)$.

Proof: this follows from the continuity of f at a.

Try it.

Similarly:

- f bounded below on a suitable $(a\text{-}\delta, a+\delta)$.
- f bounded above and below if a is the endpoint of a closed interval and f continuous at a.

Theorem 8.2
Archimedean Property

$\mathbb{N}$ is not bounded above.

Proof: (by contradiction)

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Archimedean Property

$\mathbb{N}$ is not bounded above.

Proof: (by contradiction) Suppose not.

Since $\mathbb{N}$ is not empty and bounded above, there is a least upper bound α.

So $\alpha \geq n$, for all $n \in \mathbb{N}$, in particular

$\alpha \geq n+1$. ($n+1 \in \mathbb{N}$, if $n \in \mathbb{N}$)

Thus, α-1 $\geq n$, for all $n \in \mathbb{N}$ and α is not a least upper bound. Contradiction

Corollary

For every $\varepsilon > 0$, there exists an $n \in \mathbb{N}$, such that

$$\frac{1}{n} < \varepsilon.$$

Proof: by contradiction.

Try it.

We have actually used this many times before, when dealing with limits.

Definition

A set of real numbers A is ***dense***, if every open interval contains an element of A.

Exercise

$\mathbb{Q}$ is dense in $\mathbb{R}$.

Proof: Consider the open interval (x,y), $x < y$.
Need to show that there exists $q \in \mathbb{Q}$, such that
$x < q < y$.
Try it.

Nested Interval Theorem

Consider a sequence of closed intervals
$I_1=[a_1,b_1]$, $I_2=[a_2,b_2]$,…, where
$a_1 \leq \ldots \leq a_n \leq \ldots \leq b_m \leq \ldots \leq b_{1.}$
Prove that there is a point x which is in every I_n.

Also show that this conclusion is false for open intervals.

Exercise

Suppose f is continuous on [a,b], then either

- $f\left(\frac{a+b}{2}\right) = 0$
- $f(a) < 0 < f\left(\frac{a+b}{2}\right) \qquad (*)$
- $f\left(\frac{a+b}{2}\right) < 0 < f(b)$

Suppose (*). Apply the same argument to

$[a, \left(\frac{a+b}{2}\right)]$.

Claim: This process will lead to $x \in [a,b]$ such that $f(x)=0$.

Definition

The point x is an ***accumulation point*** of the set A, if for any open set O containing x, there are infinitely many points from A in O.

x doesn't have to be in A.

Exercises

Show: Every point in [0,1] is an accumulation point.

Show: No point at all is an accumulation point of the set of natural numbers $\mathbb{N}$.

Show: every point on the real line, both rational and irrational, is an accumulation point of the set $\mathbb{Q}$.

Exercise (additional)

Show: very point of a closed interval [a, b] is an accumulation point of (a, b). No point outside can be.

Definitons

Suppose A is a set of real numbers.

The ***limit superior*** = $\overline{\lim}\ A$ = largest accumulation point of *A.*

The ***limit inferior*** = $\underline{\lim}\ A$ = smallest accumulation point of *A.*

Execises

Find the limits superior and inferior, minima, maxima, if they exist, of:

1. $A = \left\{ \frac{1}{n} \middle| n \in \mathbb{N} \right\}$
2. $A = \{ x | 0 \leq x \leq \sqrt{2}, x \text{ rational} \}$
3. $A = \left\{ \frac{1}{n} + (-1)^n \middle| n \in \mathbb{N} \right\}$

Uniform continuity

this measures how much a function oscillates i.e. how fast a function grows locally (ultimately its slope).

Example

$f(x) = x^2$ is continuous, i.e it is continuous at each a :

$$\forall a \in \mathbb{R}, \varepsilon > 0, \exists\ \delta_{a\varepsilon} > 0,$$

such that $\forall x$ satisfying $|x - a| < \delta_{a\varepsilon}$

$$|f(x) - f(a)| < \varepsilon$$

here the δ depends on a and on ε.

Draw picture.

We aim for a function where the δ depends only on on ε, and not on a.

Definition

f is ***uniformly continuous*** on an interval *I*, if for every $\varepsilon > 0$, there is a $\delta_\varepsilon > 0$,
such that for all *x,y* ∈ *I*, if $|x - y| < \delta$, then
$|f(x) - f(y)| < \varepsilon$.

Are they uniformly continuous?

1. $f(x) = \frac{1}{x}$ on $(0,1)$
2. $f(x) = \frac{1}{x}$ on $(\frac{1}{n},1)$
3. $f(x) = \sin x$ on $\mathbb{R}$
4. $f(x) = x^3$ on $[-100,100]$
5. $f(x) = x$
6. $f(x) = e^x$
7. $f(x) = e^{-x}$ on $\mathbb{R}^+$

Lemma

If f is uniformly continuous on $[a,b]$ and on $[b,c]$, then f is uniformly continuous on $[a,c]$.

Technically:

Lemma

Let $a < b < c$ and f continuous on $[a,c]$.

Let $\varepsilon > 0$ and suppose

(i) $\exists\delta_1$ such that if $x, y \in [a, b]$ and $|x - y| < \delta_1$, then $|f(x) - f(y)| < \varepsilon$

(ii) $\exists\delta_2$ such that if $x, y \in [b, c]$ and $|x - y| < \delta_2$, then $|f(x) - f(y)| < \varepsilon$

then

$\exists\delta > 0$ such that if $x, y \in [a, c]$ and $|x - y| < \delta$, then $|f(x) - f(y)| < \varepsilon$.

Theorem 8.3

If f is continuous on [a,b], then f is uniformly continuous on [a,b].

Theorem 7.2

If f is continuous on [a,b], then f is bounded on [a,b].

Proof: Take $\varepsilon = 1$, then there exists a $\delta > 0$, such that for any x, y satisfying $|x - y| < \delta$, then $|f(x) - f(y)| < 1$. Cover $[a,b]$ with overlapping subintervals

$$[a, a+\delta), (a+\frac{\delta}{2}, a+\frac{3\delta}{2}), \ldots, (a+\frac{(2n+1)\delta}{2}, b]$$

There are n such intervals. Over each of them f varies by at most 1, so altogether $|f(x) - f(y)| < n$

for any x, y in $[a,b]$.

Theorem 7.3

If f is continuous on $[a,b]$, then there are numbers $y, z \in [a,b]$ such that

for all x $\in [a,b]$: $f(x) \leq f(y)$ and

for all x $\in [a,b]$: $f(x) \geq f(z)$

Proof: see text